CHAPTER 8

NATURAL AND STEP RESPONSES OF RLC CIRCUITS

CONTENTS

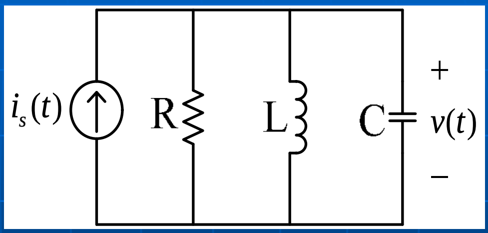
8.1 Linear Second Order Circuits

- Circuits containing two energy storage elements.
- Described by differential equations that contain second order derivatives.
- Need two initial conditions to get the unique solution.

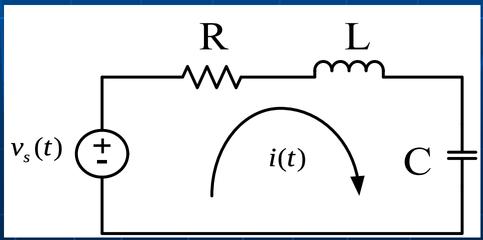
8.1 Linear Second Order Circuits

- Examples

(a) RLC parallel circuit

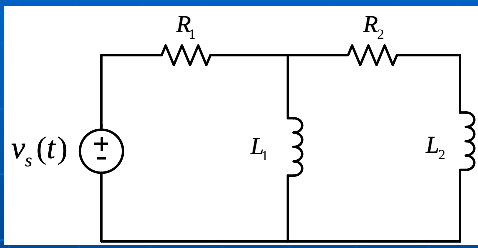


(b) RLC series circuit

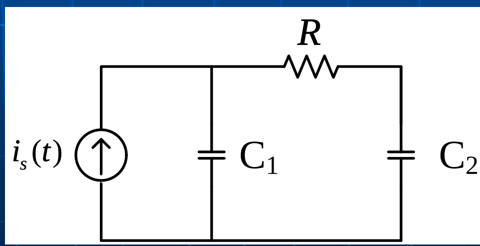


8.1 Linear Second Order Circuits

(c) 2L+R , RL circuit



(d) 2C+R , RC circuit



8.2 Solution Steps

Step1 : Choose nodal analysis or mesh analysis approach

Step2 : Differentiate the equation as many times as required to get the standard form of a second order differential equation .

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+x=y(t)$$

8.2 Solution Steps

Step 3 : Solving the differential equation

(1) homogeneous solution $\boxed{x_h(t)}$

(2) particular solution $\boxed{x_p(t)}$

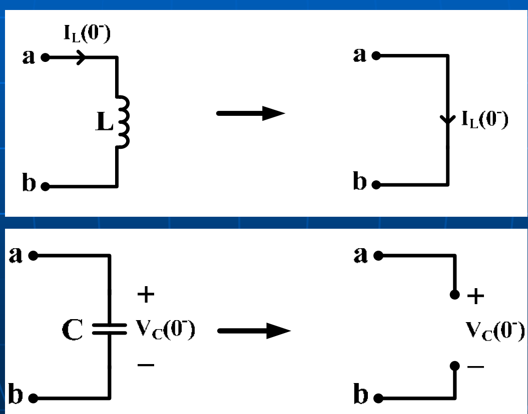
$$\boxed{x(t) = x_h(t) + x_p(t)}$$

Step 4 : Find the initial conditions

$\boxed{x(0^+)}$ and $\boxed{\frac{d}{dt}x(0^+)}$ and then get the unique solution

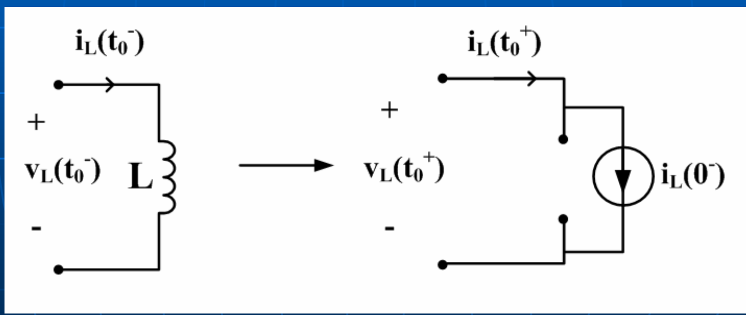
8.3 Finding Initial Values

Under DC steady state, L is like a short circuit and C is like an open circuit.



8.3 Finding Initial Values

Under transient condition, L is like an open circuit and C is like a short circuit because $i_L(t)$ and $v_C(t)$ are continuous functions if the input is bounded.



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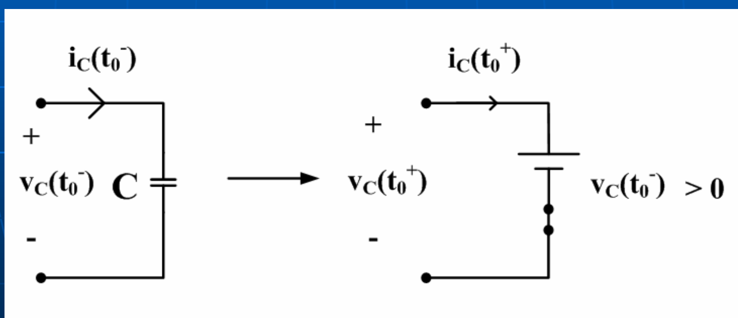


8.3 Finding Initial Values

To find $\frac{di_L(0^+)}{dt}$ and $\frac{dv_C(0^+)}{dt}$, use the following relations

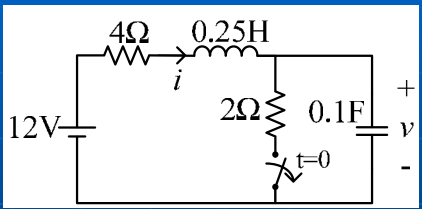
$$L\frac{di_L(0^+)}{dt} = v_L(0^+) \Rightarrow \frac{d}{dt}i_L(0^+) = \frac{v_L(0^+)}{L}$$

$$C\frac{dv_C(0^+)}{dt} = i_c(0^+) \Rightarrow \frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C}$$

One can find $v_L(0^+)$ and $i_C(0^+)$ using either nodal or mesh analysis.

8.3 Finding Initial Values

- Example 1



The circuit is under steady state.
The switch is opened at $t = 0$

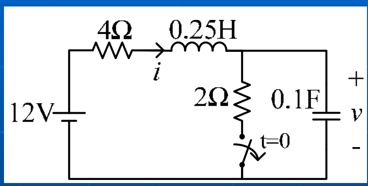
Find : (a) $i(0^+)$, $\frac{d}{dt}i(0^+)$

(b) $v(0^+)$, $\frac{d}{dt}v(0^+)$

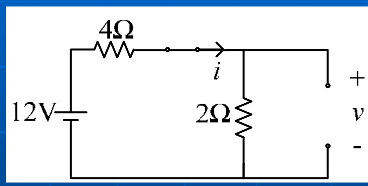
(c) $i(\infty)$, $v(\infty)$

8.3 Finding Initial Values

- Example 1 (cont.)



t < 0



$i(0^-) = 2\ A$

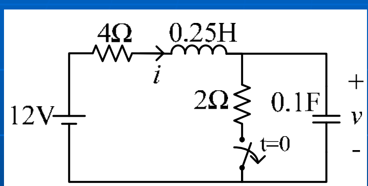
$v(0^-) = 4\ V$

$\therefore\ i(0^+) = i(0^-) = 2\ A$

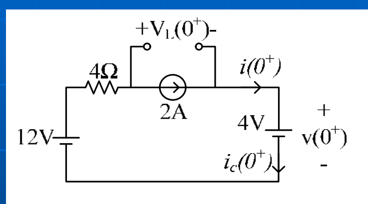
$v(0^+) = v(0^-) = 4\ V$

8.3 Finding Initial Values

- Example 1 (cont.)



$t = 0^+$

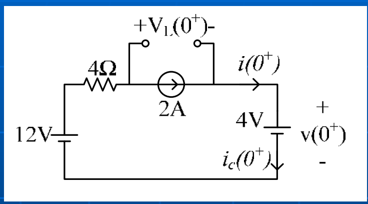


$$\because L\frac{di}{dt} = V_L \quad , \quad \therefore \frac{d}{dt}i(0^+) = \frac{V_L(0^+)}{L}$$

$$\because C\frac{dv}{dt} = i_C \quad , \quad \therefore \frac{d}{dt}v(0^+) = \frac{i_C(0^+)}{C}$$

8.3 Finding Initial Values

- Example 1 (cont.)



$$KVL: \; 2\text{A}\times 4 \; + \; v_L(0^+) + 4\text{V} = 12\text{V}$$

$$\therefore \; v_L(0^+) = 0$$

$$\therefore \frac{d}{dt} i(0^+) = 0$$

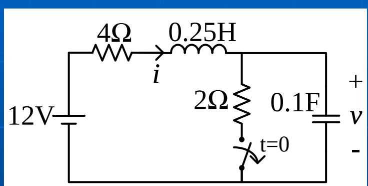
$$KCL: \; i_C(0^+) = 2\text{A}$$

$$\therefore \frac{d}{dt} v(0^+) = 0$$

$$\therefore \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20 \;\; \text{V/S}$$

8.3 Finding Initial Values

- Example 1 (cont.)



$t \to \infty$

L is short circuitd

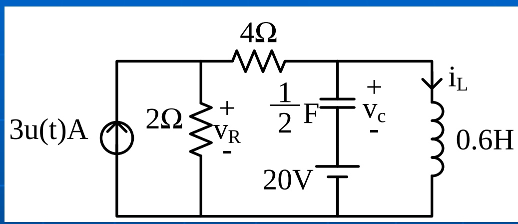
C is open

$$\therefore i(\infty) = 0$$

$$v(\infty) = 12\text{V}$$

8.3 Finding Initial Values

- Example 2



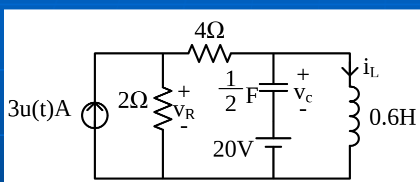
$$Find: \ (a) \ i_L(0^+) \ , \ \frac{d}{dt} i_L(0^+) \ , \ i_L(\infty)$$

$$(b) \ v_C(0^+) \ , \ \frac{d}{dt} v_C(0^+) \ , \ v_C(\infty)$$

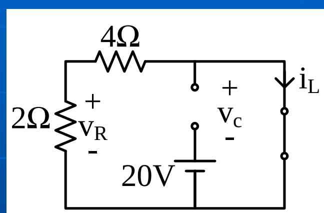
$$(c) \ v_R(0^+) \ , \ \frac{d}{dt} v_R(0^+) \ , \ v_R(\infty)$$

8.3 Finding Initial Values

- Example 2 (cont.)



t < 0



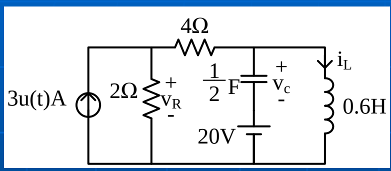
$$\therefore i_L(0^-) = 0$$

$$v_c(0^-) = -20V$$

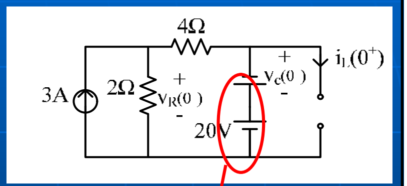
$$v_R(0^-) = 0$$

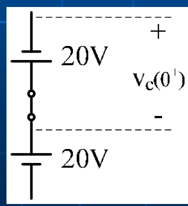
8.3 Finding Initial Values

- Example 2 (cont.)



t = 0^+



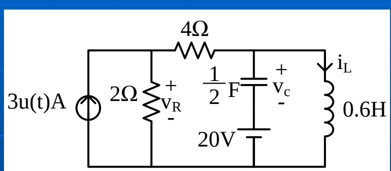


$$\therefore i_L(0^+) = i_L(0^-) = 0$$

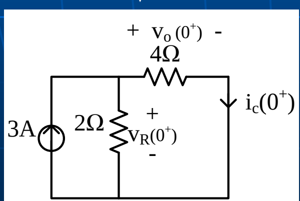
$$v_c(0^+) = v_c(0^-) = -20V$$

8.3 Finding Initial Values

- Example 2 (cont.)



t = 0^+



$$i_{2\Omega}(0^+) = 3A \times \frac{4}{2+4} = 2A$$

$$v_R(0^+) = 2A \times 2 = 4V$$

$$i_c(0^+) = 3A \times \frac{2}{2+4} = 1A$$

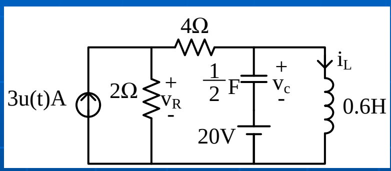
$$\therefore \frac{d}{dt} i_L(0^+) = \frac{v_L(0^+)}{L} = \frac{0}{L} = 0$$

$$\frac{d}{dt} v_c(0^+) = \frac{i_c(0^+)}{c} = \frac{1}{1/2} = 2 \ V/S$$

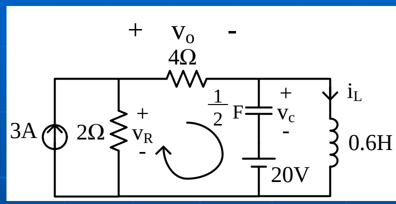
$$\frac{d}{dt} v_R(0^+) = ?$$

8.3 Finding Initial Values

- Example 2 (cont.)



$t > 0^+$



From KVL: $-v_R + v_o + v_c + 20 = 0$

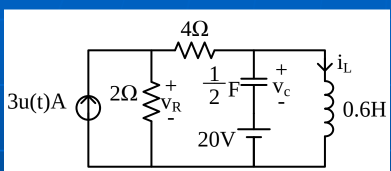
Take derivative

$$\frac{d}{dt}v_R(t) = \frac{d}{dt}v_o(t) + \frac{d}{dt}v_c(t)$$

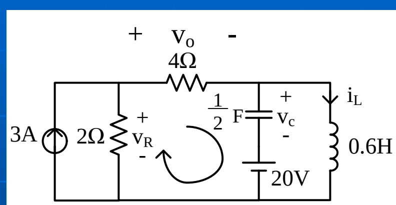
$$\therefore \frac{d}{dt}v_R(0^+) = \frac{d}{dt}v_o(0^+) + \frac{d}{dt}v_c(0^+) \cdots\cdots (A)$$

8.3 Finding Initial Values

- Example 2 (cont.)



$t > 0^+$



Also , from KCL: $3A = \frac{v_R(t)}{2} + \frac{v_o(t)}{4}$

Take derivative

$$0 = \frac{1}{2}\frac{d}{dt}v_R(t) + \frac{1}{4}\frac{d}{dt}v_o(t) \cdots\cdots (B)$$

8.3 Finding Initial Values

- Example 2 (cont.)

$$\text{From } (B) \quad \frac{d}{dt}v_o(0^+) = -2\frac{d}{dt}v_R(0^+) \cdots\cdots (C)$$

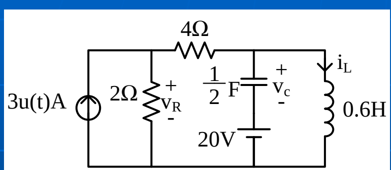
$$\text{From } (A) \text{ and } (C)$$

$$\frac{d}{dt}v_R(0^+) = -2\frac{d}{dt}v_R(0^+) + 2$$

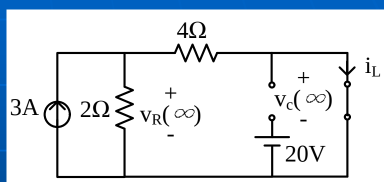
$$\therefore \frac{d}{dt}v_R(0^+) = \frac{2}{3}\ \mathrm{V/S}$$

8.3 Finding Initial Values

- Example 2 (cont.)



$t \longrightarrow \infty$



$$\therefore i_L(\infty) = 3A \times \frac{2}{2+4} = 1A$$

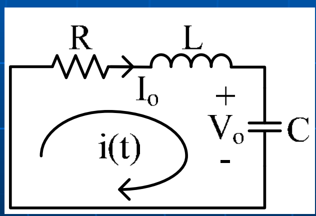
$$v_c(\infty) = -20V$$

$$v_R(\infty) = 3A \times (2\Omega \parallel 4\Omega) = 4V$$

8.4 The Natural Response of a Series/Parallel RLC Circuit

(a) The source-free series RLC circuit

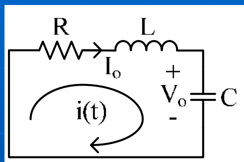
This section is an important background for studying filter design and communication networks .



initial conditions

$$\begin{cases} i(0) = I_0 \\ v(0) = V_0 \end{cases}$$

8.4 The Natural Response of a Series/Parallel RLC Circuit



initial conditions

$$\begin{cases} i(0) = I_0 \\ v(0) = V_0 \end{cases}$$

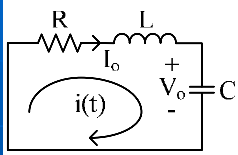
Step 1 : *Mesh analysis*

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i \; dt = 0$$

To eliminate the integral ,take derivative

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = 0$$

8.4 The Natural Response of a Series/Parallel RLC Circuit



initial conditions

$$\begin{cases} i(0) = I_0 \\ v(0) = V_0 \end{cases}$$

Step 2 : Homogeneous solution , characteristic equation

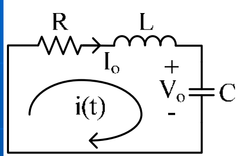
$$S^2 + \frac{R}{L}S + \frac{1}{LC} = 0$$

$$S^2 + 2\alpha S + \omega_0^{\ 2} = 0 \qquad \Rightarrow \alpha @ \frac{R}{2L}, \omega_0^{\ 2} @ \frac{1}{LC}$$

ω_0 : *undamped resonant frequency (rad/s)*

α : *damping factor or neper frequency*

8.4 The Natural Response of a Series/Parallel RLC Circuit



initial conditions

$$\begin{cases} i(0) = I_0 \\ v(0) = V_0 \end{cases}$$

characteristic roots (natural frequencies)

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^{\ 2}}$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^{\ 2}}$$

$$\therefore i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

Need two initial conditions , i.e. , $i(0^+)$ *and* $\frac{d}{di} i(0^+)$.

8.4 The Natural Response of a Series/Parallel RLC Circuit

<u>Case 1</u> *Overdamped Case* $(a > w_0)$

Two real roots

$$i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

<u>Case 2</u> *Critically Damping Case* $(a = w_0)$

Equal real roots

$$S_1 = S_2 = -\frac{R}{2L} = -a$$

$$i(t) = (A_2 + A_1 t)e^{-at}$$

8.4 The Natural Response of a Series/Parallel RLC Circuit

<u>Case 3</u> *Underdamped Case* $(a < w_0)$

Complex conjugate roots

$$S_1 = -a + jw_d$$

$$S_2 = -a - jw_d$$

$$w_d \; @ \; \sqrt{w_0^{\,2} - a^2} \quad \textit{damping frequency}$$

$$i(t) = e^{-at}(B_1 \cos w_d t + B_2 \sin w_d t)$$

Once i(t) is obtained ,solutions of other variables can be obtained from this mesh current.

8.4 The Natural Response of a Series/Parallel RLC Circuit

- The damping effect is due to the presence of resistance R .
- The damping factor α determines the rate at which the response is damped .
- If R=0 , the circuit is said to be lossless and the oscillatory response will continue .
- The damped oscillation exhibited by the underdamped response is known as ringing . It stems from the ability of the L and C to transfer energy back and forth between them .

8.4 The Natural Response of a Series/Parallel RLC Circuit

Step 3 : *Initial Condition*

$$t=0^+,\ i(0^+)=i(0^-)=I_0$$

From mesh equation , let $t=0^+$

$$Ri(0^+)+L\frac{d}{dt}i(0^+)+\frac{1}{C}\int_{-\infty}^{0^+} i dt=0$$

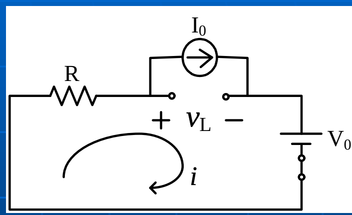
$$\therefore\ \frac{d}{dt}i(0^+)=-\frac{R}{L}i(0^+)-\frac{V_0}{L}=-\frac{R}{L}I_0-\frac{V_0}{L}$$

8.4 The Natural Response of a Series/Parallel RLC Circuit

or from equivalent circuit at $t = 0^+$

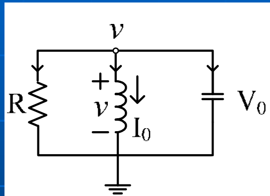
$$L\frac{d}{dt}i(0^+) = v_L = -(I_0 R + V_0)$$

$$\therefore \frac{d}{dt}i(0^+) = -\frac{I_0 R}{L} - \frac{V_0}{L}$$



8.4 The Natural Response of a Series/Parallel RLC Circuit

(b) <u>The source-free parallel RLC circuit</u>



initial inductor current I_o

initial capacitor voltage V_o

Step 1 : Nodal Equation

$$\frac{v}{R} + \frac{1}{L}\int^t v dt + C\frac{dv}{dt} = 0$$

Taking derivative to eliminate the integral

$$\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

8.4 The Natural Response of a Series/Parallel RLC Circuit

Step 2 : Homogeneous solution

Characteristic equation

$$S^2 + \frac{1}{RC}S + \frac{1}{LC} = 0$$

$$S^2 + 2a\,S + w_0^{\ 2} = 0 \qquad \Rightarrow a\ @\ \frac{1}{2RC}\ ,\ w_0^{\ 2}\ @\ \frac{1}{LC}$$

characteristic roots (natural frequencies)

$$S_1 = -a + \sqrt{a^2 - w_0^{\ 2}}$$

$$S_2 = -a - \sqrt{a^2 - w_0^{\ 2}}$$

8.4 The Natural Response of a Series/Parallel RLC Circuit

CASE 1. Overdamped Case ($\alpha > w_0$)

$$v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

CASE 2. Critically Damped Case ($\alpha = w_0$)

$$v(t) = (A_1 + A_2 t)e^{-at}$$

CASE 3. Underdamped Case ($\alpha < w_0$)

$$v(t) = e^{-at}(A_1 \cos w_d t + A_2 \sin w_d t)$$

8.4 The Natural Response of a Series/Parallel RLC Circuit

Step 3 : *Initial Condition*

$$v(0^+) = i(0^-) = V_0$$

From nodal equation

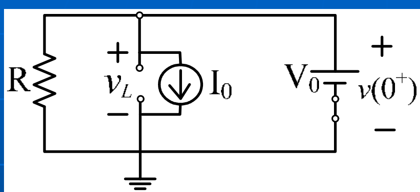
$$\frac{v\left(0^+\right)}{R} + \frac{1}{L}\int_{-\infty}^{0^+} v dt + C\frac{d}{dt}v\left(0^+\right) = 0$$

$$\therefore C\frac{d}{dt}v\left(0^+\right) = -\frac{v\left(0^+\right)}{R} - \frac{1}{L}\int_{-\infty}^{0^+} v dt$$

$$= -\frac{V_0}{R} - I_0$$

$$\therefore \frac{d}{dt}v\left(0^+\right) = -\frac{V_0}{RC} - \frac{I_0}{C}$$

8.4 The Natural Response of a Series/Parallel RLC Circuit



or from equivalent circuit at $t = 0^+$

$$C\frac{d}{dt}v(0^+) = i_C\left(0^+\right)$$

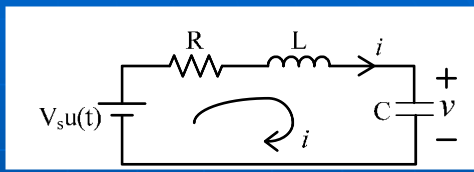
$$\therefore \frac{d}{dt}v(0^+) = \frac{i_C(0^+)}{C}$$

From KCL, $i_C(0^+) = -I_0 - \dfrac{V_0}{R}$

Once the nodal voltage is obtained, any other unknown of the circuit can be found .

8.5 The Step Response of a Series/Parallel RLC Circuit

(a) Step response of *a series RLC circuit*



Step 1. Mesh analysis ($i=i_L$)

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=V_s \quad ,t>0$$

Case(i) take derivative

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{i}{C}=0$$

Same as natural response but with $i(0^+)=I_0$

$$\frac{d}{dt}i\left(0^+\right)=\frac{V_s-I_0R-V_0}{L}$$

8.5 The Step Response of a Series/Parallel RLC Circuit

Case(ii) use v as unknown

$$i=C\frac{dv}{dt}$$

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{v}{LC}=\frac{V_s}{LC}$$

Step 2. Complete solution = v_h+v_p

$$v_p(t)=V_s$$

$$v_h(t)=\begin{cases} A_1e^{S_1t}+A_2e^{S_2t} & \textit{overdamped} \\ (A_1+A_2t)e^{-\alpha t} & \textit{critically damped} \\ e^{-\alpha t}\left(A_1\cos w_dt+A_2\sin w_dt\right) & \textit{underdamped} \end{cases}$$

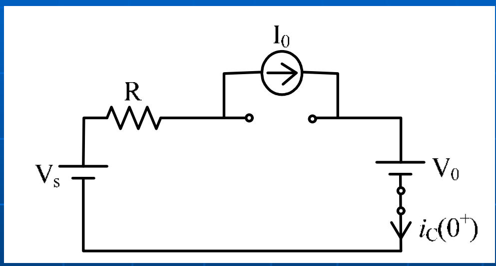
8.5 The Step Response of a Series/Parallel RLC Circuit

Step 3. Initial conditions

$$v(0^+) = v(0^-) = V_0$$

$$\frac{d}{dt}v(0^+) = ?$$

$$C\frac{dv}{dt} = i_C$$

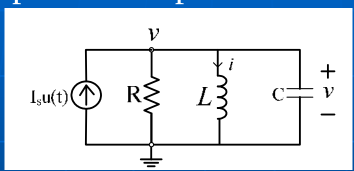


$$\therefore \frac{d}{dt}v(0^+) = \frac{i_C(0^+)}{C} = \frac{I_0}{C}$$

Then the unique solution can be determined

8.5 The Step Response of a Series/Parallel RLC Circuit

(b) Step response of a parallel RLC circuit



I_0 , V_0 Given

Step 1. Nodal equation

$$\frac{v}{R} + \frac{1}{L}\int^t v dt + C\frac{dv}{dt} = I_s \quad ,t > 0$$

Case (i) Take derivative

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

Same as natural response

8.5 The Step Response of a Series/Parallel RLC Circuit

Case (ii) Let $v = L\dfrac{di}{dt} \Rightarrow \dfrac{d^2i}{dt^2} + \dfrac{1}{RC}\dfrac{di}{dt} + \dfrac{1}{LC}i = \dfrac{I_s}{LC} \quad , t > 0$

Step 2. Complete solution = $i_h(t)+i_p(t)$

$$i_p(t) = I_s$$

$$i_h(t) = \begin{cases} A_1 e^{S_1 t} + A_2 e^{S_2 t} & \textit{overdamped} \\ (A_1 + A_2 t)e^{-\alpha t} & \textit{critically damped} \\ e^{-\alpha t}(A_1 \cos w_d t + A_2 \sin w_d t) & \textit{undererdamped} \end{cases}$$

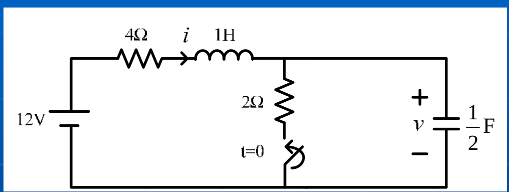
Step 3. Initial Condition

$$i(0^+) = i(0^-) = I_0$$

$$L\frac{di}{dt} = v_L \Rightarrow \frac{d}{dt}i(0^+) = \frac{v_L(0^+)}{L} = \frac{V_0}{L}$$

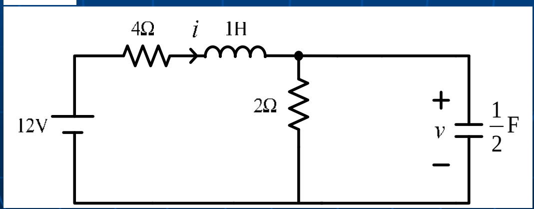
8.5 The Step Response of a Series/Parallel RLC Circuit

- Example



$t < 0 \ , \ i(0^-) = 0 \, , v(0^-) = 12\text{V}$

$t > 0$



8.5 The Step Response of a Series/Parallel RLC Circuit

Method (a)

From KCL $\Rightarrow i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt}$ (A)

From KVL $\Rightarrow 4i + 1\frac{di}{dt} + v = 12$ (B)

Substitute (A) into (B)

$$(2v + 2\frac{dv}{dt}) + (\frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2}) + v = 12$$

$$\Rightarrow \frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 24\text{V}$$

Characteristic equation

$$(s+2)(s+3) = 0$$

$$s_1 = -2, s_2 = -3$$

8.5 The Step Response of a Series/Parallel RLC Circuit

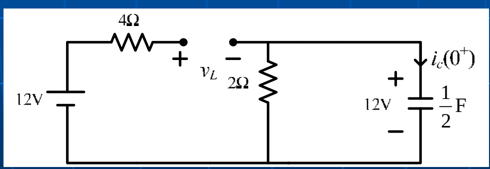
$$v_h(t) = \text{A}_1 e^{-2t} + \text{A}_2 e^{-3t}$$

$$v_p(t) = \frac{24}{6} = 4\text{V}$$

$$\therefore v(t) = 4 + \text{A}_1 e^{-2t} + \text{A}_2 e^{-3t}$$

Initial Condition $v(0^+) = v(0^-) = 12\text{V}$

$t=0^+$



$$C\frac{dv}{dt} = i_c$$

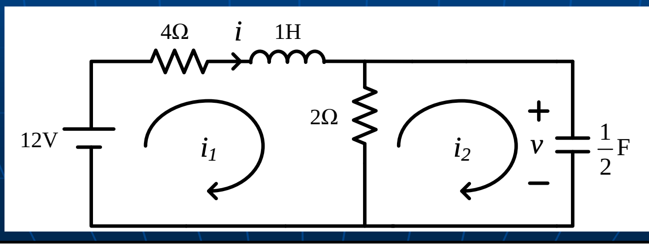
$$\therefore \frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{-6}{1/2} = -12 \text{ V/S}$$

8.5 The Step Response of a Series/Parallel RLC Circuit

$$\therefore 4 + A_1 + A_2 = 12$$
$$-2A_1 - 3A_2 = -12$$
$$\therefore A_1 = 12, \quad A_2 = -4$$
$$\therefore v(t) = 4 + 12e^{-2t} - 4e^{-3t}, t \geq 0$$

Method (b) Using Mesh Analysis

t>0



8.5 The Step Response of a Series/Parallel RLC Circuit

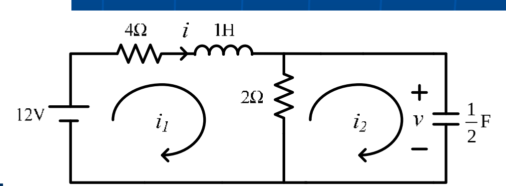
$$1\frac{di}{dt} + 4i + (i_1 - i_2)2 = 12 \qquad ......(A)$$

$$(i_2 - i_1)\times 2\Omega + \frac{1}{1/2}\int^t i_2 dt = 0 \quad ......(B)$$

From (B), $2\frac{di_2}{dt} - 2\frac{di_1}{dt} + 2i_2 = 0$

$$\frac{di}{dt} + 6i_1 - 2i_2 = 12$$

$$-2\frac{di_1}{dt} + 2\frac{di_2}{dt} + 2i_2 = 0$$



8.5 The Step Response of a Series/Parallel RLC Circuit

In matrix form with operator $D \triangleq \dfrac{d}{dt}$

$$\begin{bmatrix} D+6 & -2 \\ -2D & 2D+2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix} \qquad \text{(C)}$$

Initial Condition , $i_1(0^+) = i(0^+) = i(0^-) = 0\text{A}$

$$i_2(0^+) = i_c(0^+) = -6\text{A}$$

$$\frac{di_1(0^+)}{dt} = \frac{v_L(0^+)}{L} = 0$$

$$\frac{di_2(0^+)}{dt} = 0A/S$$

8.5 The Step Response of a Series/Parallel RLC Circuit

Eliminate i_2 variable from (C)

$$\frac{d^2 i_1}{dt^2} + 5\frac{di_1}{dt} + 6i_1 = 12$$

$$\therefore i_1(t) = 2 - 6e^{-2t} + 4e^{-3t}, t \geq 0$$

Or eliminate i_1 variable from (C)

$$\frac{d^2 i_2}{dt^2} + 5\frac{di_2}{dt} + 6i_2 = 0$$

$$\therefore i_2(t) = -12e^{-2t} + 6e^{-3t}, t \geq 0$$

8.5 The Step Response of a Series/Parallel RLC Circuit

Problem : (a) Time consuming to eliminate the other variable to get a higher order differential equation.

(b) It is also necessary to obtain the desired initial conditions.

(c) As the order gets higher when the network contains more energy storage elements, the process gets more complicated.

The difficulty can be overcome by using state equation formulation.

8.6 State Equation

When the differential equations of a circuit is written in the following form:

$$\frac{d}{dt}\underline{x} = f(\underline{x},\underline{u},\underline{t})$$

$$\underline{x}^T = [x_1 \; x_2 \ldots. x_n] \quad \text{state vector}$$

$$\underline{u}^T = [u_1 \; u_2 \ldots. u_m] \quad \text{input vector}$$

$$\underline{f}^T = [f_1 \;\; f_2 .. f_n] \quad \text{vector function}$$

8.6 State Equation

It is said that the circuit equations are in the state equation form.

(a) This form lends itself most easily to analog or digital computer programming.

(b) The extension to nonlinear and/or time varying networks is quite easy.

(c) In this form, a number of theoretic concepts of systems are readily applicable to networks.

8.6 State Equation

For a linear time-invarying circuit , a simpler form

$$\frac{d}{dt}\underline{x} = \mathrm{A}\underline{x} + \mathrm{B}\underline{u} \ , \ \text{state equation}$$

$$\underline{y} = \mathrm{C}\underline{x} + \mathrm{D}\underline{u} \ , \ \text{output equation}$$

A : $n \times n$ matrix.

B : $n \times m$ matrix.

C : $l \times n$ matrix.

D : $l \times m$ matrix.

Note that on the right hand side of the state or output equation, only $\underline{x}$ and $\underline{u}$ are allowed.

8.6 State Equation

Step1. Pick a tree which contains all the capacitors and none of inductors.

Step2. Use the tree-branch capacitor voltages and the link inductor currents as unknown (i.e. , state) variables.

Note: (a) Nodal Analysis

Every unknown of the circuit can be calculated from nodal voltages.

(b) Mesh Analysis

Every unknown of the circuit can be calculated from mesh currents.

8.6 State Equation

(c) State Equation

- *The chosen variables include both voltage and current unknown. It belongs to mixed type.*
- *Every unknown of the circuit can be calculated from the state variables by replacing each inductor with a current source and each capacitor with a voltage source and then solving the resulting resistive circuit.*

8.6 State Equation

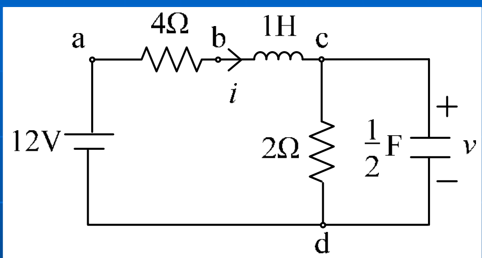
Step3. Write a fundamental cutset equation (i.e. KCL equation) for each capacitor.
Note that in these cutset equations, all branch currents must be expressed in terms of x and u.

Step4. Write a fundamental loop equation (i.e. KVL equation) for each inductor.
Note that in these loop equations, all branch voltages must be expressed in terms of x and u.

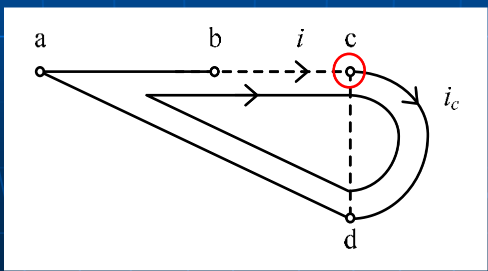
Step5. Rearrange the above equations into standard form and find the solution for the given initial condition.

8.6 State Equation

- Example 1



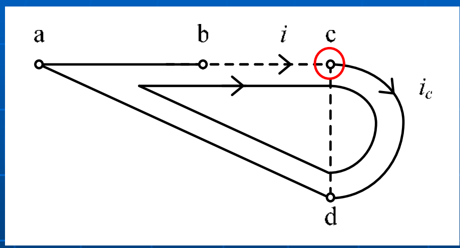
Step1 tree



8.6 State Equation

Step2 choose i and v as state variables.

Step3 fundamental cutset about the capacitor tree branch.



$$C\frac{dv}{dt} = i - \frac{v}{2}$$

Step4 fundamental loop for the inductor link.

$$L\frac{di}{dt} + v - 12V + 4i = 0$$

8.6 State Equation

Step5

$$\frac{d}{dt}\begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} \frac{1}{C} & \frac{-1}{2C} \\ \frac{-4}{L} & \frac{-1}{L} \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}(12V)$$

$$\quad [A] \qquad\qquad [B]$$

The desired solutions are v and i

$$y = \begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}(12V)$$

$$\quad [C] \qquad\qquad [D]$$

with initial condition

$$v(0^+) = 12V$$
$$i(0^+) = 0A$$

8.6 State Equation

- Example 2

$$\frac{d^3v}{dt^3}+5\frac{d^2v}{dt^2}+4\frac{dv}{dt}+3v=u(t)$$

$$\text{Let}\begin{cases} x_1 = v(t) \\ x_2 = \dfrac{dv(t)}{dt} \\ x_3 = \dfrac{d^2v(t)}{dt^3} \end{cases} \Rightarrow \begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \dfrac{dx_3}{dt} = \dfrac{d^3v}{dt^3} \end{cases}$$

$$= -5\frac{d^2v}{dt^2} - 4\frac{dv}{dt} - 3v + u(t)$$

$$= -5x_3 - 4x_2 - 3x_1 + u(t)$$

8.6 State Equation

State space representation

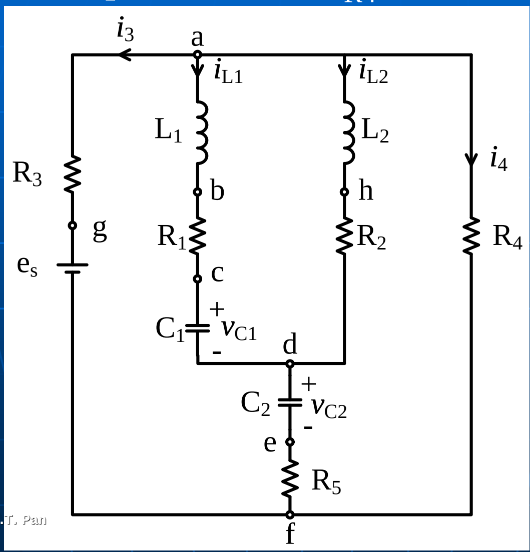
$$\therefore \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

A high order differential equation can be represented in the form of state equation.

8.6 State Equation

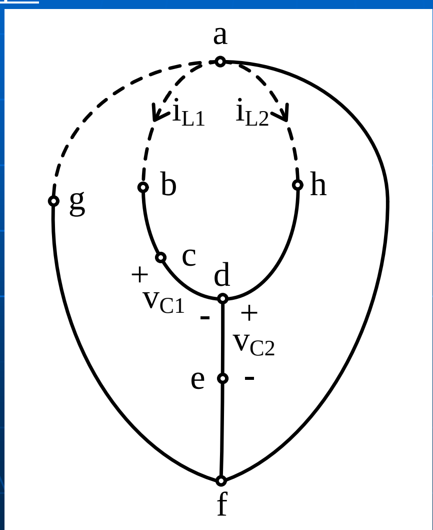
- Example 3 : Find v_{R4}



There are 8 nodes.

8.6 State Equation

Step1 Pick a tree as follows :



There are 7 tree branches and 3 links.

8.6 State Equation

Step2 Choose i_{L1}, i_{L2} , v_{C1} , v_{C2} as unknowns.

Step3 Fundamental cutsets (KCL) for capacitors.

$$C_1 \frac{dv_{C1}}{dt} = i_{L1}$$

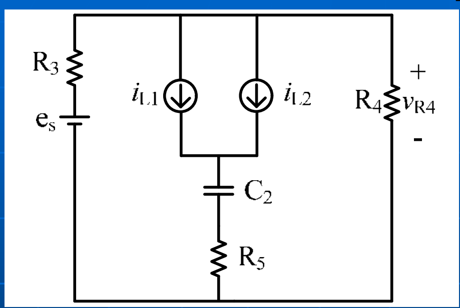
$$C_2 \frac{dv_{C2}}{dt} = i_{L1} + i_{L2}$$

Step4 Fundamental loops (KVL) for inductors.

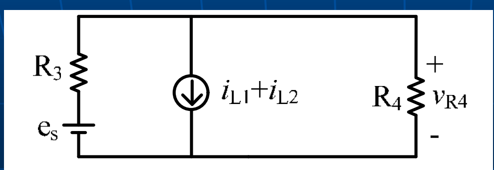
$$L_1 \frac{di_{L1}}{dt} = -v_{R1} - v_{C1} - v_{C2} - v_{R5} + v_{R4}$$
$$= -R_1 i_{L1} - v_{C1} - v_{C2} - R_5 (i_{L1} + i_{L2}) + v_{R4}$$

Note that v_{R4} should be expressed in terms of **x** and **u**

8.6 State Equation

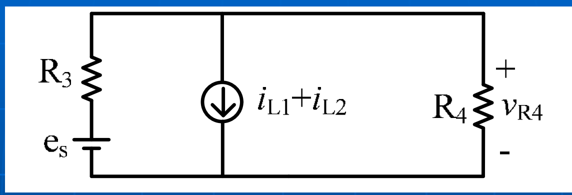


Absorb voltages v_{R1} and v_{C1} in current source i_{L1} , and v_{R2} in i_{L2}.



Absorb voltages v_{C2} and v_{R5} in (i_{L1}+ i_{L2}) current source.

8.6 State Equation



$$\therefore v_{R4} = e_s \frac{R_4}{R_3+R_4} - (i_{L1} + i_{L2})\frac{R_3R_4}{R_3+R_4}$$

Step5

8.6 State Equation

$$\frac{d}{dt}\begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L1} \\ i_{L2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & \frac{1}{C_2} & \frac{1}{C_2} \\ \frac{-1}{L_1} & \frac{-1}{L_1} & \frac{-(R_1+R_2)}{L_1} & \frac{-R}{L_1} \\ 0 & \frac{-1}{L_2} & \frac{-R}{L_2} & \frac{-(R_2+R)}{L_2} \end{bmatrix} \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L1} \\ i_{L2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_1} \\ \frac{1}{L_2} \end{bmatrix} \frac{R_4}{R_3+R_4} e_s$$

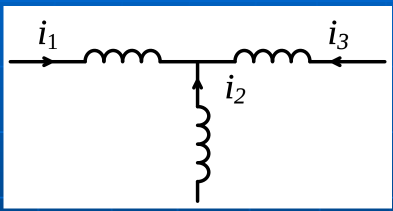
$$v_{R4} = \begin{bmatrix} 0 & 0 & \frac{-R_3R_4}{R_3+R_4} & \frac{-R_3R_4}{R_3+R_4} \end{bmatrix} \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L1} \\ i_{L2} \end{bmatrix} + \frac{R_4}{R_3+R_4} e_s$$

where $R \; @ \; R_5 + \frac{R_3R_4}{R_3+R_4}$

8.6 State Equation

Special case

(a)



From KCL

$i_1+i_2+i_3 = 0$

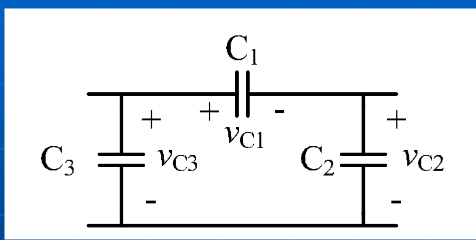
$\therefore\ i_3 = -i_1-i_2$

Inductor current i_3 is dependent on i_1 and i_2 and is no longer a state variable.

One can choose only n-1 (here 2) inductor currents as state variables.

8.6 State Equation

(b)



From KVL

$v_{C1}+v_{C2} = v_{C3}$

One can choose n-1 (here 2) capacitor voltages as state variables.

Summary

- Objective 1 : Be able to find the initial values and the initial derivative values.
- Objective 2 : Be able to determine the natural response and the step response of a series RLC circuit.
- Objective 3 : Be able to determine the natural response and the step response of a parallel RLC circuit.
- Objective 4 : Be able to obtain the state equation and output equation of a linear circuit.

Summary

Chapter Problems : 8.16
8.25
8.32
8.40
8.44

Due within one week.