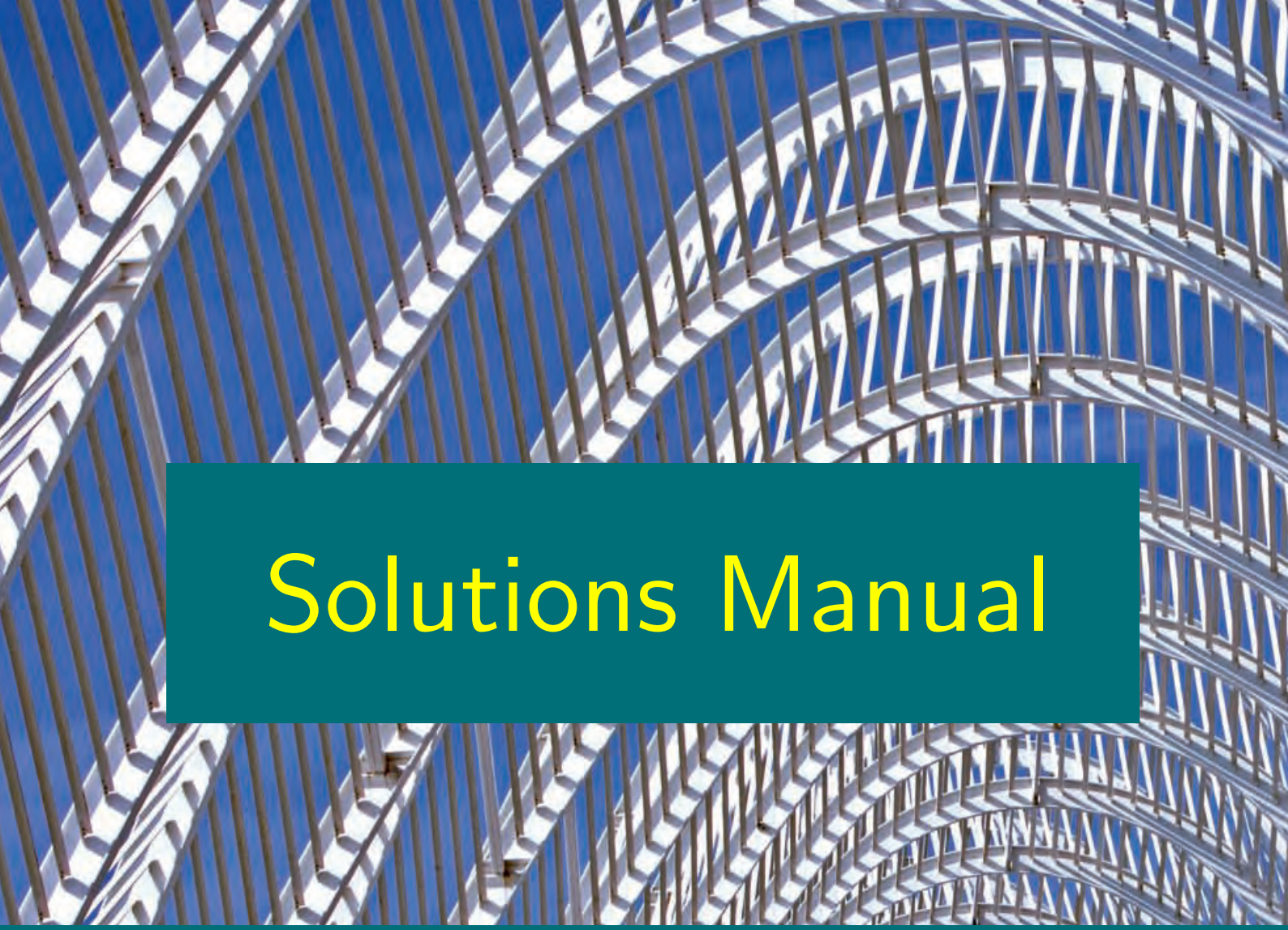

Solutions Manual

Advanced Modern Engineering Mathematics

fourth edition

Glyn James

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ISBN 978-0-273-71925-0

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This edition published 2011

ISBN: 978-0-273-71925-0

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Matrix Analysis

Exercises 1.3.3

- **1(a)** Yes, as the three vectors are linearly independent and span three-dimensional space.

1(b) No, since they are linearly dependent

$$\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1(c) No, do not span three-dimensional space. Note, they are also linearly dependent.

- **2** Transformation matrix is

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotates the $(\mathbf{e}_1, \mathbf{e}_2)$ plane through $\pi/4$ radians about the $\mathbf{e}_3$ axis.

- **3** By checking axioms (a)–(h) on p. 10 it is readily shown that all cubics $ax^3 + bx^2 + cx + d$ form a vector space. Note that the space is four dimensional.
3(a) All cubics can be written in the form

$$ax^3 + bx^2 + cx + d$$

and $\{1, x, x^2, x^3\}$ are a linearly independent set spanning four-dimensional space. Thus, it is an appropriate basis.

3(b) No, does not span the required four-dimensional space. Thus a general cubic cannot be written as a linear combination of

$$(1-x),\ (1+x),\ (1-x^3),\ (1+x^3)$$

as no term in x^2 is present.

3(c) Yes as linearly independent set spanning the four-dimensional space

$$a(1-x)+b(1+x)+c(x^2-x^3)+d(x^2+x^3)$$

$$=(a+b)+(b-a)x+(c+a)x^2+(d-c)x^3$$

$$\equiv \alpha+\beta x+\gamma x^2+\delta x^3$$

3(d) Yes as a linear independent set spanning the four-dimensional space

$$a(x-x^2)+b(x+x^2)+c(1-x^3)+d(1+x^3)$$

$$=(a+b)+(b-a)x+(c+d)x^2+(d-c)x^3$$

$$\equiv \alpha+\beta x+\gamma x^2+\delta x^3$$

3(e) No not linearly independent set as

$$(4x^3+1)=(3x^2+4x^3)-(3x^2+2x)+(1+2x)$$

■ **4** $x+2x^3$, $2x-3x^5$, $x+x^3$ form a linearly independent set and form a basis for all polynomials of the form $\alpha+\beta x^3+\gamma x^5$. Thus, S is the space of all odd quadratic polynomials. It has dimension 3.

Exercises 1.4.3

5(a) Characteristic polynomial is $\lambda^3 - p_1\lambda^2 - p_2\lambda - p_3$ with
$p_1 = \text{trace } \mathbf{A} = 12$

$$\mathbf{B}_1 = \mathbf{A} - 12\mathbf{I} = \begin{bmatrix} -9 & 2 & 1 \\ 4 & -7 & -1 \\ 2 & 3 & -8 \end{bmatrix}$$

$$\mathbf{A}_2 = \mathbf{A}\ \mathbf{B}_1 = \begin{bmatrix} -17 & -5 & -7 \\ -18 & -30 & 7 \\ 2 & -5 & -33 \end{bmatrix}$$

$p_2 = \dfrac{1}{2}\ \text{trace } \mathbf{A}_2 = -40$

$$\mathbf{B}_2 = \mathbf{A}_2 + 40\mathbf{I} = \begin{bmatrix} 23 & -5 & -7 \\ -18 & 10 & 7 \\ 2 & -5 & 7 \end{bmatrix}$$

$$\mathbf{A}_3 = \mathbf{A}\ \mathbf{B}_2 = \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$p_3 = \dfrac{1}{3}\ \text{trace } \mathbf{A}_3 = 35$

Thus, characteristic polynomial is

$$\lambda^3 - 12\lambda^2 + 40\lambda - 35$$

Note that $\mathbf{B}_3 = \mathbf{A}_3 - 35\mathbf{I} = 0$ confirming check.

5(b) Characteristic polynomial is $\lambda^4 - p_1\lambda^3 - p_2\lambda^2 - p_3\lambda - p_4$ with
$p_1 = \text{trace } \mathbf{A} = 4$

$$\mathbf{B}_1 = \mathbf{A} - 4\mathbf{I} = \begin{bmatrix} -2 & -1 & 1 & 2 \\ 0 & -3 & 1 & 0 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -4 \end{bmatrix}$$

$$\mathbf{A}_2 = \mathbf{A}\ \mathbf{B}_1 = \begin{bmatrix} -3 & 4 & 0 & -3 \\ -1 & -2 & -2 & 1 \\ 2 & 0 & -2 & -5 \\ -3 & -3 & -1 & 3 \end{bmatrix} \quad \Rightarrow p_2 = \frac{1}{2}\ \text{trace } \mathbf{A}_2 = -2$$

$$\mathbf{B}_2 = \mathbf{A}_2 + 2\mathbf{I} = \begin{bmatrix} -1 & 4 & 0 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 0 & 0 & -5 \\ -3 & -3 & -1 & 5 \end{bmatrix}$$

$$\mathbf{A}_3 = \mathbf{A}\ \mathbf{B}_2 = \begin{bmatrix} -5 & 2 & 0 & -2 \\ 1 & 0 & -2 & -4 \\ -1 & -7 & -3 & 4 \\ 0 & 4 & -2 & -7 \end{bmatrix} \quad \Rightarrow p_3 = \frac{1}{3}\ \text{trace}\ \mathbf{A}_3 = -5$$

$$\mathbf{B}_3 = \mathbf{A}_3 + 5\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 1 & 5 & -2 & -4 \\ -1 & -8 & 2 & 4 \\ 0 & 4 & -2 & -2 \end{bmatrix}$$

$$\mathbf{A}_4 = \mathbf{A}\ \mathbf{B}_3 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \Rightarrow\ p_4 = \frac{1}{4}\text{trace}\ \mathbf{A}_4 = -2$$

Thus, characteristic polynomial is $\lambda^4 - 4\lambda^3 + 2\lambda^2 + 5\lambda + 2$
Note that $\mathbf{B}_4 = \mathbf{A}_4 + 2\mathbf{I} = 0$ as required by check.

■ **6(a)** Eigenvalues given by $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$
so eigenvectors are $\lambda_1 = 2, \lambda_2 = 0$
Eigenvectors given by corresponding solutions of

$$(1 - \lambda_i)e_{i1} + e_{i2} = 0$$
$$e_{i1} + (1 - \lambda_i)e_{i2} = 0$$

Taking $i = 1, 2$ gives the eigenvectors as

$$\mathbf{e}_1 = [1\ 1]^T, \mathbf{e}_2 = [1\ -1]^T \tag{1}$$

6(b) Eigenvalues given by $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0$
so eigenvectors are $\lambda_1 = 4, \lambda_2 = -1$
Eigenvectors given by corresponding solutions of

$$(l - \lambda_i)e_{i1} + 2e_{i2} = 0$$
$$3e_{i1} + (2 - \lambda_i)e_{i2} = 0$$

Taking $i = 1, 2$ gives the eigenvectors as

$$\mathbf{e}_1 = [2\ 3]^T, \mathbf{e}_2 = [1\ -1]^T$$

6(c) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = \lambda^3 + 9\lambda^2 + 9\lambda - 81 = (\lambda - 9)(\lambda - 3)(\lambda + 3) = 0$$

So the eigenvalues are $\lambda_1 = 9, \lambda_2 = 3, \lambda_3 = -3$.
The eigenvectors are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1} + 0e_{i2} - 4e_{i3} = 0$$
$$0e_{i1} + (5-\lambda_i)e_{i2} + 4e_{i3} = 0$$
$$-4e_{i1} + 4e_{i2} + (3-\lambda_i)e_{i3} = 0$$

Taking $i = 1, \lambda_i = 9$ solution is

$$\frac{e_{11}}{8} = -\frac{e_{12}}{16} = \frac{e_{13}}{-16} = \beta_1 \qquad \Rightarrow \mathbf{e}_1 = [-1\ 2\ 2]^T$$

Taking $i = 2, \lambda_i = 3$ solution is

$$\frac{e_{21}}{-16} = -\frac{e_{22}}{16} = \frac{e_{23}}{8} = \beta_2 \qquad \Rightarrow \mathbf{e}_2 = [2\ 2\ -1]^T$$

Taking $i = 3, \lambda_i = -3$ solution is

$$\frac{e_{31}}{32} = -\frac{e_{32}}{16} = \frac{e_{33}}{32} = \beta_3 \qquad \Rightarrow \mathbf{e}_3 = [2\ -1\ 2]^T$$

6(d) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

Adding column 1 to column 2 gives

$$\begin{vmatrix} 1-\lambda & 2-\lambda & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 0 & 3-\lambda \end{vmatrix}$$

$$R_1 = R_2(2-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(3-\lambda)$$

so the eigenvalues are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$.
Eigenvectors are the corresponding solutions of $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{e}_i = 0$
When $\lambda = \lambda_1 = 3$ we have

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{11}}{-2} = -\frac{e_{12}}{2} = \frac{e_{13}}{-1} = \beta_1$$

so the eigenvector corresponding to $\lambda_2 = 3$ is $\mathbf{e}_1 = \beta_1[2\ 2\ 1]^T, \beta_1$ constant.
When $\lambda = \lambda_2 = 2$ we have

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{21}}{-2} = -\frac{e_{22}}{2} = \frac{e_{23}}{0} = \beta_3$$

so the eigenvector corresponding to $\lambda_2 = 2$ is $\mathbf{e}_2 = \beta_2[1\ 1\ 0]^T, \beta_2$ constant.
When $\lambda = \lambda_3 = 1$ we have

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{31}}{0} = -\frac{e_{32}}{2} = \frac{e_{33}}{1} = \beta_1$$

so the eigenvector corresponding to $\lambda_3 = 1$ is $\mathbf{e}_3 = \beta_3[0\ -2\ 1]^T, \beta_3$ constant.

6(e) Eigenvalues given by

$$\begin{vmatrix} 5-\lambda & 0 & 6 \\ 0 & 11-\lambda & 6 \\ 6 & 6 & -2-\lambda \end{vmatrix} = \lambda^3 - 14\lambda^2 - 23\lambda - 686 = (\lambda - 14)(\lambda - 7)(\lambda + 7) = 0$$

so eigenvalues are $\lambda_1 = 14, \lambda_2 = 7, \lambda_3 = -7$

Eigenvectors are given by the corresponding solutions of

$$\begin{aligned}(5-\lambda_i)e_{i1}+0e_{i2}+6e_{i3}&=0\\ 0e_{i1}+(11-\lambda_i)e_{i2}+6e_{i3}&=0\\ 6e_{i1}+6e_{i2}+(-2-\lambda_i)e_{i3}&=0\end{aligned}$$

When $i=1,\lambda_1=14$ solution is

$$\frac{e_{11}}{12}=\frac{-e_{12}}{-36}=\frac{e_{13}}{18}=\beta_1\ \Rightarrow \mathbf{e}_1=[2\ 6\ 3]^T$$

When $i=2,\lambda_2=7$ solution is

$$\frac{e_{21}}{-72}=\frac{-e_{22}}{-36}=\frac{e_{23}}{-24}=\beta_2\ \Rightarrow \mathbf{e}_2=[6\ -3\ 2]^T$$

When $i=3,\lambda_3=-7$ solution is

$$\frac{e_{31}}{54}=\frac{-e_{32}}{-36}=\frac{e_{33}}{-108}=\beta_3\ \Rightarrow \mathbf{e}_3=[3\ 2\ -6]^T$$

6(f) Eigenvalues given by

$$\begin{vmatrix}1-\lambda & -1 & 0\\ 1 & 2-\lambda & 1\\ -2 & 1 & -1-\lambda\end{vmatrix}\ \underline{\underline{R_1+R_2}}\ \begin{vmatrix}-1-\lambda & 0 & -1-\lambda\\ 1 & 2-\lambda & 1\\ -2 & 1 & -1-\lambda\end{vmatrix}$$

$$=(1+\lambda)\begin{vmatrix}-1 & 0 & 0\\ 1 & 2-\lambda & 0\\ -2 & 1 & 1-\lambda\end{vmatrix}=0,\ \text{i.e. } (1+\lambda)(2-\lambda)(1-\lambda)=0$$

so eigenvalues are $\lambda_1=2,\lambda_2=1,\lambda_3=-1$
Eigenvectors are given by the corresponding solutions of

$$\begin{aligned}(1-\lambda_i)e_{i1}-e_{i2}+0e_{i3}&=0\\ e_{i1}+(2-\lambda_i)e_{i2}+e_{i3}&=0\\ -2e_{i1}+e_{i2}-(1+\lambda_i)e_{i3}&=0\end{aligned}$$

Taking $i=1,2,3$ gives the eigenvectors as

$$\mathbf{e}_1=[-1\ 1\ 1]^T,\mathbf{e}_2=[1\ 0\ -1]^T,\mathbf{e}_3=[1\ 2\ -7]^T$$

6(g) Eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 2 & 5-\lambda & 4 \\ -1 & -1 & -\lambda \end{vmatrix} \underline{\underline{R_1 + (R_2 + R_3)}} \begin{vmatrix} 5-\lambda & 5-\lambda & 5-\lambda \\ 2 & 5-\lambda & 4 \\ -1 & -1 & -\lambda \end{vmatrix}$$
$$= (5-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3-\lambda & 2 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda)(1-\lambda) = 0$$

so eigenvalues are $\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 1$
Eigenvectors are given by the corresponding solutions of

$$\begin{aligned} (4-\lambda_i)e_{i1} + e_{i2} + e_{i3} &= 0 \\ 2e_{i1} + (5-\lambda_i)e_{i2} + 4e_{i3} &= 0 \\ -e_{i1} - e_{i2} - \lambda_i e_{i3} &= 0 \end{aligned}$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [2\ 3\ -1]^T, \mathbf{e}_2 = [1\ -1\ 0]^T, \mathbf{e}_3 = [0\ -1\ 1]^T$$

6(h) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & -4 & -2 \\ 0 & 3-\lambda & 1 \\ 1 & 2 & 4-\lambda \end{vmatrix} \underline{\underline{R_1 \pm 2R_2}} \begin{vmatrix} 1-\lambda & 2-2\lambda & 0 \\ 0 & 3-\lambda & 1 \\ 1 & 2 & 4-\lambda \end{vmatrix}$$
$$= (1-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 1 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)(4-\lambda) = 0$$

so eigenvalues are $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 1$
Eigenvectors are given by the corresponding solutions of

$$\begin{aligned} (1-\lambda_i)e_{i1} - 4e_{i2} - 2e_{i3} &= 0 \\ 2e_{i1} + (3-\lambda_i)e_{i2} + e_{i3} &= 0 \\ e_{i1} + 2e_{i2} + (4-\lambda_i)e_{i3} &= 0 \end{aligned}$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [2 \ -1 \ -1]^T, \mathbf{e}_2 = [2 \ -1 \ 0]^T, \mathbf{e}_3 = [4 \ -1 \ -2]^T$$

Exercises 1.4.5

- **7(a)** Eigenvalues given by

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} \underset{=}{R_1 - R_2} \begin{vmatrix} 1-\lambda & -1+\lambda & 0 \\ 0 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 4-\lambda & 1 \\ 1 & 3 & 2-\lambda \end{vmatrix} = (1-\lambda)[\lambda^2 - 6\lambda + 5] = (1-\lambda)(\lambda - 1)(\lambda - 5) = 0$$

so eigenvalues are $\lambda_1 = 5, \lambda_2 = \lambda_3 = 1$
The eigenvectors are the corresponding solutions of

$$\begin{aligned} (2-\lambda_i)e_{i1} + 2e_{i2} + e_{i3} &= 0 \\ e_{i1} + (3-\lambda_i)e_{i2} + e_{i3} &= 0 \\ e_{i1} + 2e_{i2} + (2-\lambda_i)e_{i3} &= 0 \end{aligned}$$

When $i = 1, \lambda_1 = 5$ and solution is

$$\frac{e_{11}}{4} = \frac{-e_{12}}{-4} = \frac{e_{13}}{4} = \beta_1 \ \Rightarrow \ \mathbf{e}_1 = [1 \ 1 \ 1]^T$$

When $\lambda_2 = \lambda_3 = 1$ solution is given by the single equation

$$e_{21} + 2e_{22} + e_{23} = 0$$

Following the procedure of Example 1.6 we can obtain two linearly independent solutions. A possible pair are

$$\mathbf{e}_2 = [0 \ 1 \ 2]^T \ , \ \mathbf{e}_3 = [1 \ 0 \ -1]^T$$

7(b) Eigenvalues given by

$$\begin{vmatrix} -\lambda & -2 & -2 \\ -1 & 1-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda+1)(\lambda-2)^2 = 0$$

so eigenvalues are $\lambda_1 = \lambda_2 = 2, \lambda_3 = -1$
The eigenvectors are the corresponding solutions of

$$\begin{aligned} -\lambda_i e_{i1} - 2e_{i2} - 2e_{i3} &= 0 \\ -e_{i1} + (1-\lambda_i)e_{i2} + 2e_{i3} &= 0 \\ -e_{i1} - e_{i2} + (2-\lambda_i)e_{i3} &= 0 \end{aligned}$$

When $i = 3, \lambda_3 = -1$ corresponding solution is

$$\frac{e_{31}}{8} = \frac{-e_{32}}{-1} = \frac{e_{33}}{3} = \beta_3 \;\Rightarrow\; \mathbf{e}_3 = [8\ 1\ 3]^T$$

When $\lambda_1 = \lambda_2 = 2$ solution is given by

$$-2e_{21} - 2e_{22} - 2e_{23} = 0 \tag{1}$$

$$-e_{21} - e_{22} + 2e_{23} = 0 \tag{2}$$

$$-e_{21} - e_{22} = 0 \tag{3}$$

From (1) and (2) $e_{23} = 0$ and it follows from (3) that $e_{21} = -e_{22}$. We deduce that there is only one linearly independent eigenvector corresponding to the repeated eigenvalues $\lambda = 2$. A possible eigenvector is

$$\mathbf{e}_2 = [1\ -1\ 0]^T$$

7(c) Eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} \underline{\underline{R_1 - 3R_3}} \begin{vmatrix} 1-\lambda & -3+3\lambda & 0 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1 & -3 & 0 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 1 & 6-\lambda & 2 \\ 1 & -8 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2 + \lambda + 4) = (1-\lambda)(\lambda-2)^2 = 0$$

so eigenvalues are $\lambda_1 = \lambda_2 = 2,\ \lambda_3 = 1$.
The eigenvectors are the corresponding solutions of

$$\begin{aligned}(4-\lambda_i)e_{i1} + 6e_{i2} + 6e_{i3} &= 0\\ e_{i1} + (3-\lambda_i)e_{i2} + 2e_{i3} &= 0\\ -e_{i1} - 5e_{i2} - (2+\lambda_i)e_{i3} &= 0\end{aligned}$$

When $i = 3, \lambda_3 = 1$ corresponding solution is

$$\frac{e_{31}}{4} = \frac{-e_{32}}{-1} = \frac{e_{33}}{-3} = \beta_3 \ \Rightarrow\ \mathbf{e}_3 = [4\ 1\ -3]^T$$

When $\lambda_1 = \lambda_2 = 2$ solution is given by

$$\begin{aligned}2e_{21} + 6e_{22} + 6e_{23} &= 0\\ e_{21} + e_{22} + 2e_{23} &= 0\\ -e_{21} - 5e_{22} - 4e_{23} &= 0\end{aligned}$$

so that

$$\frac{e_{21}}{6} = \frac{-e_{22}}{-2} = \frac{e_{23}}{-4} = \beta_2$$

leading to only one linearly eigenvector corresponding to the eigenvector $\lambda = 2$. A possible eigenvector is

$$\mathbf{e}_2 = [3\ 1\ -2]^T$$

7(d) Eigenvalues given by

$$\begin{vmatrix} 7-\lambda & -2 & -4 \\ 3 & -\lambda & -2 \\ 6 & -2 & -3-\lambda \end{vmatrix} \underline{\underline{R_1 - 2R_2}} \begin{vmatrix} 1-\lambda & -2+2\lambda & 0 \\ 3 & -\lambda & -2 \\ 6 & -2 & -3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1 & -2 & 0 \\ 3 & -\lambda & -2 \\ 6 & -2 & -3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & 6-\lambda & -2 \\ 6 & 10 & -3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(\lambda-2)(\lambda-1) = 0$$

so eigenvalues are $\lambda_1 = 2,\ \lambda_2 = \lambda_3 = 1$.
The eigenvectors are the corresponding solutions of

$$\begin{aligned}(7-\lambda_i)e_{i1} - 2e_{i2} - 4e_{i3} &= 0\\ 3e_{i1} - \lambda_i e_{i2} - 2e_{i3} &= 0\\ 6e_{i1} - 2e_{i2} - (3+\lambda_i)e_{i3} &= 0\end{aligned}$$

When $i = 1, \lambda_2 = 2$ and solution is

$$\frac{e_{11}}{6} = \frac{-e_{12}}{-3} = \frac{e_{13}}{6} = \beta_1 \;\Rightarrow\; \mathbf{e}_1 = [2\ 1\ 2]^T$$

When $\lambda_2 = \lambda_3 = 1$ the solution is given by the single equation

$$3e_{21} - e_{22} - 2e_{23} = 0$$

Following the procedures of Example 1.6 we can obtain two linearly independent solutions. A possible pair are

$$\mathbf{e}_2 = [0\ 2\ -1]^T\ , \mathbf{e}_3 = [2\ 0\ 3]^T$$

■ **8**

$$(\mathbf{A} - \mathbf{I}) = \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Performing a series of row and column operators this may be reduced to the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ indicating that $(\mathbf{A} - \mathbf{I})$ is of rank 2. Thus, the nullity $q = 3 - 2 = 1$ confirming that there is only one linearly independent eigenvector associated with the eigenvalue $\lambda = 1$. The eigenvector is given by the solution of

$$\begin{aligned} -4e_{11} - 7e_{12} - 5e_{13} &= 0 \\ 2e_{11} + 3e_{12} + 3e_{13} &= 0 \\ e_{11} + 2e_{12} + e_{13} &= 0 \end{aligned}$$

giving

$$\frac{e_{11}}{-3} = \frac{-e_{12}}{-1} = \frac{e_{13}}{1} = \beta_1 \;\Rightarrow\; \mathbf{e}_1 = [-3\ 1\ 1]^T$$

■ **9**

$$(\mathbf{A} - \mathbf{I}) = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

Performing a series of row and column operators this may be reduced to the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ indicating that $(\mathbf{A}-\mathbf{I})$ is of rank 1. Then, the nullity of $q = 3-1 = 2$ confirming that there are two linearly independent eigenvectors associated with the eigenvalue $\lambda = 1$. The eigenvectors are given by the single equation

$$e_{11} + e_{12} - e_{13} = 0$$

and two possible linearly independent eigenvectors are

$$\mathbf{e}_1 = [1\ 0\ 1]^T \text{ and } \mathbf{e}_2 = [0\ 1\ 1]^T$$

Exercises 1.4.8

- **10** These are standard results.

- **11(a)** (i) Trace $\mathbf{A} = 4 + 5 + 0 = 9 =$ sum eigenvalues;

(ii) $\det \mathbf{A} = 15 = 5 \times 3 \times 1 =$ product eigenvalues;

(iii) $\mathbf{A^{-1}} = \dfrac{1}{15}\begin{bmatrix} 4 & -1 & -1 \\ -4 & 1 & -14 \\ 3 & 3 & 18 \end{bmatrix}$. Eigenvalues given by

$$\begin{vmatrix} 4-15\lambda & -1 & -1 \\ -4 & 1-15\lambda & -14 \\ 3 & 3 & 18-15\lambda \end{vmatrix} \underset{=}{C_3 = C_2} \begin{vmatrix} 4-15\lambda & -1 & 0 \\ -4 & 1-15\lambda & -15+15\lambda \\ 3 & 3 & 15-15\lambda \end{vmatrix}$$

$$= (15-15\lambda)\begin{vmatrix} 4-15\lambda & -1 & 0 \\ -4 & 1-15\lambda & -1 \\ 3 & 3 & 1 \end{vmatrix} = (15-15\lambda)(15\lambda-5)(15\lambda-3) = 0$$

confirming eigenvalues as $1, \frac{1}{3}, \frac{1}{5}$.

(iv) $A^T = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 5 & -1 \\ 1 & 4 & 0 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 2 & -1 \\ 1 & 5-\lambda & -1 \\ 1 & 4 & -\lambda \end{vmatrix} = (\lambda-5)(\lambda-3)(\lambda-1) = 0$$

that is, eigenvalue as for $\mathbf{A}$.

11(b) (i) $2\mathbf{A} = \begin{bmatrix} 8 & 2 & 2 \\ 4 & 10 & 8 \\ -2 & -2 & 0 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 8-\lambda & 2 & 2 \\ 4 & 10-\lambda & 8 \\ -2 & -2 & -\lambda \end{vmatrix} \underset{=}{C_1 = C_2} \begin{vmatrix} 6-\lambda & 2 & 2 \\ -6+\lambda & 10-\lambda & 8 \\ 0 & -2 & -\lambda \end{vmatrix}$$

$$= (6-\lambda)\begin{vmatrix} 1 & 2 & 2 \\ -1 & 10-\lambda & 8 \\ 0 & -2 & -\lambda \end{vmatrix} = (6-\lambda)\begin{vmatrix} 1 & 2 & 2 \\ 0 & 12-\lambda & 10 \\ 0 & -2 & -\lambda \end{vmatrix}$$

$$= (6-\lambda)(\lambda-10)(\lambda-2) = 0$$

Thus eigenvalues are 2 times those of $\mathbf{A}$; namely 6, 10 and 2.

(ii) $\mathbf{A} + 2\mathbf{I} = \begin{bmatrix} 6 & 1 & 1 \\ 2 & 7 & 4 \\ -1 & -1 & 2 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 6-\lambda & 1 & 1 \\ 2 & 7-\lambda & 4 \\ -1 & -1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 15\lambda^2 - 71\lambda + 105 = -(\lambda-7)(\lambda-5)(\lambda-3) = 0$$

confirming the eigenvalues as $5+2, 3+2, 1+2$.

Likewise for $\mathbf{A} - 2\mathbf{I}$

(iii) $A^2 = \begin{bmatrix} 17 & 8 & 8 \\ 14 & 23 & 22 \\ -6 & -6 & -5 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 17-\lambda & 8 & 8 \\ 14 & 23-\lambda & 22-\lambda \\ -6 & -6 & -5-\lambda \end{vmatrix} \underline{\underline{R_1 + (R_2) + R_3)}} \begin{vmatrix} 25-\lambda & 25-\lambda & 25-\lambda \\ 14 & 23-\lambda & 22 \\ -6 & -6 & -5-\lambda \end{vmatrix}$$

$$= (25-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 14 & 9-\lambda & 8 \\ -6 & 0 & 1-\lambda \end{vmatrix} = (25-\lambda)(9-\lambda)(1-\lambda) = 0$$

that is, eigenvalues $\mathbf{A}^2$ are 25, 9, 1 which are those of $\mathbf{A}$ squared.

■ **12** Eigenvalues of $\mathbf{A}$ given by

$$\begin{vmatrix} -3-\lambda & -3 & -3 \\ -3 & 1-\lambda & -1 \\ -3 & -1 & 1-\lambda \end{vmatrix} \underline{\underline{R_3 + R_2}} \begin{vmatrix} -3-\lambda & -3 & -3 \\ -3 & 1-\lambda & -1 \\ 0 & -2+\lambda & 2-\lambda \end{vmatrix}$$

$$= (\lambda-2)\begin{vmatrix} -3-\lambda & -3 & -3 \\ -3 & 1-\lambda & -1 \\ 0 & 1 & -1 \end{vmatrix} \underline{\underline{C_3 + C_2(\lambda-2)}} \begin{vmatrix} -3-\lambda & -3 & -6 \\ -3 & (1-\lambda) & -\lambda \\ 0 & 1 & 0 \end{vmatrix}$$

$$= -(\lambda-2)(\lambda+6)(\lambda-3) = 0$$

so eigenvalues are $\lambda_1 = -6, \lambda_2 = 3, \lambda_3 = 2$

Eigenvectors are given by corresponding solutions of

$$\begin{aligned} (-3-\lambda_i)e_{i1} - 3e_{i2} - 3e_{i3} &= 0 \\ -3e_{i1} + (1-\lambda_i)e_{i2} - e_{i3} &= 0 \\ -3e_{i1} - e_{i2} + (1-\lambda_i)e_{i3} &= 0 \end{aligned}$$

Taking $i = 1, 2, 3$ gives the eigenvectors as

$$\mathbf{e}_1 = [2\ 1\ 1]^T,\ \mathbf{e}_2 = [-1\ 1\ 1]^T,\ \mathbf{e}_3 = [0\ 1\ -1]^T$$

It is readily shown that

$$\mathbf{e}_1^T\mathbf{e}_2 = \mathbf{e}_1^T\mathbf{e}_3 = \mathbf{e}_2^T\mathbf{e}_3 = 0$$

so that the eigenvectors are mutually orthogonal.

■ **13** Let the eigenvector be $\mathbf{e} = [a\ b\ c]^T$ then since the three vectors are mutually orthogonal

$$a + b - 2c = 0$$

$$a + b - c = 0$$

giving $c = 0$ and $a = -b$ so an eigenvector corresponding to $\lambda = 2$ is $\mathbf{e} = [1\ -1\ 0]^T$.

Exercises 1.5.3

■ **14** Taking $x^{(0)} = [1\ 1\ 1]^T$ iterations may then be tabulated as follows:

Iteration k	0	1	2	3	4
$\mathbf{x}^{(k)}$	1	0.9	0.874	0.869	0.868
	1	1	1	1	1
	1	0.5	0.494	0.493	0.492
$\mathbf{A}\ \mathbf{x}^{(k)}$	9	7.6	7.484	7.461	7.457
	10	8.7	8.61	8.592	8.589
	5	4.3	4.242	4.231	4.228
$\lambda\ \simeq$	10	8.7	8.61	8.592	8.589

Thus, estimate of dominant eigenvalue is $\lambda \simeq 8.59$ and corresponding eigenvector $\mathbf{x} \simeq [0.869\ 1\ 0.493]^T$ or $\mathbf{x} \simeq [0.61\ 0.71\ 0.35]^T$ in normalised form.

■ **15(a)** Taking $\mathbf{x}^{(0)} = [1\ 1\ 1]^T$ iterations may then be tabulated as follows:

Iteration k	0	1	2	3	4	5	**6**
$\mathbf{x}^{(k)}$	1	0.75	0.667	0.636	0.625	0.620	0.619
	1	1	1	1	1	1	1
	1	1	1	1	1	1	1
$\mathbf{A}\ \mathbf{x}^{(k)}$	3	2.5	2.334	2.272	2.250	2.240	
	4	3.75	3.667	3.636	3.625	3.620	
	4	3.75	3.667	3.636	3.625	3.620	
$\lambda\ \simeq$	4	3.75	3.667	3.636	3.625	3.620	

Thus, correct to two decimal places dominant eigenvalue is 3.62 having corresponding eigenvectors $[0.62\ 1\ 1]^T$.

15(b) Taking $\mathbf{x}^{(0)} = [1\ 1\ 1]^T$ iterations may be tabulated as follows:

Iteration k	0	1	2	3	4	5
$\mathbf{x}^{(k)}$	1	0.364	0.277	0.257	0.252	0.251
	1	0.545	0.506	0.501	0.493	0.499
	1	1	1	1	1	1
$\mathbf{A}\ \mathbf{x}^{(k)}$	4	2.092	1.831	1.771	1.756	
	6	3.818	3.566	3.561	3.49	
	11	7.546	7.12	7.03	6.994	
$\lambda \simeq$	11	7.546	7.12	7.03	6.994	

Thus, correct to two decimal places dominant eigenvalue is 7 having corresponding eigenvector $[0.25\ 0.5\ 1]^T$.

15(c) Taking $\mathbf{x}^{(0)} = [1\ 1\ 1\ 1]^T$ iterations may then be tabulated as follows:

Iteration k	0	1	2	3	4	5	6
$\mathbf{x}^{(k)}$	1	1	1	1	1	1	1
	1	0	−0.5	−0.6	−0.615	−0.618	− 0.618
	1	1	−0.5	−0.6	−0.615	−0.618	−0.618
	1	1	1	1	1	1	1
$\mathbf{A}\ \mathbf{x}^{(k)}$	1	2	2.5	2.6	2.615	2.618	
	0	−1	−1.5	−1.6	−1.615	−1.618	
	0	−1	−1.5	−1.6	−1.615	−1.618	
	1	2	2.5	2.6	2.615	2.618	
$\lambda \simeq$	1	2	2.5	2.6	2.615	2.618	

Thus, correct to two decimal places dominant eigenvalue is 2.62 having corresponding eigenvector $[1\ -0.62\ -0.62\ \ 1]^T$.

■ **16** The eigenvalue λ_1 corresponding to the dominant eigenvector $\mathbf{e}_1 = [1\ 1\ 2]^T$ is such that $\mathbf{A}\ \mathbf{e}_1 = \lambda_1 \mathbf{e}_1$ so

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

so $\lambda_1 = 6$.

Then

$$\mathbf{A}_1 = \mathbf{A} - 6\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1^T \text{ where } \hat{\mathbf{e}}_1 = \left[\frac{1}{\sqrt{6}}\,\frac{1}{\sqrt{6}}\,\frac{2}{\sqrt{6}}\right]^T$$

so

$$\mathbf{A}_1 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Applying the power method with $\mathbf{x}^{(0)} = [1\ 1\ 1]^T$

$$\mathbf{y}^{(1)} = \mathbf{A}_1\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{x}^{(1)}$$

$$\mathbf{y}^{(2)} = \mathbf{A}_1\mathbf{x}^{(1)} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Clearly, $\lambda_2 = 3$ and $\hat{\mathbf{e}}_2 = \dfrac{1}{\sqrt{3}}[1\ 1\ -1]^T$.

Repeating the process

$$\mathbf{A}_2 = \mathbf{A}_1 - \lambda_2\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2^T = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking $\mathbf{x}^{(0)} = [1\ -1\ 0]^T$ the power method applied to $\mathbf{A}_2$ gives

$$\mathbf{y}^{(1)} = \mathbf{A}_2\mathbf{x}^{(0)} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and clearly, $\lambda_3 = 2$ with $\hat{\mathbf{e}}_3 = \dfrac{1}{\sqrt{2}}[1\ -1\ 0]^T$.

■ **17** The three Gerschgorin circles are

$$|\ \lambda - 5\ | = 2,\ |\ \lambda\ | = 2,\ |\ \lambda + 5\ | = 2$$

which are three non-intersecting circles. Since the given matrix $\mathbf{A}$ is symmetric its three eigenvalues are real and it follows from Theorem 1.2 that

$$3 < \lambda_1 < 7\ ,\ -2 < \lambda_2 < 2\ ,\ -7 < \lambda_3 < 7$$

(i.e., an eigenvalue lies within each of the three circles).

■ **18** The characteristic equation of the matrix $\mathbf{A}$ is

$$\begin{vmatrix} 10-\lambda & -1 & 0 \\ -1 & 2-\lambda & 2 \\ 0 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\text{that is} \quad (10-\lambda)[(2-\lambda)(3-\lambda)-4]-(3-\lambda)=0$$
$$\text{or} \quad f(\lambda)=\lambda^3-15\lambda^2+51\lambda-17=0$$

Taking $\lambda_0 = 10$ as the starting value the Newton–Raphson iterative process produces the following table:

i	λ_i	$f(\lambda_i)$	$f'(\lambda_i)$	$-\dfrac{f(\lambda_i)}{f'(\lambda_i)}$
0	10	7	−51.00	0.13725
1	10.13725	−0.28490	−55.1740	−0.00516
2	10.13209	−0.00041	−55.0149	−0.000007

Thus to three decimal places the largest eigenvalue is $\lambda = 10.132$

Using Properties 1.1 and 1.2 of section 1.4.6 we have

$$\sum_{i=1}^{3}\lambda_i = \text{ trace } \mathbf{A} = 15 \text{ and } \prod_{i=1}^{3}\lambda_i = |\ \mathbf{A}\ | = 17$$

Thus,

$$\begin{aligned} \lambda_2 + \lambda_3 &= 15 - 10.132 = 4.868 \\ \lambda_2\lambda_3 &= 1.67785 \\ \text{so} \quad \lambda_2(4.868 - \lambda_2) &= 1.67785 \\ \lambda_2^2 - 4.868\lambda_2 + 1.67785 &= 0 \\ \lambda_2 &= 2.434 \pm 2.0607 \\ \text{that is} \quad \lambda_2 &= 4.491 \text{ and } \quad \lambda_3 = 0.373 \end{aligned}$$

■ **19(a)** If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the corresponding eigenvectors to $\lambda_1, \lambda_2, \ldots, \lambda_n$ then $(K\mathbf{I} - \mathbf{A})\mathbf{e}_i = (K - \lambda_i)\mathbf{e}_i$ so that $\mathbf{A}$ and $(K\mathbf{I} - \mathbf{A})$ have the same eigenvectors and eigenvalues differ by K.
Taking $\mathbf{x}^{(o)} = \sum_{i=1}^{n} \alpha_r \mathbf{e}_i$ then

$$\mathbf{x}^{(p)} = (K\mathbf{I} - \mathbf{A})\mathbf{x}^{(p-1)} = (K\mathbf{I} - \mathbf{A})^2\mathbf{x}^{(p-2)} = \ldots = \sum_{r=1}^{n} \alpha_r (K - \lambda_r)^p \mathbf{e}_r$$

Now $K - \lambda_n > K - \lambda_{n-1} > \ldots > K - \lambda_1$ and

$$\begin{aligned} \mathbf{x}^{(p)} &= \alpha_n (K - \lambda_n)^p \mathbf{e}_n + \sum_{r=1}^{n} \alpha_r (K - \lambda_r)^p \mathbf{e}_r \\ &= (K - \lambda_n)^p [\alpha_n \mathbf{e}_n + \sum_{r=1}^{n-1} \alpha_r \big[\frac{K - \lambda_r}{K - \lambda_n}\big]^p \mathbf{e}_r] \\ &\to (K - \lambda_n)^p \alpha_n \mathbf{e}_n = K\mathbf{e}_n \text{ as } p \to \infty \end{aligned}$$

Also

$$\frac{x_i^{(p+1)}}{x_i^{(p)}} \;\to\; \frac{(K - \lambda_n)^{p+1}}{(K - \lambda_n)^p} \; \frac{\alpha_n e_n}{\alpha_n e_n} \; = K - \lambda_n$$

Hence, we can find λ_n

19(b) Since $\mathbf{A}$ is a symmetric matrix its eigenvalues are real. By Gerschgorin's theorem the eigenvalues lie in the union of the intervals

$$\begin{aligned} & | \lambda - 2 | \leq 1, \; | \lambda - 2 | \leq 2, \; | \lambda - 2 | \leq 1 \\ \text{i.e.} \quad & | \lambda - 2 | \leq 2 \text{ or } 0 \leq \lambda \leq 4. \end{aligned}$$

Taking $K = 4$ in (a)

$$K\mathbf{I} - \mathbf{A} = 4\mathbf{I} - \mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Taking $\mathbf{x}^{(0)} = [1\ 1\ 1]^T$ iterations using the power method are tabulated as follows:

Iteration k	0	1	2	3	4	5
$\mathbf{x}^{(k)}$	1	0.75	0.714	0.708	0.707	0.707
	1	1	1	1	1	1
	1	0.75	0.714	0.708	0.707	0.707
$\mathbf{A}\ \mathbf{x}^{(k)}$	3	2.5	2.428	2.416	2.414	
	4	3.5	3.428	3.416	3.414	
	3	2.5	2.428	2.416	2.414	
$\lambda \simeq$	4	3.5	3.428	3.416	3.414	

Thus $\lambda_3 = 4 - 3.41 = 0.59$ correct to two decimal places.

Exercises 1.6.3

■ **20** Eigenvalues given by

$$\Delta = \begin{vmatrix} -1-\lambda & 6 & -12 \\ 0 & -13-\lambda & 30 \\ 0 & -9 & 20-\lambda \end{vmatrix} = 0$$

Now $\Delta = (-1-\lambda)\begin{vmatrix} -13-\lambda & 30 \\ -9 & 20-\lambda \end{vmatrix} = (-1-\lambda)(\lambda^2 - 7\lambda + 10)$
$= (-1-\lambda)(\lambda-5)(\lambda-2)$ so $\Delta = 0$ gives $\lambda_1 = 5, \lambda_2 = 2, \lambda_3 = -1$
Corresponding eigenvectors are given by the solutions of

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{e}_i = 0$$

When $\lambda = \lambda_1 = 5$ we have

$$\begin{bmatrix} -6 & 6 & -12 \\ 0 & -18 & 30 \\ 0 & -9 & 15 \end{bmatrix} \quad \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{11}}{-36} = \frac{-e_{12}}{-180} = \frac{e_{13}}{108} = \beta_1$$

so the eigenvector corresponding to $\lambda_1 = 5$ is $\mathbf{e}_1 = \beta_1[1\ -5\ -3]^T$
When $\lambda = \lambda_2 = 2$, we have

$$\begin{bmatrix} -3 & 6 & -12 \\ 0 & -15 & 30 \\ 0 & -9 & 18 \end{bmatrix} \quad \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{21}}{0} = \frac{-e_{22}}{-90} = \frac{e_{23}}{45} = \beta_2$$

so the eigenvector corresponding to $\lambda_2 = 2$ is $\mathbf{e}_2 = \beta_2[0\ 2\ 1]^T$
When $\lambda = \lambda_3 = -1$, we have

$$\begin{bmatrix} 0 & 6 & -12 \\ 0 & -12 & 30 \\ 0 & -9 & 21 \end{bmatrix} \quad \begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{31}}{18} = \frac{-e_{32}}{0} = \frac{e_{33}}{0} = \beta_3$$

so the eigenvector corresponding to $\lambda_3 = -1$ is $\mathbf{e}_3 = \beta_3[1\ 0\ 0]^T$
A modal matrix $\mathbf{M}$ and spectral matrix $\mathbf{\Lambda}$ are

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 2 & 0 \\ -3 & 1 & 0 \end{bmatrix} \qquad \mathbf{\Lambda} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 3 & -5 \\ 1 & -1 & 2 \end{bmatrix} \quad \text{and matrix multiplication confirms } \mathbf{M}^{-1}\mathbf{A}\ \mathbf{M} = \mathbf{\Lambda}$$

■ **21** From Example 1.9 the eigenvalues and corresponding normalised eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 1$$

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{5}}[1\ 2\ 0]^T, \hat{\mathbf{e}}_2 = [0\ 0\ 1]^T, \hat{\mathbf{e}}_3 = \frac{1}{\sqrt{5}}[-2\ 1\ 0]^T,$$

$$\hat{\mathbf{M}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & 1 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$$

$$\hat{\mathbf{M}}^T\ \mathbf{A}\ \mathbf{M} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & \sqrt{5} \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & 1 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} 6 & 12 & 0 \\ 0 & 0 & 3\sqrt{5} \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & 1 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} 30 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{\Lambda}$$

■ **22** The eigenvalues of $\mathbf{A}$ are given by

$$\begin{vmatrix} 5-\lambda & 10 & 8 \\ 10 & 2-\lambda & -2 \\ 8 & -2 & 11-\lambda \end{vmatrix} = -(\lambda^3 - 18\lambda^2 - 81\lambda + 1458) = -(\lambda-9)(\lambda+9)(\lambda-18) = 0$$

so eigenvalues are $\lambda_1 = 18, \lambda_2 = 9, \lambda_3 = -9$

The eigenvectors are given by the corresponding solutions of

$$(5-\lambda_i)e_{i1} + 10e_{i2} + 8e_{i3} = 0$$
$$10e_{i1} + (2-\lambda_i)e_{i2} - 2e_{i3} = 0$$
$$8e_{i1} - 2e_{i2} + (11-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [2\ 1\ 2]^T,\ \mathbf{e}_2 = [1\ 2\ -2]^T, \mathbf{e}_3 = [-2\ 2\ 1]^T$$

Corresponding normalised eigenvectors are

$$\hat{\mathbf{e}}_1 = \frac{1}{3}[2\ 1\ 2]^T,\ \hat{\mathbf{e}}_2 = \frac{1}{3}[1\ 2\ -2]^T,\ \hat{\mathbf{e}}_3 = \frac{1}{3}[-2\ 2\ 1]^T$$

$$\hat{\mathbf{M}} = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}, \quad \hat{\mathbf{M}}^T = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\hat{\mathbf{M}}^T\ \mathbf{A}\ \mathbf{M} = \frac{1}{9}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 10 & 8 \\ 10 & 2 & -2 \\ 8 & -2 & 11 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} 36 & 18 & 36 \\ 9 & 18 & -18 \\ 18 & -18 & -9 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 2 & 4 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix} = \mathbf{\Lambda}$$

■ **23**

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Eigenvalues given by

$$0 = \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = -(\lambda^3 - 2\lambda^2 - \lambda + 2) = -(\lambda - 1)(\lambda - 2)(\lambda + 1) = 0$$

so eigenvalues are $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$.
The eigenvectors are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1} + e_{i2} - 2e_{i3} = 0$$
$$-e_{i1} + (2-\lambda_i)e_{i2} + e_{i3} = 0$$
$$0e_{i1} + e_{i2} - (1+\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [1\ 3\ 1]^T, \mathbf{e}_2 = [3\ 2\ 1]^T, \mathbf{e}_3 = [1\ 0\ 1]^T$$

$$\mathbf{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$\mathbf{M}^{-1} = -\frac{1}{6}\begin{bmatrix} 2 & -2 & -2 \\ -3 & 0 & -3 \\ 1 & 2 & -7 \end{bmatrix}$$

Matrix multiplication then confirms

$$\mathbf{M}^{-1}\,\mathbf{A}\,\mathbf{M} = \mathbf{\Lambda} \quad \text{and} \quad \mathbf{A} = \mathbf{M}\,\mathbf{\Lambda}\,\mathbf{M}^{-1}$$

■ **24** Eigenvalues given by

$$\begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & -2-\lambda & 6 \\ 4 & 6 & -1-\lambda \end{vmatrix} = -\lambda^3 + 63\lambda - 162 = -(\lambda+9)(\lambda-6)(\lambda-3) = 0$$

so the eigenvalues are $\lambda_1 = -9, \lambda_2 = 6, \lambda_3 = 3$. The eigenvectors are the corresponding solutions of

$$\begin{aligned} (3-\lambda_i)e_{i1} - 2e_{i2} + 44e_{i3} &= 0 \\ -2e_{i1} - (2+\lambda_i)e_{i2} + 6e_{i3} &= 0 \\ 4e_{i1} + 6e_{i2} - (1+\lambda_i)e_{i3} &= 0 \end{aligned}$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [1\ 2\ -2]^T, \mathbf{e}_2 = [2\ 1\ 2]^T, \mathbf{e}_3 = [-2\ 2\ 1]^T$$

Since $\mathbf{e}_1^T\mathbf{e}_2 = \mathbf{e}_1^T\mathbf{e}_3 = \mathbf{e}_2^T\mathbf{e}_3 = 0$ the eigenvectors are orthogonal

$$\mathbf{L} = [\hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2\ \hat{\mathbf{e}}_3] = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned} \hat{\mathbf{L}}\,\mathbf{A}\,\mathbf{L} &= \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \\ &= \frac{1}{9}\begin{bmatrix} -9 & -18 & 18 \\ 12 & 6 & 12 \\ -6 & 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \\ &= \frac{1}{9}\begin{bmatrix} -81 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 27 \end{bmatrix} = \begin{bmatrix} -9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \mathbf{\Lambda} \end{aligned}$$

■ **25** Since the matrix $\mathbf{A}$ is symmetric the eigenvectors

$$\mathbf{e}_1 = [1\ 2\ 0]^T,\ \mathbf{e}_2 = [-2\ 1\ 0]^T,\ \mathbf{e}_3 = [e_{31}\ e_{32}\ e_{33}]^T$$

are orthogonal. Hence,

$$\mathbf{e}_1^T\mathbf{e}_3 = e_{31} + 2e_{32} = 0 \text{ and } \mathbf{e}_2^T\mathbf{e}_3 = -2e_{31} + e_{32} = 0$$

Thus, $e_{31} = e_{32} = 0$ and e_{33} arbitrary so a possible eigenvector is $\mathbf{e}_3 = [0\ 0\ 1]^T$. Using $\mathbf{A} = \hat{\mathbf{M}}\ \mathbf{\Lambda}\ \hat{\mathbf{M}}^T$ where $\mathbf{\Lambda} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ gives

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

■ **26** $\mathbf{A} - \mathbf{I} = \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is of rank 2

Nullity $(\mathbf{A} - \mathbf{I}) = 3 - 2 = 1$ so there is only one linearly independent vector corresponding to the eigenvalue 1. The corresponding eigenvector $\mathbf{e}_1$ is given by the solution of $(\mathbf{A} - \mathbf{I})\mathbf{e}_1 = 0$ or

$$-4e_{11} - 7e_{12} - 5e_{13} = 0$$
$$2e_{11} + 3e_{12} + 3e_{13} = 0$$
$$e_{11} + 2e_{12} + 2_{12} = 0$$

that is, $\mathbf{e}_1 = [-3\ 1\ 1]^T$. To obtain the generalised eigenvector $\mathbf{e}_1^*$ we solve

$$(\mathbf{A} - \mathbf{I})\mathbf{e}_1^* = \mathbf{e}_1 \text{ or}$$
$$\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} e_{11}^* \\ e_{12}^* \\ e_{13}^* \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

giving $\mathbf{e}_1^* = [-1\ 1\ 0]^T$. To obtain the second generalised eigenvector $\mathbf{e}_1^{**}$ we solve

$$(\mathbf{A} - \mathbf{I})\mathbf{e}_1^{**} = \mathbf{e}_1^* \text{ or}$$
$$\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} e_{11}^{**} \\ e_{12}^{**} \\ e_{13}^{**} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

giving $\mathbf{e}_1^{**} = [2\ -1\ 0]^T$.

$$\mathbf{M} = [\mathbf{e}_1\ \mathbf{e}_1^*\ \mathbf{e}_1^{**}] = \begin{bmatrix} -3 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\det \mathbf{M} = -1 \text{ and } \mathbf{M}^{-1} = - \begin{bmatrix} 0 & 0 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Matrix multiplication then confirms

$$\mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

■ **27** Eigenvalues are given by

$$|\ \mathbf{A} - \lambda \mathbf{I}\ | = 0$$

that is, $\lambda^4 - 4\lambda^3 - 12\lambda^2 + 32\lambda + 64 = (\lambda + 2)^2(\lambda - 4)^2 = 0$ so the eigenvalues are -2, -2, 4 and 4 as required.

Corresponding to the repeated eigenvalue $\lambda_1, \lambda_2 = -2$

$$(\mathbf{A} + 2\mathbf{I}) = \begin{bmatrix} 3 & 0 & 0 & -3 \\ 0 & 3 & -3 & 0 \\ -0.5 & -3 & 3 & 0.5 \\ -3 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is of rank 2}$$

Thus, nullity $(\mathbf{A}+2\mathbf{I})$ is $4-2 = 2$ so there are two linearly independent eigenvectors corresponding to $\lambda = -2$.

Corresponding to the repeated eigenvalues $\lambda_3, \lambda_4 = 4$

$$(\mathbf{A} - 4\mathbf{I}) = \begin{bmatrix} -3 & 0 & 0 & -3 \\ 0 & -3 & -3 & 0 \\ -0.5 & -3 & -3 & 0.5 \\ -3 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ is of rank 3}$$

Thus, nullity $(\mathbf{A} - 4\mathbf{I})$ is $4 - 3 = 1$ so there is only one linearly independent eigenvector corresponding to $\lambda = 4$.

When $\lambda = \lambda_1 = \lambda_2 = -2$ the eigenvalues are given by the solution of $(\mathbf{A}+2\mathbf{I})\mathbf{e} = 0$ giving $\mathbf{e}_1 = [0\ 1\ 1\ 0]^T, \mathbf{e}_2 = [1\ 0\ 0\ 1]^T$ as two linearly independent solutions. When $\lambda = \lambda_3 = \lambda_4 = 8$ the eigenvectors are given by the solution of

$$(\mathbf{A} - 4\mathbf{I})\mathbf{e} = 0$$

giving the unique solution $\mathbf{e}_3 = [0\ 1\ -1\ 0]^T$. The generalised eigenvector $\mathbf{e}_3^*$ is obtained by solving

$$(\mathbf{A} - 4\mathbf{I})\mathbf{e}_3^* = \mathbf{e}_3$$

giving $\mathbf{e}_3^* = (6\ \ -1\ \ \ 0\ \ -6]^T$. The Jordan canonical form is

$$\mathbf{J} = \left[\begin{array}{cc|cc} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ \hline 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

Exercises 1.6.5

■ **28** The quadratic form may be written in the form $V = \mathbf{x}^T\mathbf{A}\mathbf{x}$ where $\mathbf{x} = [\,x_1 \quad x_2 \quad x_3\,]^T$ and

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

The eigenvalues of $\boldsymbol{A}$ are given by

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 2 & 5-\lambda & 2 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 7\lambda + 6) + 4(\lambda - 1) + (\lambda - 1) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - 8\lambda + 7) = 0 \Rightarrow (\lambda - 1)^2(\lambda - 7) = 0$$

giving the eigenvalues as $\lambda_1 = 7, \lambda_2 = \lambda_3 = 1$
Normalized eigenvector corresponding to $\lambda_1 = 7$ is

$$\hat{\mathbf{e}}_1 = \left[\ \tfrac{1}{\sqrt{6}} \quad \tfrac{2}{\sqrt{6}} \quad \tfrac{1}{\sqrt{6}}\ \right]^T$$

and two orthogonal linearly independent eigenvectors corresponding to $\lambda - 1$ are

$$\hat{\mathbf{e}}_2 = \left[\, \tfrac{1}{\sqrt{2}} \quad 0 \quad -\tfrac{1}{\sqrt{2}} \,\right]^T$$
$$\hat{\mathbf{e}}_3 = \left[\, -\tfrac{1}{\sqrt{3}} \quad \tfrac{1}{\sqrt{3}} \quad -\tfrac{1}{\sqrt{3}} \,\right]^T$$

Note that $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ are automatically orthogonal to $\hat{\mathbf{e}}_1$. The normalized orthogonal modal matrix $\hat{\mathbf{M}}$ and spectral matrix $\mathbf{\Lambda}$ are

$$\hat{\mathbf{M}} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}, \mathbf{\Lambda} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that $\hat{\mathbf{M}}^T \mathbf{A} \hat{\mathbf{M}} = \mathbf{\Lambda}$.

Under the orthogonal transformation $\mathbf{x} = \hat{\mathbf{M}}\mathbf{y}$ the quadratic form V reduces to

$$\begin{aligned} V &= \mathbf{y}^T \hat{\mathbf{M}}^T \mathbf{A} \hat{\mathbf{M}} \mathbf{y} = \mathbf{y}^T \mathbf{\Lambda} \mathbf{y} \\ &= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\ &= 7y_1^2 + y_2^2 + y_3^2 \end{aligned}$$

■ **29(a)** The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 7 \end{bmatrix}$ and its leading principal minors are

$$1, \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1, \det \mathbf{A} = 2$$

Thus, by Sylvester's condition (a) the quadratic form is positive definite.

29(b) Matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 5 \end{bmatrix}$ and its leading principal minors are

$$1, \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1, \det \mathbf{A} = 0$$

Thus, by Sylvester's condition (c) the quadratic form is positive semidefinite.

29(c) Matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 4 \end{bmatrix}$ and its leading principal minors are

$$1, \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1, \det \mathbf{A} = -1.$$

Thus, none of Sylvester's conditions are satisfied and the quadratic form is indefinite.

■ **30(a)** The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}$ and its leading principal minors are a and $ac - b^2$. By Sylvester's condition (a) in the text the quadratic form is positive definite if and only if

$$a > 0 \text{ and } ac - b^2 > 0$$
$$\text{that is, } a > 0 \text{ and } ac > b^2$$

30(b) The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & a & b \\ 0 & b & 3 \end{bmatrix}$ having principal minors $2, 2a - 1$ and $\det \mathbf{A} = 6a - 2b^2 - 3$. Thus, by Sylvester's condition (a) in the text the quadratic form is positive definite if and only if

$$2a - 1 > 0 \text{ and } 6a - 2b^2 - 3 > 0$$
$$\text{or } 2a > 1 \text{ and } 2b^2 < 6a - 3$$

■ **31** The eigenvalues of the matrix $\mathbf{A}$ are given by

$$0 = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix} \underline{\underline{\mathrm{R}_1 + \mathrm{R}_3}} \begin{vmatrix} 3-\lambda & 3-\lambda & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix}$$
$$= (3-\lambda) \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix}$$
$$= (3-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 3\lambda)$$

so the eigenvalues are $3, 3, 0$ indicating that the matrix is positive semidefinite.
The principal minors of $\mathbf{A}$ are

$$2, \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3, \ \det \mathbf{A} = 0$$

confirming, by Sylvester's condition (a), that the matrix is positive semidefinite.

■ **32** The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} K & 1 & 1 \\ 1 & K & -1 \\ 1 & -1 & 1 \end{bmatrix}$ having principal minors

$$K, \begin{vmatrix} K & 1 \\ 1 & K \end{vmatrix} = K^2 - 1 \text{ and } \det \mathbf{A} = K^2 - K - 3$$

Thus, by Sylvester's condition (a) the quadratic form is positive definite if and only if

$$K^2 - 1 = (K-1)(K+1) > 0 \text{ and } K^2 - K - 3 = (K-2)(K+1) > 0$$

i.e. $K > 2$ or $K < -1$.
If $K = 2$ then $\det \mathbf{A} = 0$ and the quadratic form is positive semidefinite.

■ **33** Principal minors of the matrix are

$$3 + a, \begin{vmatrix} 3+a & 1 \\ 1 & a \end{vmatrix} = a^2 + 3a - 1, \det \mathbf{A} = a^3 + 3a^2 - 6a - 8$$

Thus, by Sylvester's condition (a) the quadratic form is positive definite if and only if

$$3 + a > 0, a^2 + 3a - 1 > 0 \text{ and } a^3 + 3a^2 - 6a - 8 > 0$$
$$\text{or } (a+1)(a+4)(a-2) > 0$$

$$3 + a > 0 \ \Rightarrow \ a > -3$$
$$a^2 + 3a - 1 > 0 \ \Rightarrow \ a < -3.3 \text{ or } a > 0.3$$
$$(a+1)(a+4)(a-2) > 0 \ \Rightarrow \ a > 2 \text{ or } -4 < a < -1$$

Thus, minimum value of a for which the quadratic form is positive definite is $a = 2$.

■ **34** $\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & \lambda & -3 \\ -2 & -3 & \lambda \end{bmatrix}$

Principal minors are

$$1,\ \begin{vmatrix} 1 & 2 \\ 2 & \lambda \end{vmatrix} = \lambda - 4,\ \det \mathbf{A} = \lambda^2 - 8\lambda + 15 = 0$$

Thus, by Sylvester's condition (a) the quadratic form is positive definite if and only if

$$\lambda - 4 > 0 \ \Rightarrow\ \lambda > 4$$
$$\text{and } (\lambda - 5)(\lambda - 3) > 0 \ \Rightarrow\ \lambda < 3 \text{ or } \lambda > 5$$

Thus, it is positive definite if and only if $\lambda > 5$.

Exercises 1.7.1

■ **35** The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 5 - \lambda & 6 \\ 2 & 3 - \lambda \end{vmatrix} = \lambda^2 - 8\lambda + 3 = 0$$

$$\text{Now } \mathbf{A}^2 = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 27 & 48 \\ 16 & 21 \end{bmatrix} \text{ so}$$

$$\mathbf{A}^2 - 8\mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 37 & 48 \\ 16 & 21 \end{bmatrix} - \begin{bmatrix} 40 & 48 \\ 16 & 24 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so that $\mathbf{A}$ satisfies its own characteristic equation.

■ **36** The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 1 - \lambda & 2 \\ 1 & 1 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda - 1 = 0$$

By Cayley–Hamilton theorem

$$\mathbf{A}^2 - 2\mathbf{A} - \mathbf{I} = 0$$

36(a) Follows that $\mathbf{A}^2 = 2\mathbf{A} + \mathbf{I} = \begin{bmatrix} 2 & 4 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

36(b) $\mathbf{A}^3 = 2\mathbf{A}^2 + \mathbf{A} = \begin{bmatrix} 6 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix}$

36(c) $\mathbf{A}^4 = 2\mathbf{A}^3 + \mathbf{A}^2 = \begin{bmatrix} 14 & 20 \\ 10 & 14 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 24 \\ 12 & 17 \end{bmatrix}$

■ **37(a)** The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\text{that is,} \quad \lambda^2 - 4\lambda + 3 = 0$$

Thus, by the Cayley–Hamilton theorem

$$\mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I} = 0$$

$$\mathbf{I} = \frac{1}{3}[4\mathbf{A} - \mathbf{A}^2]$$

$$\text{so that} \quad \mathbf{A}^{-1} = \frac{1}{3}[4\mathbf{I} - \mathbf{A}]$$

$$= \frac{1}{3}\left\{ \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

37(b) The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 3 & 1-\lambda & 1 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\text{that is,} \quad \lambda^3 - 3\lambda^2 - 7\lambda - 11 = 0$$

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{bmatrix}$$

Using (1.44)

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{11}(\mathbf{A}^2 - 3\mathbf{A} - 7\mathbf{I}) \\ &= \frac{1}{11} \begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix} \end{aligned}$$

■ **38** $\mathbf{A}^2 = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 11 & 11 \\ 11 & 14 & 11 \\ 11 & 11 & 14 \end{bmatrix}$

The characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 6\lambda^2 - 3\lambda + 18 = 0$$

so by the Cayley–Hamilton theorem

$$\mathbf{A}^3 = 6\mathbf{A}^2 + 3\mathbf{A} - 18\mathbf{I}$$

giving

$$\begin{aligned} \mathbf{A}^4 &= 6(6\mathbf{A}^2 + 3\mathbf{A} - 18\mathbf{I}) + 3\mathbf{A}^2 - 18\mathbf{A} = 39\mathbf{A}^2 - 108\mathbf{I} \\ \mathbf{A}^5 &= 39(6\mathbf{A}^2 + 3\mathbf{A} - 18\mathbf{I}) + 108\mathbf{A} = 234\mathbf{A}^2 + 9\mathbf{A} - 702\mathbf{I} \\ \mathbf{A}^6 &= 234(6\mathbf{A}^2 + 3\mathbf{A} - 18\mathbf{I}) + 9\mathbf{A}^2 - 702\mathbf{A} = 1413\mathbf{A}^2 - 4212\mathbf{I} \\ \mathbf{A}^7 &= 1413(6\mathbf{A}^2 + 3\mathbf{A} - 18\mathbf{I}) + 4212\mathbf{A} = 8478\mathbf{A}^2 + 27\mathbf{A} - 25434\mathbf{I} \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{A}^7 - 3\mathbf{A}^6 + \mathbf{A}^4 + 3\mathbf{A}^3 - 2\mathbf{A}^2 + 3\mathbf{I} &= 4294\mathbf{A}^2 + 36\mathbf{A} - 12957\mathbf{I} \\ &= \begin{bmatrix} 47231 & 47342 & 47270 \\ 47342 & 47195 & 47306 \\ 47270 & 47306 & 47267 \end{bmatrix} \end{aligned}$$

■ **39(a)** Eigenvalues $\mathbf{A}$ are $\lambda = 1$ (repeated). Thus,

$$e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \text{ with}$$

$$\left.\begin{array}{ll} e^t & = \alpha_0 + \alpha_1 \\ te^t & = \alpha_1 \end{array}\right\} \Rightarrow \alpha_1 = te^t,\ \alpha_0 = (1-t)e^t$$

$$\text{so } e^{\mathbf{A}t} = (1-t)e^t\mathbf{I} + te^t\mathbf{A} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

39(b) Eigenvalues $\mathbf{A}$ are $\lambda = 1$ and $\lambda = 2$. Thus,

$$e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \text{ with}$$

$$\left.\begin{array}{ll} e^t & = \alpha_0 + \alpha_1 \\ e^{2t} & = \alpha_0 + 2\alpha_1 \end{array}\right\} \Rightarrow \alpha_0 = 2e^t - e^{2t},\ \alpha_1 = e^{2t} - e^t$$

$$\text{so } e^{\mathbf{A}t} = (2e^t - e^{2t})\mathbf{I} + (e^{2t} - e^t)\mathbf{A} = \begin{bmatrix} e^t & 0 \\ e^{2t} - e^t & e^{2t} \end{bmatrix}$$

■ **40** Eigenvalues of $\mathbf{A}$ are $\lambda_1 = \pi, \lambda_2 = \dfrac{\pi}{2}, \lambda_3 = \dfrac{\pi}{2}$.
Thus,

$$\sin\mathbf{A} = \alpha_0\mathbf{A} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2 \text{ with}$$

$$\sin\pi = 0 = \alpha_0 + \alpha_1\pi + \alpha_2\pi^2$$

$$\sin\frac{\pi}{2} = 1 = \alpha_0 + \alpha_1\frac{\pi}{2} + \alpha_2\frac{\pi^2}{4}$$

$$\cos\frac{\pi}{2} = 0 = \alpha_1 + \pi\alpha_2$$

Solving gives $\alpha_0 = 0,\ \alpha_1 = \dfrac{4}{\pi},\ \alpha_2 = -\dfrac{4}{\pi^2}$ so that

$$\sin\mathbf{A} = \frac{4}{\pi}\mathbf{A} - \frac{4}{\pi^2}\mathbf{A}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■ **41(a)**

$$\frac{d\mathbf{A}}{dt} = \begin{bmatrix} \frac{d}{dt}(t^2+1) & \frac{d}{dt}(2t-3) \\ \frac{d}{dt}(5-t) & \frac{d}{dt}(t^2-t+3) \end{bmatrix} = \begin{bmatrix} 2t & 2 \\ -1 & 2t-1 \end{bmatrix}$$

41(b)

$$\int_1^2 \mathbf{A}dt = \begin{bmatrix} \int_1^2 (t^2+1)dt & \int_1^2 (2t-3)dt \\ \int_1^2 (5-t)dt & \int_1^2 (t^2-t+3)dt \end{bmatrix} = \begin{bmatrix} \frac{10}{3} & 0 \\ \frac{7}{2} & \frac{23}{6} \end{bmatrix}$$

■ **42**

$$\begin{aligned} \mathbf{A}^2 &= \begin{bmatrix} t^2+1 & t-1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} t^2+1 & t-1 \\ 5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} t^4+2t^2+5t-4 & t^3-t^2+t-1 \\ 5t^2+5 & 5t-5 \end{bmatrix} \\ \frac{d}{dt}(\mathbf{A}^2) &= \begin{bmatrix} 4t^3+4t+5 & 3t^2-2t+1 \\ 10t & 5 \end{bmatrix} \\ 2\mathbf{A}\frac{d\mathbf{A}}{dt} &= \begin{bmatrix} 4t^3+4t & 2t^2+1 \\ 20t & 0 \end{bmatrix} \end{aligned}$$

Thus, $\dfrac{d}{dt}(\mathbf{A}^2) \neq 2\mathbf{A}\dfrac{d\mathbf{A}}{dt}$.

Exercises 1.8.4

■ **43(a)** *row rank*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 7 & 10 \\ 2 & 1 & 5 & 7 \end{bmatrix} \begin{matrix} \text{row2} - 3\text{row1} \\ \rightarrow \\ \text{row3} - 2\text{row1} \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -2 \\ 0 & -3 & -1 & -1 \end{bmatrix}$$

$$\begin{matrix} -\frac{1}{2}\text{row2} \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -1 & -1 \end{bmatrix} \quad \begin{matrix} \text{row3} + 3\text{row2} \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

echelon form, row rank 3

column rank

$$\mathbf{A} \begin{matrix} \text{col2} - 2\text{col1} \\ \rightarrow \\ \text{col3} - 3\text{col1} \\ \text{col4} - 4\text{col1} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & -2 & 2 \\ 2 & -3 & 2 & 0 \end{bmatrix} \begin{matrix} \text{col3} - \text{col2} \\ \rightarrow \\ \text{col4} - \text{col2} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & -3 & 2 & 2 \end{bmatrix}$$

$$\begin{matrix} \text{col4} - \text{col3} \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 2 & -3 & 2 & 0 \end{bmatrix}$$

echelon form, column rank3

Thus row rank($\mathbf{A}$) = column rank($\mathbf{A}$) = 3

(b) $\mathbf{A}$ is of full rank since rank($\mathbf{A}$)=min(m, n)=min(3,4)= 3

■ **44(a)** $\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix} = 9 \begin{bmatrix} 37 & 9 \\ 9 & 13 \end{bmatrix}$

The eigenvalues λ_i of $\boldsymbol{A}\boldsymbol{A}^T$ are given by the solutions of the equations

$$\left|\mathbf{A}\mathbf{A}^T - \lambda\mathbf{I}\right| = \begin{vmatrix} 333-\lambda & 8 \\ 81 & 117-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 450\lambda + 32400 = 0$$

$$\Rightarrow (\lambda - 360)(\lambda - 90) = 0$$

giving the eigenvalues as $\lambda_1 = 360, \lambda_2 = 90$. Solving the equations.

$$(\mathbf{A}\mathbf{A}^T - \lambda_i\mathbf{I})\mathbf{u}_i = 0$$

gives the corresponding eigenvectors as

$$\mathbf{u}_1 = [\,3 \quad 1\,]^T, \mathbf{u}_2 = [\,1 \quad -2\,]^T$$

with the corresponding normalized eigenvectors being

$$\hat{\mathbf{u}}_1 = \left[\,\tfrac{3}{\sqrt{10}} \quad \tfrac{1}{\sqrt{10}}\,\right]^T, \hat{\mathbf{u}}_2 = \left[\,\tfrac{1}{\sqrt{10}} \quad -\tfrac{3}{\sqrt{10}}\,\right]^T$$

leading to the orthogonal matrix

$$\hat{\mathbf{U}} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}$$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

$$\text{Solving } \left|\mathbf{A}^T\mathbf{A} - \mu\mathbf{I}\right| = \begin{vmatrix} 80-\mu & 100 & 40 \\ 100 & 170-\mu & 140 \\ 40 & 140 & 200-\mu \end{vmatrix} = 0$$

gives the eigenvalues $\mu_1 = 360, \mu_2 = 90, \mu_3 = 0$ with corresponding normalized eigenvectors

$$\hat{\mathbf{v}}_1 = \left[\,\tfrac{1}{3} \quad \tfrac{2}{3} \quad \tfrac{2}{3}\,\right]^T, \hat{\mathbf{v}}_2 = \left[\,-\tfrac{2}{3} \quad -\tfrac{1}{3} \quad \tfrac{2}{3}\,\right]^T, \hat{\mathbf{v}}_3 = \left[\,\tfrac{2}{3} \quad -\tfrac{2}{3} \quad \tfrac{1}{3}\,\right]^T$$

leading to the orthogonal matrix

$$\hat{\mathbf{V}} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

The singular values of $\mathbf{A}$ are $\sigma_1 = \sqrt{360} = 6\sqrt{10}$ and $\sigma_2 = \sqrt{90} = 3\sqrt{10}$ giving

$$\boldsymbol{\Sigma} = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

Thus, the SVD form of $\mathbf{A}$ is

$$\mathbf{A} = \hat{\mathbf{U}}\boldsymbol{\Sigma}\hat{\mathbf{V}}^T = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

(Direct multiplication confirms $\mathbf{A} = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$)

(b) Using (1.55) the pseudo inverse of $\boldsymbol{A}$ is

$$\mathbf{A}^\dagger = \hat{\mathbf{V}}\boldsymbol{\Sigma}^*\hat{\mathbf{U}},\ \boldsymbol{\Sigma}^* = \begin{bmatrix} \frac{1}{6\sqrt{10}} & 0 \\ 0 & \frac{2}{3\sqrt{10}} \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{6\sqrt{10}} & 0 \\ 0 & \frac{1}{3\sqrt{10}} \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} \Rightarrow \mathbf{A}^\dagger = \frac{1}{180} \begin{bmatrix} -1 & 13 \\ 4 & 8 \\ 10 & -10 \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^\dagger = \frac{1}{180} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} -1 & 13 \\ 4 & 8 \\ 10 & -10 \end{bmatrix} = \frac{1}{180} \begin{bmatrix} 180 & 0 \\ 0 & 180 \end{bmatrix} = \mathbf{I}$$

(c) Rank($\boldsymbol{A}$) = 2 so $\boldsymbol{A}$ is of full rank. Since number of rows is less than the number of columns $\mathbf{A}^\dagger$ may be determined using (1.58b) as

$$\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1} = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}^{-1} = \frac{1}{180} \begin{bmatrix} -1 & 13 \\ 4 & 8 \\ 10 & -10 \end{bmatrix}$$

which confirms with the value determined in (b).

■ **45** $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ -2 & 1 \\ 0 & 2 \\ -1 & 2 \end{bmatrix} \begin{matrix} \text{row2} - 3\text{row1} \\ \text{row3} + 2\text{row1} \\ \rightarrow \\ \text{row5} + \text{row1} \end{matrix} \begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 3 \\ 0 & 2 \\ 0 & 3 \end{bmatrix} \begin{matrix} \text{row3} + \text{row2} \\ \text{row4} + \frac{2}{3}\text{row2} \\ \rightarrow \\ \text{row5} + \text{row2} \end{matrix} \begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

echelon form so row rank = 2 = column rank

Thus, rank $\boldsymbol{A} = 2 = \min(5,2)$ and so $\boldsymbol{A}$ is of full rank.

Since $\boldsymbol{A}$ is of full rank and number of rows is greater than number of columns we can determine the pseudo inverse using result (1.58a)

$$\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \begin{bmatrix} 15 & -3 \\ -3 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 & -2 & 0 & -1 \\ 1 & 0 & 1 & 2 & 2 \end{bmatrix}$$

$$= \tfrac{1}{141} \begin{bmatrix} 10 & 3 \\ 3 & 15 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 & 0 & -1 \\ 1 & 0 & 1 & 2 & 2 \end{bmatrix}$$

$$= \tfrac{1}{141} \begin{bmatrix} 13 & 30 & -17 & 6 & -4 \\ 18 & 9 & 9 & 30 & 27 \end{bmatrix}$$

$$\mathbf{A}^\dagger\mathbf{A} = \tfrac{1}{141} \begin{bmatrix} 13 & 30 & -17 & 6 & -4 \\ 18 & 9 & 9 & 30 & 27 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ -2 & 1 \\ 0 & 2 \\ -1 & 2 \end{bmatrix} = \tfrac{1}{141} \begin{bmatrix} 141 & 0 \\ 0 & 141 \end{bmatrix} = \mathbf{I}$$

■ **46(a)** $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{matrix} \text{row2} + 2\text{row1} \\ \rightarrow \\ \text{row3} - 2\text{row1} \end{matrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus, rank $\mathbf{A} = 1$and is not of full rank

(b) $\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix}$

The eigenvalues λ_i are given by

$$\begin{vmatrix} 2-\lambda & -4 & 4 \\ -4 & 2-\lambda & -8 \\ 4 & -8 & 8-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2(-\lambda + 18) = 0$$

giving the eigenvalues as $\lambda_1 = 18, \lambda_2 = 0, \lambda_3 = 0$. The corresponding eigenvectors and normalized eigenvectors are

$$\mathbf{u}_1 = [\,1 \quad -2 \quad 2\,]^T \Rightarrow \hat{\mathbf{u}}_1 = [\,\tfrac{1}{3} \quad -\tfrac{2}{3} \quad \tfrac{2}{3}\,]^T$$
$$\mathbf{u}_2 = [\,0 \quad 1 \quad 1\,]^T \Rightarrow \hat{\mathbf{u}}_2 = \left[\,0 \quad \tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}}\,\right]^T$$
$$\mathbf{u}_3 = [\,2 \quad 1 \quad 0\,]^T \Rightarrow \hat{\mathbf{u}}_3 = \left[\,\tfrac{2}{\sqrt{5}} \quad \tfrac{1}{\sqrt{5}} \quad 0\,\right]^T$$

leading to the orthogonal matrix

$$\hat{\mathbf{U}} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = 9 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

having eigenvalues $\mu_1 = 18$ and $\mu_2 = 0$ and corresponding eigenvectors

$$\mathbf{v}_1 = [\,1 \quad -1\,]^T \Rightarrow \hat{\mathbf{v}}_1 = \left[\,\tfrac{1}{\sqrt{2}} \quad -\tfrac{1}{\sqrt{2}}\,\right]^T$$
$$\mathbf{v}_2 = [\,1 \quad 1\,]^T \Rightarrow \hat{\mathbf{v}}_2 = \left[\,\tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}}\,\right]^T$$

leading to the orthogonal matrix

$$\hat{\mathbf{V}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\boldsymbol{A}$ has the single (equal to its rank) singular value $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ so that $\mathbf{\Sigma} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and the SVD form of $\boldsymbol{A}$ is

$$\mathbf{A} = \hat{\mathbf{U}}\mathbf{\Sigma}\hat{\mathbf{V}}^T = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Direct multiplication confirms that $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

(c) Pseudo inverse is given by

$$\mathbf{A}^\dagger = \hat{\mathbf{V}}\mathbf{\Sigma}^*\hat{\mathbf{U}}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{3\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

Direct multiplication confirms $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}and\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$

(d) Equations may be written as

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \equiv \mathbf{A}\mathbf{x} = \mathbf{b}$$

The least squares solution is $\mathbf{x} = \mathbf{A}^\dagger\mathbf{b} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{6} \end{bmatrix}$ giving $x = \frac{1}{6}$ and $y = -\frac{1}{6}$

(e) Minimize $L = (x - y - 1)^2 + (-2x + 2y - 2)^2 + (2x - 2y - 3)^2$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 2(x - y - 1) - 4(-2x + 2y - 2) + 4(2x - 2y - 3) = 18x - 18y - 6 = 0$$
$$\Rightarrow 3x - 3y - 1 = 0$$
$$\frac{\partial L}{\partial y} = 0 \Rightarrow -2(x - y - 1) + 4(-2x + 2y - 2) - 4(2x - 2y - 3) = -18x + 18y + 6 = 0$$
$$\Rightarrow -3x + 3y + 1 = 0$$

Solving the two simultaneous equations gives the least squares solution $x = \frac{1}{6}$, $y = -\frac{1}{6}$ confirming the answer in (d)

- **47(a)** Equations may be written as

$$\begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \equiv \mathbf{A}\mathbf{x} = \mathbf{b}$$

Using the pseudo inverse obtained in Example 1.39, the least squares solution is

$$\mathbf{x} = \mathbf{A}^\dagger\mathbf{b} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 17 & 4 & 5 \\ -7 & 16 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

giving $x = y = \frac{2}{3}$

(b) Minimize $L = (3x - y - 1)^2 + (x + 3y - 2)^2 + (x + y - 3)^2$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 6(3x - y - 1) + 2(x + 3y - 2) + 2(x + y - 3) = 0$$
$$\Rightarrow 11x + y - 8 = 0$$
$$\frac{\partial L}{\partial y} = 0 \Rightarrow -2(3x - y - 1) + 6(x + 3y - 2) + 2(x + y - 3) = 0$$
$$\Rightarrow x + 11y - 8 = 0$$

Solving the two simultaneous equations gives the least squares solution $x = y = \frac{2}{3}$ confirming the answer in (a)

- **48(a)**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix} \begin{matrix} \text{row3} + \text{row1} \\ \rightarrow \\ \text{row4} - 2\text{row1} \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 6 \end{bmatrix} \begin{matrix} \text{row3} - \text{row2} \\ \rightarrow \\ \text{row4} + \text{row2} \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Thus, $\boldsymbol{A}$ is of rank 3 and is of full rank as 3=min(4,3)

(b) Since $\boldsymbol{A}$ is of full rank

$$\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \begin{bmatrix} 6 & -3 & 1 \\ -3 & 3 & -2 \\ 1 & -2 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ -2 & -1 & 1 & 2 \end{bmatrix}$$
$$\Rightarrow \mathbf{A}^\dagger = \tfrac{1}{75} \begin{bmatrix} 26 & 28 & 3 \\ 28 & 59 & 9 \\ 3 & 9 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ -2 & -1 & 1 & 2 \end{bmatrix} = \tfrac{1}{15} \begin{bmatrix} 4 & 5 & 1 & 6 \\ 2 & 10 & 8 & 3 \\ -3 & 0 & 3 & 3 \end{bmatrix}$$

(c) Direct multiplication confirms that $\mathbf{A}^\dagger$ satisfies the conditions

$$\mathbf{A}\mathbf{A}^T \text{ and } \mathbf{A}^T\mathbf{A} \text{ are symmetric, } \mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A} \text{ and } \mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$$

- **49(a)** $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$ is of full rank 2 so pseudo inverse is

$$\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \begin{bmatrix} 0.6364 & -0.3636 & 0.0909 \\ -0.3636 & 0.6364 & 0.0909 \end{bmatrix}$$

Equations (i) are consistent with unique solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \Rightarrow x = y = 1$$

Equations (ii) are inconsistent with least squares solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \Rightarrow x = 1.0909, y = 1.0909$$

(b) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 10 & 10 \end{bmatrix}$ with pseudo inverse $\mathbf{A}^{\dagger} = \begin{bmatrix} 0.5072 & -0.4928 & 0.0478 \\ -0.4928 & 0.5072 & 0.0478 \end{bmatrix}$
Equations (i) are consistent with unique solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} 3 \\ 3 \\ 20 \end{bmatrix} \Rightarrow x = y = 1$$

Equations (ii) are inconsistent and have least squares solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} 3 \\ 3 \\ 30 \end{bmatrix} \Rightarrow x = y = 1.4785$$

(c) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 100 & 100 \end{bmatrix}$ with pseudo inverse $\mathbf{A}^{\dagger} = \begin{bmatrix} 0.5001 & -0.4999 & 0.0050 \\ -o.4999 & 0.5001 & 0.0050 \end{bmatrix}$
Equations (i) are consistent with unique solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} 3 \\ 3 \\ 200 \end{bmatrix} \Rightarrow x = y = 1$$

Equations (ii) are inconsistent with least squares solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A} \begin{bmatrix} 3 \\ 3 \\ 300 \end{bmatrix} \Rightarrow x = y = 1.4998$$

Since the sets of equations (i) are consistent weighting the last equation has no effect on the least squares solution which is unique. However, since the sets of equations (ii) are inconsistent the solution given is not unique but is the best in the least squares sense. Clearly as the weighting of the third equation increases from (a) to (b) to (c) the better is the matching to the third equation, and the last case (c) does not bother too much with the first two equations.

■ **50** Data may be represented in the matrix form

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

$Az = Y$
MATLAB gives the pseudo inverse

$$\mathbf{A}^{\dagger} = \begin{bmatrix} -0.2 & -0.1 & 0 & 0.1 & 0.2 \\ 0.8 & 0.4 & 0.2 & 0 & -0.2 \end{bmatrix}$$

and, the least squares solution

$$\begin{bmatrix} m \\ c \end{bmatrix} = \mathbf{A}^{\dagger}\mathbf{y} = \begin{bmatrix} 0.5 \\ 0.8 \end{bmatrix}$$

leads to the linear model

$$y = 0.5x + 0.8$$

Exercises 1.9.3

■ **51(a)** Taking $x_1 = y$

$$\dot{x}_1 = x_2 = \frac{dy}{dt}$$
$$\dot{x}_2 = x_3 = \frac{d^2y}{dt^2}$$
$$\dot{x}_3 = \frac{d^3y}{dt^3} = u(t) - 4x_1 - 5x_2 - 4x_3$$

Thus, state space form is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y = x_1 = [1\ 0\ 0]\ [x_1\ x_2\ x_3]^T$$

51(b)

$$x_1 = y$$

$$x_2 = \dot{x}_1 = \frac{dy}{dt}$$

$$x_3 = \dot{x}_2 = \frac{d^2y}{dt^2}$$

$$x_4 = \dot{x}_3 = \frac{d^3y}{dt^3}$$

$$\dot{x}_4 = \frac{d^4y}{dt^4} = -4x_2 - 2x_3 + 5u(t)$$

Thus, state space form is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} u(t)$$

$$y = x_1 = [1\ 0\ 0\ 0]\ [x_1\ x_2\ x_3\ x_4]^T$$

■ **52(a)** Taking $\boldsymbol{A}$ to be the companion matrix of the LHS

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -5 & -6 \end{bmatrix}$$

and taking $\mathbf{b} = [\,0 \quad 0 \quad 1\,]^T$ and then using (1.67) in the text $\mathbf{c} = [\,5 \quad 3 \quad 1\,]$. Then from (1.84) the state-space form of the dynamic model is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \mathrm{y} = \mathbf{c}\mathbf{x}$$

(b) Taking $\boldsymbol{A}$ to be the companion matrix of the LHS

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

and taking $\mathbf{b} = [\,0 \quad 0 \quad 1\,]^T$ then using (1.67) in the text $\mathbf{c} = [\,2 \quad 3 \quad 1\,]$. Then from (1.84) the state-space form of the dynamic model is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \mathrm{y} = \mathbf{c}\mathbf{x}$$

■ **53** Applying Kirchhoff's second law to the individual loops gives

$$e = R_1(i_1 + i_2) + v_c + L_1\frac{di_1}{dt}\ ,\ \dot{v}_c = \frac{1}{C}(i_1 + i_2)$$
$$e = R_1(i_1 + i_2) + v_c + L_2\frac{di_2}{dt} + R_2 i_2$$

so that,

$$\frac{di_1}{dt} = -\frac{R_1}{L_1}i_1 - \frac{R_1}{L_1}i_2 - \frac{v_c}{L_1} + \frac{e}{L_1}$$
$$\frac{di_2}{dt} = -\frac{R_1}{L_2}i_1 - \frac{(R_1 + R_2)}{L_2}i_2 - \frac{v_c}{L_2} + \frac{e}{L_2}$$
$$\frac{dv_c}{dt} = \frac{1}{C}(i_1 + i_2)$$

Taking $x_1 = i_1, x_2 = i_2, x_3 = v_c, u = e(t)$ gives the state equation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & -\frac{R_2}{L_1} & -\frac{1}{L_1} \\ -\frac{R_1}{L_2} & -\frac{(R_1+R_2)}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \\ 0 \end{bmatrix} u(t) \qquad (1)$$

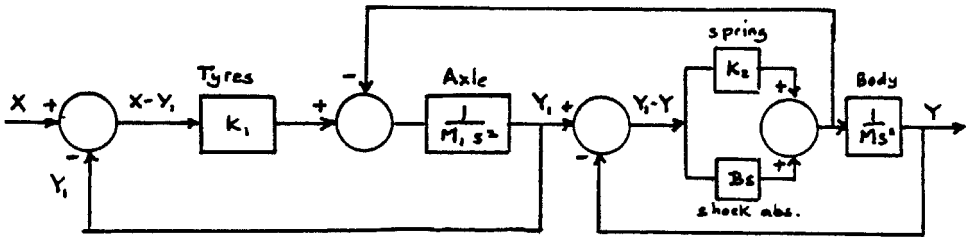
The output $y =$ voltage drop across $R_2 = R_2 i_2 = R_2 x_2$ so that

$$y = [0\ R_2\ 0]\ [x_1\ x_2\ x_3]^T \qquad (2)$$

Equations (1) and (2) are then in the required form

$$\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x} + \mathbf{b}u\ ,\ y = \mathbf{c}^T\mathbf{x}$$

■ **54** The equations of motion, using Newton's second law, may be written down for the body mass and axle/wheel mass from which a state-space model can be deduced. Alternatively a block diagram for the system, which is more informative for modelling purposes, may be drawn up as follows



where s denotes the Laplace 's' and upper case variables X, Y, Y_1 denote the corresponding Laplace transforms of the corresponding lower case time domain variables $x(t), y(t), y_1(t); y_1(t)$ is the vertical displacement of the axle/wheel mass. Using basic block diagram rules this block diagram may be reduced to the input/output transfer function model

$$\xrightarrow{X} \boxed{\frac{K_1(K+Bs)}{(M_1s^2+K_1)(Ms^2+Bs+K)+Ms^2(K+Bs)}} \xrightarrow{Y}$$

or the time domain differential equation model

$$M_1M\frac{d^4y}{dt^4}+B(M_1+M)\frac{d^3y}{dt^3}+(K_1M+KM_1+KM)\frac{d^2y}{dt^2} + K_1B\frac{dy}{dt}+K_1Ky=K_1K_2x+K_1B\frac{dx}{dt}$$

A possible state space model is

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{bmatrix}=\begin{bmatrix}-B(M_1+M) & 1 & 0 & 0\\ \frac{-(K_1M+KM_1+KM)}{MM_1} & 0 & 1 & 0\\ \frac{-K_1B}{M_1M} & 0 & 0 & 1\\ \frac{-K_1K}{M_1M} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{K_1B}{M_1M}\\ \frac{K_1K_2}{MM_1}\end{bmatrix}x(t)$$

$$y=[1\ 0\ 0\ 0]\mathbf{z}(t),\ \mathbf{z}=[z_1\ z_2\ z_3\ z_4]^T.$$

Clearly alternative forms may be written down, such as, for example, the companion form of equation (1.66) in the text. Disadvantage is that its output y is not one of the state variables.

■ **55** Applying Kirchhoff's second law to the first loop gives

$$x_1 + R_3(i - i_1) + R_1 i = u$$
$$\text{that is, } (R_1 + R_3)i - R_3 i_1 + x_1 = u$$

Applying it to the outer loop gives

$$x_2 + (R_4 + R_2)i_1 + R_1 i = u$$

Taking $\alpha = R_1R_3 + (R_1 + R_3)(R_4 + R_2)$ then gives

$$\alpha i = (R_2 + R_3 + R_4)u - (R_2 + R_4)x_1 - R_3 x_2$$
$$\text{and} \quad \alpha i_1 = R_3 u + R_1 x_1 - (R_1 + R_3)x_2$$

Thus,

$$\alpha(i - i_1) = (R_4 + R_2)u - (R_1 + R_2 + R_4)x_1 + R_1 x_2$$

$$\text{Voltage drop across } C_1 : \dot{x}_1 = \frac{1}{C_1}(i - i_1)$$
$$= \frac{1}{\alpha C_1}[-(R_1 + R_2 + R_4)x_1 + R_1 x_2 + (R_4 + R_2)u] \quad (1)$$

$$\text{Voltage drop across } C_2 : \ \dot{x}_2 = \frac{1}{C_2} i_1$$
$$= \frac{1}{\alpha C_2}[R_1 x_1 - (R_1 + R_3)x_2 + R_3 u] \quad (2)$$

$$y_1 = i_1 = \frac{R_1}{\alpha} x - \frac{(R_1 + R_3)}{\alpha} x_2 + \frac{R_3}{\alpha} u \quad (3)$$

$$y_2 = R_2(i - i_1) = -\frac{R_3}{\alpha}(R_1 + R_2 + R_4)x_1 + \frac{R_3 R_1}{\alpha} x_2 + R_3 \frac{(R_4 + R_2)}{\alpha} u \quad (4)$$

Equations (1)–(4) give the required state space model.
Substituting the given values for R_1, R_2, R_3, R_4, C_1 and C_2 gives the state matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{bmatrix} \frac{-9}{35.10^{-3}} & \frac{1}{35.10^{-3}} \\ \frac{1}{35.10^{-3}} & \frac{-4}{35.10^{-3}} \end{bmatrix} = \frac{10^3}{35} \begin{bmatrix} -9 & 1 \\ 1 & -4 \end{bmatrix}$$

Let $\beta = \dfrac{10^3}{35}$ then eigenvalues are solutions of

$$\begin{vmatrix} -9\beta - \lambda & \beta \\ \beta & -4\beta - \lambda \end{vmatrix} = \lambda^2 + 13\beta\lambda + 35\beta^2 = 0$$

giving

$$\lambda = \frac{-13 \pm \sqrt{29}}{2}\beta \simeq -2.6 \times 10^2 \text{ or } -1.1 \times 10^2$$

Exercises 1.10.4

■ **56** $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ where $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Eigenvalues of $\mathbf{A}$ are $\lambda = 1, \lambda = 1$ so

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A}$$

where α_0, α_1 satisfy

$$e^{\lambda t} = \alpha_0 + \alpha_1\lambda, \quad \lambda = 1$$
$$te^{\lambda t} = \alpha_1$$

giving $\alpha_1 = te^t$, $\alpha_0 = e^t - te^t$

Thus,

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} e^t - te^t & 0 \\ 0 & e^t - te^t \end{bmatrix} + \begin{bmatrix} te^t & 0 \\ te^t & te^t \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

56(a) $\mathbf{\Phi}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$

56(b)

$$\mathbf{\Phi}(t_2 - t_1)\mathbf{\Phi}(t_1) = \begin{bmatrix} e^{t_2}e^{-t_1} & 0 \\ (t_2 - t_1)e^{t_2}e^{-t_1} & e^{t_2}e^{-t_1} \end{bmatrix} \begin{bmatrix} e^{t_1} & 0 \\ t_1e^{t_1} & e^{t_1} \end{bmatrix}$$
$$= \begin{bmatrix} e^{t_2} & 0 \\ (t_2 - t_1)e^{t_2} + t_1e^{t_2} & e^{t_2} \end{bmatrix} = \begin{bmatrix} e^{t_2} & 0 \\ t_2e^{t_2} & e^{t_2} \end{bmatrix} = \mathbf{\Phi}(t_2)$$

56(c) $\boldsymbol{\Phi}^{-1} = \dfrac{1}{e^{2t}} \begin{bmatrix} e^t & 0 \\ -te^t & e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ -te^{-t} & e^{-t} \end{bmatrix} = \boldsymbol{\Phi}(-t)$

■ **57** Take $x_1 = y,\ x_2 = \dot{x}_1 = \dfrac{dy}{dt},\ \dot{x}_2 = \dfrac{d^2y}{dt^2} = -x_1 - 2x_2$ so in vector–matrix form the differential equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x},\ y = [1\ 0]\mathbf{A}$$

Taking $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ its eigenvalues are $\lambda = -1, \lambda = -1$

$$e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \text{ where } \alpha_0, \alpha_1 \text{ satisfy}$$
$$e^{\lambda t} = \alpha_0 + \alpha_1\lambda,\ \lambda = -1$$
$$te^{\lambda t} = \alpha_1$$

giving $\alpha_0 = e^{-t} + te^{-t}, \alpha_1 = te^{-t}$. Thus,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Thus, solution of differential equation is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0),\ \mathbf{x}(0) = [1\ 1]^T$$
$$= \begin{bmatrix} e^{-t} + 2te^{-t} \\ e^{-t} - 2te^{-t} \end{bmatrix}$$

giving $y(t) = x_1(t) = e^{-t} + 2te^{-t}$

The differential equation may be solved directly using the techniques of Chapter 10 of the companion text *Modern Engineering Mathematics* or using Laplace transforms. Both approaches confirm the solution

$$y = (1 + 2t)e^{-2t}$$

■ **58** Taking $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then from Exercise 56

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

and the required solution is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t \\ (1+t)e^t \end{bmatrix}$$

■ **59** Taking $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ its eigenvalues are $\lambda_1 = -3,\ \lambda_2 = -2$.
Thus, $e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A}$ where α_0, α_1 satisfy

$$e^{-3t} = \alpha_0 - 3\alpha_1, \quad e^{-2t} = \alpha_0 - 2\alpha_1$$
$$\alpha_0 = 3e^{-2t} - 2e^{-3t},\ \alpha_1 = e^{-2t} - e^{-3t}$$

so

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ 6e^{-3t} - te^{-2t} & 3e^{-3t} - 2e^{-2t} \end{bmatrix}$$

Thus, the first term in (6.73) becomes

$$e^{\mathbf{A}t}\mathbf{x}(0) = e^{\mathbf{A}t}[1\ \ -1]^T = \begin{bmatrix} 2e^{-2t} - e^{-3t} \\ 3e^{-3t} - 4e^{-2t} \end{bmatrix}$$

and the second term is

$$\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau = \int_0^t 2 \begin{bmatrix} 6e^{-2(t-\tau)} - 6e^{-3(t-\tau)} \\ 18e^{-3(t-\tau)} - 12e^{-2(t-\tau)} \end{bmatrix} d\tau$$
$$= 2 \begin{bmatrix} 3e^{-2(t-\tau)} - 2e^{-3(t-\tau)} \\ 6e^{-3(t-\tau)} - 6e^{-2(t-\tau)} \end{bmatrix}_0^t$$
$$= 2 \begin{bmatrix} 1 - 3e^{-2t} + 2e^{-3t} \\ 6e^{-2t} - 6e^{-3t} \end{bmatrix}$$

Thus, required solution is

$$\mathbf{x}(t) = \begin{bmatrix} 2e^{-2t} - e^{-3t} + 2 - 6e^{-3t} + 4e^{-3t} \\ 3e^{-3t} - 4e^{-2t} + 12e^{-2t} - 12e^{-3t} \end{bmatrix}$$
$$= \begin{bmatrix} 2 - 4e^{-2t} + 3e^{-3t} \\ 8e^{-2t} - 9e^{-3t} \end{bmatrix}$$

that is, $x_1 = 2 - 4e^{-2t} + 3e^{-3t},\ x_2 = 8e^{-2t} - 9e^{-3t}$

■ **60** In state space form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t),\ u(t) = e^{-t},\ x(0) = [0\ 1]^T$$

Taking $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ its eigenvalues are $\lambda_1 = -2, \lambda_2 = -1$ so

$$e^{\mathbf{A}t} = \alpha_0 I + \alpha_1 \mathbf{A} \text{ where } \alpha_0, \alpha_1 \text{ satisfy}$$
$$e^{-2t} = \alpha_0 - 2\alpha_1,\ e^{-t} = \alpha_0 - \alpha_1 \ \Rightarrow\ \alpha_0 = 2e^{-t} - e^{-2t},\ \alpha_1 = e^{-t} - e^{-2t}$$

Thus,

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

and $e^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix}$

$$\begin{aligned} \int_0^t \mathbf{A}^{(t-\tau)}\mathbf{b}u(\tau)d\tau &= \int_0^t \begin{bmatrix} 4e^{-(t-\tau)} - 2e^{-2(t-\tau)} \\ -4e^{-(t-\tau)} + 4e^{-2(t-\tau)} \end{bmatrix} e^{-\tau} d\tau \\ &= \int_0^t \begin{bmatrix} 4e^{-t} - 2e^{-2t}e^{\tau} \\ -4e^{-t} + 4e^{-2t}e^{\tau} \end{bmatrix} d\tau \\ &= \begin{bmatrix} 4\tau e^{-t} - 2e^{-2t}e^{\tau} \\ -4\tau e^{-t} + 4e^{-2t}e^{\tau} \end{bmatrix}_0^t \\ &= \begin{bmatrix} 4te^{-t} - 2e^{-t} + 2e^{-2t} \\ -4te^{-t} + 4e^{-t} - 4e^{-2t} \end{bmatrix} \end{aligned}$$

We therefore have the solution

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}\mathbf{b}u(\tau)d\tau \\ &= \begin{bmatrix} 4te^{-t} + e^{-2t} - e^{-t} \\ -4te^{-t} + 3e^{-t} - 2e^{-2t} \end{bmatrix} \end{aligned}$$

that is,

$$x_1 = 4te^{-t} + e^{-2t} - e^{-t},\ x_2 = -4te^{-t} + 3e^{-t} - 2e^{-2t}$$

■ **61** Taking $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ its eigenvalues are $\lambda_1 = 5, \lambda_2 = -1$.

$$e^{\mathbf{A}t} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A} \text{ where } \alpha_0, \alpha_1 \text{ satisfy}$$
$$e^{5t} = \alpha_0 + 5\alpha_1,\ e^{-t} = \alpha_0 - \alpha_1 \ \Rightarrow\ \alpha_0 = \frac{1}{6}e^{5t} + \frac{5}{6}e^{-t},\ \alpha_1 = \frac{1}{6}e^{5t} + \frac{1}{6}e^{-t}$$

Thus, transition matrix is

$$e^{\mathbf{A}t} = \begin{bmatrix} \frac{1}{3}e^{-t} + \frac{2}{3}e^{5t} & \frac{2}{3}e^{5t} - \frac{2}{3}e^{-t} \\ \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} \end{bmatrix}$$

and $e^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{e}^{\mathbf{A}t}[1\ 2]^T = \begin{bmatrix} 2e^{5t} - e^{-t} \\ e^{5t} + e^{-t} \end{bmatrix}$

$$\begin{aligned} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau &= \int_0^t e^{\mathbf{A}(t-\tau)} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} d\tau \\ &= \int_0^t \mathbf{A}^{t-\tau} \begin{bmatrix} 3 \\ 7 \end{bmatrix} d\tau \\ &= \int_0^t \begin{bmatrix} \frac{20}{3}e^{5(t-\tau)} - \frac{11}{3}e^{-(t-\tau)} \\ \frac{10}{3}e^{5(t-\tau)} + \frac{11}{3}e^{-(t-\tau)} \end{bmatrix} d\tau \\ &= \begin{bmatrix} -\frac{4}{3}e^{5(t-\tau)} - \frac{11}{3}e^{-(t-\tau)} \\ -\frac{2}{3}e^{5(t-\tau)} + \frac{11}{3}e^{-(t-\tau)} \end{bmatrix}_0^t \\ &= \begin{bmatrix} -5 + \frac{11}{3}e^{-t} + \frac{4}{3}e^{5t} \\ 3 - \frac{11}{3}e^{-t} + \frac{2}{3}e^{5t} \end{bmatrix} \end{aligned}$$

Thus, solution is

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(t)d\tau \\ &= \begin{bmatrix} -5 + \frac{8}{3}e^{-t} + \frac{10}{3}e^{5t} \\ 3 - \frac{8}{3}e^{-t} + \frac{5}{3}e^{5t} \end{bmatrix} \end{aligned}$$

Exercises 1.10.7

■ **62** Eigenvalues of matrix $\mathbf{A} = \begin{bmatrix} -\frac{3}{2} & \frac{3}{4} \\ 1 & -\frac{5}{2} \end{bmatrix}$ are given by

$$| A - \lambda I | = \lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1) = 0$$

that is, $\lambda_1 = -1,\ \lambda_2 = -3$
having corresponding eigenvectors $\mathbf{e}_1 = [3\ 2]^T,\ \mathbf{e}_2 = [1\ -2]^T$.
Denoting the reciprocal basis vectors by

$$\mathbf{r}_1 = [r_{11}\ r_{12}]^T,\ \mathbf{r}_2 = [r_{21}\ r_{22}]^T$$

and using the relationships $\mathbf{r}_i^T\mathbf{e}_j = \delta_{ij}(i,j=1,2)$ we have

$$\left.\begin{array}{l} 3r_{11}+2r_{12}=1 \\ r_{11}-2r_{12}=0 \end{array}\right\} \quad \mathbf{r}_1 = [\tfrac{1}{4}\ \ \tfrac{1}{8}]^T$$
$$\left.\begin{array}{l} 3r_{21}+2r_{22}=0 \\ r_{21}-2r_{22}=1 \end{array}\right\} \quad \mathbf{r}_2 = [\tfrac{1}{4}\ \ -\tfrac{3}{8}]^T$$

Thus,

$$\mathbf{r}_1^T\mathbf{x}(0) = \frac{1}{2}+\frac{1}{2} = 1,\ \mathbf{r}_2^T\mathbf{x}(0) = \frac{1}{2}-\frac{3}{2} = -1$$

so the spectral form of solution is

$$\mathbf{x}(t) = e^{-t}\mathbf{e}_1 - e^{-3t}\mathbf{e}_2$$

The trajectory is readily drawn showing that it approaches the origin along the eigenvector $\mathbf{e}_1$ since $e^{-3t}\to 0$ faster than e^{-t}. See Figure 1.9 in the text.

■ **63** Taking $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix}$ eigenvalues are $\lambda_1 = -6$, $\lambda_2 = -1$ having corresponding eigenvectors $\mathbf{e}_1 = [1\ \ -2]^T$, $\mathbf{e}_2 = [2\ 1]^T$.
Denoting the reciprocal basis vectors by

$$\mathbf{r}_1 = [r_{11}\ r_{12}]^T,\ \mathbf{r}_2 = [r_{21}\ r_{22}]^T$$

and using the relationships $\mathbf{r}_i^T\mathbf{e}_j = \delta_{ij}(i,j=1,2)$ we have

$$\left.\begin{array}{l} r_{11}-2r_{12}=1 \\ 2r_{11}+r_{12}=0 \end{array}\right\} \quad \Rightarrow\ r_{11}=\tfrac{1}{5}, r_{12}=-\tfrac{2}{5}\ \Rightarrow\ \mathbf{r}_1 = \tfrac{1}{5}[1\ \ -2]^T$$
$$\left.\begin{array}{l} r_{21}-2r_{22}=0 \\ 2r_{21}+r_{22}=1 \end{array}\right\} \quad \Rightarrow\ r_{21}=\tfrac{2}{5}, r_{22}=-\tfrac{1}{5}\ \Rightarrow\ \mathbf{r}_2 = \tfrac{1}{5}[2\ 1]^T$$

Thus,

$$\mathbf{r}_1^T\mathbf{x}(0) = \frac{1}{5}[1\ -2]\ \begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\frac{4}{5}$$
$$\mathbf{r}_2^T\mathbf{x}(0) = \frac{1}{5}[2\ 1]\ \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{7}{5}$$

then response is

$$\begin{aligned}\mathbf{x}(t) &= \sum_{i=1}^{2} \mathbf{r}_i^T \mathbf{x}(0) e^{\lambda i t} \mathbf{e}_i \\ &= -\frac{4}{5} e^{-6t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{7}{5} e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -4e^{-6t} + 14e^{-t} \\ 8e^{-6t} + 7e^{-t} \end{bmatrix}\end{aligned}$$

Again, following Figure 1.9 in the text, the trajectory is readily drawn and showing that it approaches the origin along the eigenvector $\mathbf{e}_2$ since $e^{-6t} \to 0$ faster than e^{-t}.

■ **64** Taking $\mathbf{A} = \begin{bmatrix} 0 & -4 \\ 2 & -4 \end{bmatrix}$ eigenvalues are $\lambda_1 = -2 + j2,\ \lambda_2 = -2 - j2$ having corresponding eigenvectors $\mathbf{e}_1 = [2 \ \ 1 - j]^T, \mathbf{e}_2 = [2 \ \ 1 + j]^T$.

Let $\mathbf{r}_1 = \mathbf{r}_1' + j\mathbf{r}_1''$ be reciprocal base vector to $\mathbf{e}_1$ then

$$\mathbf{r}_1^T \mathbf{e}_1 = 1 = [r' + j\mathbf{r}_1'']^T [\mathbf{e}_1' + je_1'']^T \text{ where } \mathbf{e}_1 = \mathbf{e}_1' + j\mathbf{e}_1''$$
$$\mathbf{r}_1^T \mathbf{e}_2 = 0 = [\mathbf{r}_1' + j\mathbf{r}_1'']^T [\mathbf{e}_1' - j\mathbf{e}_1'']^T \text{ since } \mathbf{e}_2 = \text{conjugate } \ \mathbf{e}_1$$

Thus,

$$[(\mathbf{r}_1')^T \mathbf{e}_1' - (r_1'')^T \mathbf{e}_1''] + j[(r_1'')^T \mathbf{e}_1' + (r_1')^T \mathbf{e}_1''] = 1$$

and

$$[(\mathbf{r}_1')^T \mathbf{e}_1' - (r_1'')^T \mathbf{e}_1'] + j[(r_1')^T \mathbf{e}_1^1 - (r_1')^T \mathbf{e}_1'] = 0$$

giving

$$(\mathbf{r}_1')^T\mathbf{e}_1' = \frac{1}{2}\ ,\ (\mathbf{r}_1')^T\mathbf{e}_1' = \frac{1}{2}\ ,\ (\mathbf{r}_1')^T\mathbf{e}_1' = (\mathbf{r}_1')^T\mathbf{e}_1'' = 0$$

Now $\mathbf{e}_1' = [2\ 1]^T, \mathbf{e}_1'' = [0\ -1]^T$

Let $\mathbf{r}_1' = [a\ b]^T$ and $\mathbf{r}_1'' = [c\ d]^T$ then from above

$2a + b = \frac{1}{2}, -b = 0$ and $-d = -\frac{1}{2}, 2c + d = 0$

giving $a = \frac{1}{4}, b = 0, c = -\frac{1}{4}, d = \frac{1}{2}$ so that

$$\mathbf{r}_1 = \mathbf{r}_1' + j\mathbf{r}_1'' = \frac{1}{4}[1 - j\ \ 2j]^T$$

Since $\mathbf{r}_2$ is the complex conjugate of $\mathbf{r_1}$

$$\mathbf{r}_2 = \frac{1}{4}[1 + j\ \ -2j]^T$$

so the solution is given by

$$\mathbf{x}(t) = \mathbf{r}_1^T\mathbf{x}(0)e^{\lambda_1 t}\mathbf{e}_1 + \mathbf{r}_2^T\mathbf{x}(0)e^{\lambda_2 t}\mathbf{e}_2$$

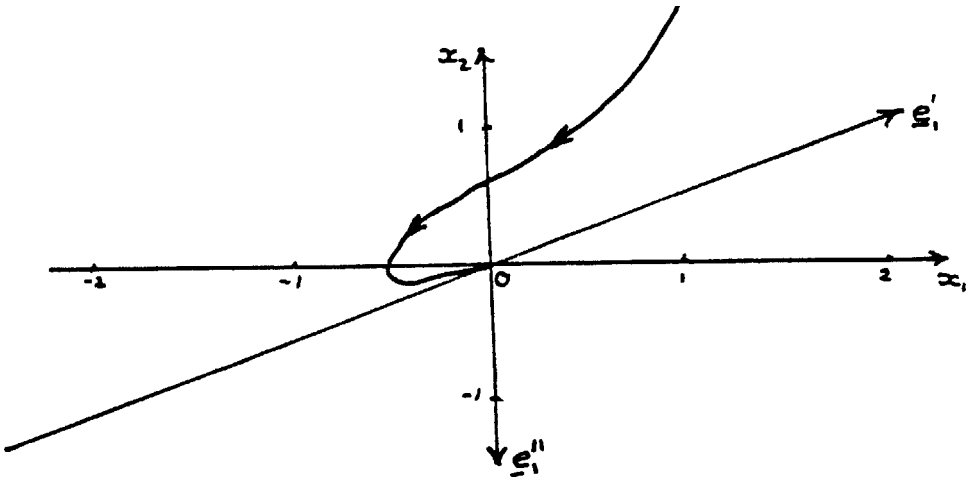
and since $\mathbf{r}_1^T\mathbf{x}(0) = \frac{1}{2}(1 + j),\ \mathbf{r}_2^T\mathbf{x}(0) = \frac{1}{2}(1 - j)$

$$x(t) = e^{-2t}\left\{\frac{1}{2}(1+j)e^{2jt}\ \begin{bmatrix}2\\1-j\end{bmatrix} + \frac{1}{2}(1-j)e^{-2jt}\ \begin{bmatrix}2\\1+j\end{bmatrix}\right\}$$

$$= e^{-2t}\left\{(\cos 2t - \sin 2t)\ \begin{bmatrix}2\\1\end{bmatrix} - (\cos 2t + \sin 2t)\ \begin{bmatrix}0\\-1\end{bmatrix}\right\}$$

$$= e^{-2t}\left\{(\cos 2t - \sin 2t)\mathbf{e}_1' - (\cos 2t + \sin 2t)\mathbf{e}_1''\right\} \text{ where } \mathbf{e}_1 = \mathbf{e}_1' + j\mathbf{e}_1''$$

To plot the trajectory, first plot $\mathbf{e}_1', \mathbf{e}_1''$ in the plane and then using these as a frame of reference plot the trajectory. A sketch is as follows



■ **65** Following section 1.10.6 if the equations are representative of

$$\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x} + \mathbf{b}u\ ,\ y = \mathbf{c}^T\mathbf{x}$$

then making the substitution $\mathbf{x} = \mathbf{M}\ \boldsymbol{\xi}$, where $\mathbf{M}$ is the modal matrix of $\mathbf{A}$, reduces the system to the canonical form

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\Lambda}\ \boldsymbol{\xi} + (\mathbf{M}^{-1}\mathbf{b})u\ ,\ y = (\mathbf{c}^T\mathbf{M})\boldsymbol{\xi}$$

where $\boldsymbol{\Lambda}$ is the spectral matrix of $\mathbf{A}$.
Eigenvalues of $\mathbf{A}$ are given by

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda-1)(\lambda+2)(\lambda+1) = 0$$

so the eigenvalues are $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$. The corresponding eigenvectors are readily determined as

$$\mathbf{e}_1 = [1\ 3\ 1]^T,\ \mathbf{e}_2 = [3\ 2\ 1]^T,\ \mathbf{e}_3 = [1\ 0\ 1]^T$$

Thus, $\mathbf{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $\boldsymbol{\Lambda} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\mathbf{M}^{-1} = \dfrac{1}{\det \mathbf{M}} \text{ adj } \mathbf{M} = -\dfrac{1}{6}\begin{bmatrix} 2 & -2 & -2 \\ -3 & 0 & 3 \\ 1 & 2 & -7 \end{bmatrix}$ so required canonical form is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{4}{3} \end{bmatrix} u$$

$$y = [1 \ -4 \ -2]\ [\xi_1 \ \xi_2 \ \xi_3]^T$$

■ **66** Let $\mathbf{r}_1 = [r_{11} \ r_{12} \ r_{13}]^T, \mathbf{r}_2 = [r_{21} \ r_{22} \ r_{23}]^T, \mathbf{r}_3 = [r_{31} \ r_{32} \ r_{33}]^T$ be the reciprocal base vectors to $\mathbf{e}_1 = [1 \ 1 \ 0]^T$, $\mathbf{e}_2 = [0 \ 1 \ 1]^T$, $\mathbf{e}_3 = [1 \ 2 \ 3]^T$.

$$\left.\begin{aligned} \mathbf{r}_1^T\mathbf{e}_1 = r_{11} + r_{12} = 1 \\ \mathbf{r}_1^T\mathbf{e}_2 = r_{11} + r_{13} = 0 \\ \mathbf{r}_1^T\mathbf{e}_3 = r_{11} + 2r_{12} + 3r_{13} = 0 \end{aligned}\right\} \Rightarrow \mathbf{r}_1 = \frac{1}{2}[1 \ 1 \ -1]^T$$

$$\left.\begin{aligned} \mathbf{r}_2^T\mathbf{e}_1 = r_{21} + r_{22} = 0 \\ \mathbf{r}_2^T\mathbf{e}_2 = r_{22} + r_{23} = 1 \\ \mathbf{r}_2^T\mathbf{e}_3 = r_{21} + 2r_{22} + 3r_{23} = 0 \end{aligned}\right\} \Rightarrow \mathbf{r}_2 = \frac{1}{2}[-3 \ 3 \ 1]^T$$

$$\left.\begin{aligned} \mathbf{r}_3^T\mathbf{e}_1 = r_{31} + r_{32} = 0 \\ \mathbf{r}_3^T\mathbf{e}_2 = r_{32} + r_{33} = 0 \\ \mathbf{r}_3^T\mathbf{e}_3 = r_{31} + 2r_{32} + 3r_{33} = 1 \end{aligned}\right\} \Rightarrow \mathbf{r}_3 = \frac{1}{2}[1 \ -1 \ 1]^T$$

Then using the fact that $\mathbf{x}(0) = [1 \ 1 \ 1]^T$

$$\alpha_0 = \mathbf{r}_1^T\mathbf{x}(0) = -\tfrac{1}{2}\ ,\ \alpha_1 = \mathbf{r}_2^T\mathbf{x}(0) = \tfrac{1}{2}\ ,\ \alpha_3 = \mathbf{r}_3^T\mathbf{x}(0) = \tfrac{1}{2}$$

■ **67** The eigenvectors of $\mathbf{A}$ are given by

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - 6)(\lambda - 1) = 0$$

so the eigenvalues are $\lambda_1 = 6, \lambda_2 = 1$. The corresponding eigenvectors are readily determined as $\mathbf{e}_1 = [4 \ 1]^T, \mathbf{e}_2 = [1 \ -1]^T$.
Taking $\mathbf{M}$ to be the modal matrix $\mathbf{M} = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$ then substituting $\mathbf{x} = M\boldsymbol{\xi}$ into $\dot{\mathbf{x}} = \mathbf{A}x(t)$ reduces it to the canonical form

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\Lambda}\,\boldsymbol{\xi}$$

where $\mathbf{\Lambda} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, the decoupled canonical form is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \text{ or } \dot{\xi}_1 = 6\xi_1 \text{ and } \dot{\xi}_2 = \xi_2$$

which may be individually solved to give

$$\xi_1 = \alpha e^{6t} \text{ and } \xi_1 = \beta e^{t}$$

Now $\boldsymbol{\xi}(0) = \mathbf{M}^{-1}\mathbf{x}(0) = -\frac{1}{5}\begin{bmatrix} -1 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$
so $\xi_1(0) = 1 = \alpha$ and $\xi_2(0) = -3 = \beta$
giving the solution of the uncoupled system as

$$\boldsymbol{\xi} = \begin{bmatrix} e^{6t} \\ -3e^{t} \end{bmatrix}$$

The solution for $\mathbf{x}(t)$ as

$$\mathbf{x} = \mathbf{M}\,\boldsymbol{\xi} = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{6t} \\ -3e^{t} \end{bmatrix} = \begin{bmatrix} 4e^{6t} - 3e^{t} \\ e^{6t} + 3e^{t} \end{bmatrix}$$

■ **68** Taking $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ its eigenvalues are $\lambda_1 = 5, \lambda_2 = -1$ having corresponding eigenvectors $\mathbf{e}_1 = [2\ 1]^T$, $\mathbf{e}_2 = [1\ -1]^T$.
Let $\mathbf{M} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ be the modal matrix of $\mathbf{A}$, then $\dot{\mathbf{x}} = \mathbf{M}\,\boldsymbol{\xi}$ reduces the equation to

$$\dot{\boldsymbol{\xi}}(t) = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}\boldsymbol{\xi} + \mathbf{M}^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{u}(t)$$

Since $\mathbf{M}^{-1} = \dfrac{1}{\det \mathbf{M}} \text{ adj } \mathbf{M} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ we have,

$$\dot{\boldsymbol{\xi}}(t) = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}\boldsymbol{\xi} + \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}\mathbf{u}(t)$$

With $\mathbf{u}(t) = [4\ 3]^T$ the decoupled equations are

$$\dot{\xi}_1 = 5\xi_1 + \frac{10}{3}$$
$$\dot{\xi}_2 = -\xi_2 - \frac{11}{3}$$

which can be solved independently to give

$$\xi_1 = \alpha e^{5t} - \frac{2}{3}\ ,\ \xi_2 = \beta e^{-t} - \frac{11}{3}$$

We have that $\boldsymbol{\xi}(0) = \boldsymbol{M}^{-1}\mathbf{x}(0) = \frac{1}{3}\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$ so

$$1 = \alpha - \frac{2}{3} \Rightarrow \alpha = \frac{5}{3}$$
$$-1 = \beta - \frac{11}{3} \Rightarrow \beta = \frac{8}{3}$$

giving

$$\boldsymbol{\xi} = \begin{bmatrix}\frac{5}{3}e^{5t} - \frac{2}{3}\\ \frac{8}{3}e^{-t} - \frac{11}{3}\end{bmatrix}$$

and $\mathbf{x} = \boldsymbol{M}\boldsymbol{\xi} = \begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{5}{3}e^{5t} - \frac{2}{3}\\ \frac{8}{3}e^{-t} - \frac{11}{3}\end{bmatrix} = \begin{bmatrix}-5 + \frac{8}{3}e^{-t} + \frac{10}{3}e^{5t}\\ 3 - \frac{8}{3}e^{-t} + \frac{5}{3}e^{5t}\end{bmatrix}$
which confirms Exercises 57 and 58.

Exercises 1.11.1 (Lyapunov)

■ **69** Take tentative Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ giving

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x} = -\mathbf{x}^T\mathbf{Q}\mathbf{x} \text{ where}$$

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \qquad \text{(i)}$$

Take $\mathbf{Q} = \mathbf{I}$ so that $\dot{V}(\mathbf{x}) = -(x_1^2 + x_2^2)$ which is negative definite. Substituting in (i) gives

$$\begin{bmatrix}-4 & 3\\ 2 & -2\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix} + \begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}-4 & 2\\ 3 & -2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

Equating elements gives

$$-8p_{11} + 6p_{12} = -1, 4p_{12} - 4p_{22} = -1, 2p_{11} - 6p_{12} + 3p_{22} = 0$$

Solving gives $p_{11} = \frac{5}{8}, p_{12} = \frac{2}{3}, p_{22} = \frac{11}{12}$ so that, $\mathbf{P} = \begin{bmatrix}\frac{5}{8} & \frac{2}{3}\\ \frac{2}{3} & \frac{11}{12}\end{bmatrix}$ Principal minors of $\mathbf{P}$ are: $\frac{5}{8} > 0$ and $\det\mathbf{P} = (\frac{55}{96} - \frac{4}{9}) > 0$ so $\mathbf{P}$ is positive definite and the system is asymptotically stable

Note that, in this case, we have $V(\mathbf{x}) = \frac{5}{8}x_1^2 + \frac{4}{3}x_1x_2 + \frac{11}{12}x_2^2$ which is positive definite and $\dot{V}(\mathbf{x}) = \frac{5}{4}x_1\dot{x}_1 + \frac{4}{3}\dot{x}_1x_2 + \frac{4}{3}x_1\dot{x}_2 + \frac{11}{6}x_2\dot{x}_2 = -x_1^2 - x_2^2$ which is negative definite.

■ **70** Take tentative Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ giving

$$\begin{gathered}\dot{V}(\mathbf{x}) = \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x} = -\mathbf{x}^T\mathbf{Q}\mathbf{x} \text{ where} \\ \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}\end{gathered} \tag{i}$$

Take $\mathbf{Q} = \mathbf{I}$ so that $\dot{V}(\mathbf{x}) = -(x_1^2 + x_2^2)$ which is negative definite. Substituting in (i) gives

$$\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Equating elements gives

$$-6p_{11} - 2p_{12} = -1, 4p_{12} - 2p_{22} = -1, 2p_{11} - 4p_{12} - p_{22} = 0$$

Solving gives $p_{11} = \frac{7}{40}, p_{12} = -\frac{1}{40}, p_{22} = \frac{18}{40}$ so that $\mathbf{P} = \begin{bmatrix} \frac{7}{40} & -\frac{1}{40} \\ -\frac{1}{40} & \frac{18}{40} \end{bmatrix}$
Principal minors of $\mathbf{P}$ are: $\frac{7}{40} > 0$ and $\det \mathbf{P} = \frac{5}{64} > 0$ so $\mathbf{P}$ is positive definite and the system is asymptotically stable.

■ **71** Take tentative Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ giving

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x} = -\mathbf{x}^T\mathbf{Q}\mathbf{x} \text{ where}$$

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{i}$$

Take $\mathbf{Q} = \mathbf{I}$ so that $\dot{V}(\mathbf{x}) = -(x_1^2 + x_2^2)$ which is negative definite. Substituting in (i) gives

$$\begin{bmatrix} 0 & -a \\ 1 & -b \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Equating elements gives

$$-8p_{12} = -1, 2p_{12} - 2bp_{22} = -1, p_{11} - bp_{12} - ap_{22} = 0$$

Solving gives $p_{12} = \frac{1}{2a}, p_{22} = \frac{a+1}{2ab}, p_{11} = \frac{b^2+a^2+a}{2ab}$ so that, $\mathbf{P} = \begin{bmatrix} \frac{b^2+a^2+a}{2ab} & \frac{1}{2a} \\ \frac{1}{2a} & \frac{a+1}{2ab} \end{bmatrix}$

For asymptotic stability the principal minors of $\mathbf{P}$ must be positive. Thus,

$$\frac{\mathrm{b}^2 + a^2 + a}{2ab} > 0 \tag{ii}$$

$$\text{and } (b^2 + a^2 + a)(a + 1) > b^2 \tag{iii}$$

Case 1 $ab > 0$

$$\text{(ii)} \Rightarrow a^2 + b^2 + a > 0 \text{ so (iii)} \Rightarrow a + 1 > \frac{b^2}{b^2 + a^2 + a}$$
$$\Rightarrow a[a^2 + (a + 1)^2] > 0 \Rightarrow a > 0.$$

Since $ab > 0 \Rightarrow b > 0$ it follows that (ii) and (iii) are satisfied if $a, b > 0$

Case 2 $ab < 0$ No solution to (ii) and (iii) in this case.

Thus, system is asymptotically stable when both $a > 0$ and $b > 0$.

Note: This example illustrates the difficulty in interpretating results when using the Lyapunov approach. It is a simple task to confirm this result using the Routh–Hurwitz criterion developed in Section 5.6.2.

- **72(a)**

$$\dot{x}_1 = x_2 \tag{i}$$

$$\dot{x}_2 = -2x_2 + x_3 \tag{ii}$$

$$\dot{x}_3 = -kx_1 - x_3 \tag{iii}$$

If $\dot{V}(\mathbf{x})$ is identically zero then x_3 is identically zero $\Rightarrow x_1$ is identically zero from (iii)

$$\Rightarrow x_2 \text{is identically zero from (i)}$$

Hence $\dot{V}(\mathbf{x})$ is identically zero only at the origin.

(b) $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \Rightarrow$

$$\begin{bmatrix} 0 & 0 & -k \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -k & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equating elements and solving for the elements of $\boldsymbol{P}$ gives the matrix

$$\mathbf{P} = \begin{bmatrix} \frac{k^2+12k}{12-2k} & \frac{6k}{12-2k} & 0 \\ \frac{6k}{12-2k} & \frac{3k}{12-2k} & \frac{k}{12-2k} \\ 0 & \frac{k}{12-2k} & \frac{6}{12-2k} \end{bmatrix}$$

(c) Principal minors of $\mathbf{P}$ are:

$$\Delta_1 = \frac{k^2+12k}{12-2k} > 0 \text{ if } k > 0 \text{and} (12-2k) > 0 \Rightarrow 0 < k < 6$$
$$\Delta_2 = \left[\frac{k^2+12k}{12-2k}\right]\left[\frac{3k}{12-2k}\right] - \frac{36k^2}{12-2k} = \frac{3k^3}{(12-2k)^2} > 0 \text{ if } k > 0$$
$$\Delta_3 = \frac{(k^2+12k)(8k-k^2)}{(12-2k)^3} - \frac{216k^2}{(12-2k)^3} > 0 \text{if } (6k^3-k^4) > 0 \Rightarrow 0 < k < 6$$

Thus system asymptotically stable for $0 < k < 6$.

■ **73** State-space form is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{i}$$

Take $V(\mathbf{x}) = kx_1^2 + (x_2 + ax_1)^2$ then

$$\begin{aligned} \dot{V}(\mathbf{x}) &= 2kx_1\dot{x}_1 + 2(x_2+ax_1)(\dot{x}_2 + a\dot{x}_1) \\ &= 2kx_1(x_2) + 2(x_2+ax_1)(-kx_1 - ax_2 + ax_1) \text{using (i)} \\ &= -2kax_1^2 \end{aligned}$$

Since k>0 and a>0 then $\dot{V}(\mathbf{x})$ is negative semidefinite but is not identically zero along any trajectory of (i). Consequently, this choice of Lyapunov function assures asymptotic stability.

Review Exercises 1.13

■ **1(a)** Eigenvalues given by

$$\begin{vmatrix} -1-\lambda & 6 & 12 \\ 0 & -13-\lambda & 30 \\ 0 & -9 & 20-\lambda \end{vmatrix} = (1+\lambda)[(-13-\lambda)(20-\lambda)+270] = 0$$

that is, $(1+\lambda)(\lambda-5)(\lambda-2)=0$
so eigenvalues are $\lambda_1=5, \lambda_2=2, \lambda_3=-1$
Eigenvectors are given by corresponding solutions of

$$\begin{bmatrix} -1-\lambda_i & 6 & 12 \\ 0 & -13-\lambda_i & 30 \\ 0 & -9 & 20-\lambda_i \end{bmatrix} \begin{bmatrix} e_{i1} \\ e_{i2} \\ e_{i3} \end{bmatrix} = 0$$

When $i=1, \lambda_i=5$ and solution given by

$$\frac{e_{11}}{198} = \frac{-e_{12}}{-90} = \frac{e_{13}}{54} = \beta_1$$

so $\mathbf{e}_1 = [11\ 5\ 3]^T$
When $i=2, \lambda_i=2$ and solution given by

$$\frac{e_{21}}{216} = \frac{-e_{22}}{-54} = \frac{e_{23}}{27} = \beta_2$$

so $\mathbf{e}_2 = [8\ 2\ 1]^T$
When $i=3, \lambda_i=-1$ and solution given by

$$\frac{e_{31}}{1} = \frac{-e_{32}}{0} = \frac{e_{33}}{0} = \beta_3$$

so $\mathbf{e}_3 = [1\ 0\ 0]^T$

1(b) Eigenvalues given by

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ -1 & 4-\lambda & -1 \\ -1 & 2 & 0-\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} + \begin{vmatrix} -1 & 4-\lambda \\ -1 & 2 \end{vmatrix} = 0$$

$$\begin{aligned} \text{that is, } 0 &= (2-\lambda)[(4-\lambda)(-\lambda)+2] + [-2+(4-\lambda)] \\ &= (2-\lambda)(\lambda^2-4\lambda+3) = (2-\lambda)(\lambda-3)(\lambda-1) = 0 \end{aligned}$$

so eigenvalues are

$$\lambda_1=3, \lambda_2=2, \lambda_3=1$$

Eigenvectors are given by the corresponding solutions of

$$\begin{aligned} (2-\lambda_i)e_{i1} + 0e_{i2} + e_{i3} &= 0 \\ -e_{i1} + (4-\lambda_i)e_{i2} - e_{i3} &= 0 \\ -e_{i1} + 2e_{i2} - \lambda_i e_{i3} &= 0 \end{aligned}$$

Taking $i = 1, 2, 3$ gives the eigenvectors as

$$\mathbf{e}_1 = [1\ 2\ 1]^T, \mathbf{e}_2 = [2\ 1\ 0]^T, \mathbf{e}_3 = [1\ 0\ -1]^T$$

1(c) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \underline{\underline{R_1 + (R_2 + R_3)}} \begin{vmatrix} -\lambda & -\lambda & -\lambda \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\text{that is, } \lambda \begin{vmatrix} -1 & -1 & -1 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = \lambda \begin{vmatrix} -1 & 0 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & -1 & 1-\lambda \end{vmatrix} = \lambda(3-\lambda)(1-\lambda) = 0$$

so eigenvalues are $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$

Eigenvalues are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1} - e_{i2} - 0e_{i3} = 0$$
$$-e_{i1} + (2-\lambda_i)e_{i2} - e_{i3} = 0$$
$$0e_{i1} - e_{i2} + (1-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ gives the eigenvectors as

$$\mathbf{e}_1 = [1\ -2\ 1]^T, \mathbf{e} = [1\ 0\ -1]^T, \mathbf{e}_3 = [1\ 1\ 1]^T$$

■ **2** Principal stress values (eigenvalues) given by

$$\begin{vmatrix} 3-\lambda & 2 & 1 \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} \underline{\underline{R_1 + (R_2 + R_3)}} \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix}$$

$$= (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$\text{that is, } (6-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1-\lambda & -1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (6-\lambda)(1-\lambda)(3-\lambda) = 0$$

so the principal stress values are $\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 1$.

Corresponding principal stress direction $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are given by the solutions of

$$(3-\lambda_i)e_{i1} + 2e_{i2} + e_{i3} = 0$$
$$2e_{i1} + (3-\lambda_i)e_{i2} + e_{i3} = 0$$
$$e_{i1} + e_{i2} + (4-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ gives the principal stress direction as

$$\mathbf{e}_1 = [1\ 1\ 1]^T,\ \mathbf{e}_2 = [1\ 1\ -2]^T,\ \mathbf{e}_3 = [1\ -1\ 0]^T$$

It is readily shown that $\mathbf{e}_1^T\mathbf{e}_2 = \mathbf{e}_1^T\mathbf{e}_3 = \mathbf{e}_2^T\mathbf{e}_3 = 0$ so that the principal stress directions are mutually orthogonal.

■ **3** Since $[1\ 0\ 1]^T$ is an eigenvector of $\mathbf{A}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & b \\ 0 & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

so $2 = \lambda, -1 + b = 0, c = \lambda$
giving $b = 1$ and $c = 2$.
Taking these values $\mathbf{A}$ has eigenvalues given by

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - (2-\lambda)$$
$$= (2-\lambda)(\lambda-1)(\lambda-4) = 0$$

that is, eigenvalues are $\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 1$
Corresponding eigenvalues are given by the solutions of

$$(2-\lambda_i)e_{i1} - e_{i2} + 0e_{i3} = 0$$
$$-e_{i1} + (3-\lambda_i)e_{i2} + e_{i3} = 0$$
$$0e_{i1} + e_{i2} + (2-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ gives the eigenvectors as

$$\mathbf{e}_1 = [1\ -2\ -1]^T, \mathbf{e}_2 = [1\ 0\ 1]^T, \mathbf{e}_3 = [1\ 1\ -1]^T$$

■ **4** The three Gerschgorin circles are

$$|\lambda - 4| = |-1| + |0| = 1$$
$$|\lambda - 4| = |-1| + |-1| = 2$$
$$|\lambda - 4| = 1$$

Thus, $|\lambda - 4| \leq 1$ and $|\lambda - 4| \leq 2$ so $|\lambda - 4| \leq 2$ or $2 \leq \lambda \leq 6$.
Taking $\mathbf{x}^{(o)} = [-1\ 1\ -1]^T$ iterations using the power method may be tabulated as follows

Iteration k	**0**	**1**	**2**	**3**	**4**	**5**	**6**
$\mathbf{x}^{(k)}$	-1	-0.833	-0.765	-0.734	-0.720	-0.713	-0.710
	1	1	1	1	1	1	1
	-1	-0.833	-0.765	-0.734	-0.720	-0.713	-0.710
$\mathbf{A}\ \mathbf{x}^{(k)}$	-5	-4.332	-4.060	-3.936	-3.88	-3.852	
	6	5.666	5.530	5.468	5.44	5.426	
	-5	-4.332	-4.060	-3.936	3.88	-3.852	
$\lambda \simeq$	6	5.666	5.530	5.468	5.44	5.426	

Thus, correct to one decimal place the dominant eigenvalue is $\lambda = 5.4$

■ **5(a)** Taking $\boldsymbol{x}^{(o)} = [1\ 1\ 1]^7$ iterations may be tabulated as follows

Iteration k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$\mathbf{x}^{(k)}$	1	0.800	0.745	0.728	0.722	0.720	0.719	0.719
	1	0.900	0.862	0.847	0.841	0.838	0.837	0.837
	1	1	1	1	1	1	1	1
$\mathbf{A}\ \mathbf{x}^{(k)}$	4	3.500	3.352	3.303	3.285	3.278	3.275	
	4.5	4.050	3.900	3.846	3.825	3.815	3.812	
	5	4.700	4.607	4.575	4.563	4.558	4.556	
$\lambda \simeq$	5	4.700	4.607	4.575	4.563	4.558	4.556	

Thus, estimate of dominant eigenvalues is $\lambda \simeq 4.56$ with associated eigenvector $\mathbf{x} = [0.72\ 0.84\ 1]^T$

5(b) $\sum_{i=1}^{3} \lambda_i = \text{trace } \mathbf{A} \Rightarrow 7.5 = 4.56 + 1.19 + \lambda_3 \Rightarrow \lambda_3 = 1.75$

5(c) (i) $\det \mathbf{A} = \prod_{i=1}^{3} \lambda_i = 9.50$ so $\mathbf{A}^{-1}$ exists and has eigenvalues

$$\frac{1}{1.19}, \frac{1}{1.75}, \frac{1}{4.56}$$

so power method will generate the eigenvalue 1.19 corresponding to $\mathbf{A}$.

(ii) $\mathbf{A} - 3\mathbf{I}$ has eigenvalues

$$1.19 - 3,\ 1.75 - 3,\ 4.56 - 3$$
$$\text{that is, } -1.91,\ -1.25,\ 1.56$$

so applying the power method on $\mathbf{A} - 3\mathbf{I}$ generates the eigenvalues corresponding to 1.75 of $\mathbf{A}$.

■ **6** $\dot{x} = \alpha\lambda e^{\lambda t},\ \dot{y} = \beta\lambda e^{\lambda t},\ \dot{z} = \gamma\lambda e^{\lambda t}$ so the differential equations become

$$\alpha\lambda e^{\lambda t} = 4\alpha e^{\lambda t} + \beta e^{\lambda t} + \gamma e^{\lambda t}$$
$$\beta\lambda e^{\lambda t} = 2\alpha e^{\lambda t} + 5\beta e^{\lambda t} + 4\gamma e^{\lambda t}$$
$$\gamma\lambda e^{\lambda t} = -\alpha e^{\lambda t} - \beta e^{\lambda t}$$

Provided $e^{\lambda t} \neq 0$ (i.e. non-trivial solution) we have the eigenvalue problem

$$\begin{bmatrix} 4 & 1 & 1 \\ 2 & 5 & 4 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 2 & 5-\lambda & 4 \\ -1 & -1 & 0 \end{vmatrix} \underset{C_2 = C_3}{} \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 1-\lambda & 4 \\ -1 & \lambda-1 & -\lambda \end{vmatrix} = (\lambda-1) \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & -1 & 4 \\ -1 & 1 & -\lambda \end{vmatrix}$$
$$= -(\lambda-1)(\lambda-5)(\lambda-3)$$

so its eigenvalues are 5, 3 and 1.
When $\lambda = 1$ the corresponding eigenvector is given by

$$3e_{11} + e_{12} + e_{13} = 0$$
$$2e_{11} + 4e_{12} + 4e_{13} = 0$$
$$-e_{11} - e_{12} - e_{13} = 0$$

having solution $\dfrac{e_{11}}{0} = \dfrac{-e_{12}}{2} = \dfrac{e_{13}}{2} = \beta_1$
Thus, corresponding eigenvector is $\beta[0\ -1\ 1]^T$

■ **7** Eigenvalues are given by

$$|\ \mathbf{A} - \lambda\mathbf{I}\ | = \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

Row 1 − (Row 2 + Row 3) gives

$$|\ \mathbf{A} - \lambda\mathbf{I}\ | = \begin{vmatrix} 1-\lambda & -1+\lambda & -1+\lambda \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & -1 & -1 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 4 & 1-\lambda & 2 \\ 3 & -1 & 4-\lambda \end{vmatrix} = (1-\lambda)[(1-\lambda)(4-\lambda)+2]$$

$$= (1-\lambda)(\lambda-2)(\lambda-3)$$

Thus, eigenvalues are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$.
Corresponding eigenvectors are given by

$$(8-\lambda)e_{i1} - 8e_{i2} - 2e_{i3} = 0$$
$$4e_{i1} - (3+\lambda)e_{i2} - 2e_{i3} = 0$$
$$3e_{i1} - 4e_{i2} + (1-\lambda)e_{i3} = 0$$

When $i = 1, \lambda_i = \lambda_1 = 3$ and solution given by

$$\frac{e_{11}}{4} = \frac{-e_{12}}{-2} = \frac{e_{13}}{2} = \beta_1$$

so a corresponding eigenvector is $\mathbf{e}_1 = [2\ 1\ 1]^T$.
When $i = 2, \lambda_i = \lambda_2 = 2$ and solution given by

$$\frac{e_{21}}{-3} = \frac{-e_{22}}{2} = \frac{e_{23}}{-1} = \beta_2$$

so a corresponding eigenvector is $\mathbf{e}_2 = [3\ 2\ 1]^T$.
When $i = 3, \lambda_i = \lambda_3 = 1$ and solution given by

$$\frac{e_{31}}{-8} = \frac{-e_{32}}{6} = \frac{e_{33}}{-4} = \beta_3$$

so a corresponding eigenvector is $\mathbf{e}_3 = [4\ 3\ 2]^T$.
Corresponding modal and spectral matrices are

$$\mathbf{M} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{\Lambda} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -2 \\ -1 & 1 & 1 \end{bmatrix} \text{ and matrix multiplication confirms } \mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M} = \mathbf{\Lambda}$$

■ **8** Eigenvectors of $\mathbf{A}$ are given by

$$\begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$$

that is, $\lambda^3 - 9\lambda^2 - 9\lambda + 81 = (\lambda - 9)(\lambda - 3)(\lambda + 3) = 0$
so the eigenvalues are $\lambda_1 = 9, \lambda_2 = 3$ and $\lambda_3 = -3$.
The eigenvectors are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1} + 0e_{i2} - 4e_{i3} = 0$$
$$0e_{i1} + (5-\lambda_i)e_{i2} + 4e_{i3} = 0$$
$$-4e_{i1} + 4e_{i2} + (3-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ the normalized eigenvectors are given by

$$\hat{\mathbf{e}}_1 = [\tfrac{1}{3}\ \tfrac{-2}{3}\ \tfrac{-2}{3}]^T, \hat{\mathbf{e}}_2 = [\tfrac{2}{3}\ \tfrac{2}{3}\ \tfrac{-1}{3}]^T, \hat{\mathbf{e}}_3 = [\tfrac{2}{3}\ \tfrac{-1}{3}\ \tfrac{2}{3}]^T$$

The normalised modal matrix

$$\hat{\mathbf{M}} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

so

$$\hat{\mathbf{M}}^T\mathbf{A}\ \hat{\mathbf{M}} = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \mathbf{\Lambda}$$

■ **9** $\dot{\mathbf{N}} = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 6 & -4 & 0 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \mathbf{N},\ \mathbf{N} = [N_1\ N_2\ N_3\ N_4]^T$

Since the matrix $\mathbf{A}$ is a triangular matrix its eigenvalues are the diagonal elements. Thus, the eigenvalues are

$$\lambda_1 = -6,\ \lambda_2 = -4, \lambda_3 = -2, \lambda_4 = 0$$

The eigenvectors are the corresponding solutions of

$$\begin{aligned} (-6-\lambda_i)e_{i1} + 0e_{i2} + 0e_{i3} + 0e_{i4} &= 0 \\ 6e_{i1} + (-4-\lambda_i)e_{i2} + 0e_{i3} + 0e_{i4} &= 0 \\ 0e_{i1} + 4e_{i2} + (-2-\lambda_i)e_{i3} + 0e_{i4} &= 0 \\ 0e_{i1} + 0e_{i2} + 2e_{i3} - \lambda_i e_{i4} &= 0 \end{aligned}$$

Taking $i = 1, 2, 3, 4$ and solving gives the eigenvectors as

$$\begin{aligned} \mathbf{e}_1 &= [1\ -3\ 3\ -1]^T,\ \mathbf{e}_2 = [0\ 1\ -2\ 1]^T \\ \mathbf{e}_3 &= [0\ 0\ 1\ -1]^T,\ \mathbf{e}_4 = [0\ 0\ 0\ 1]^T \end{aligned}$$

Thus, spectral form of solution to the equation is

$$\mathbf{N} = \alpha e^{-6t}\mathbf{e}_1 + \beta e^{-4t}\mathbf{e}_2 + \gamma e^{-2t}\mathbf{e}_3 + \delta \mathbf{e}_4$$

Using the given initial conditions at $t = 0$ we have

$$\begin{bmatrix} C \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \delta \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so $C = \alpha,\ 0 = -3\alpha + \beta,\ 0 = 3\alpha - 2\beta + \gamma,\ 0 = -\alpha + \beta - \gamma + \delta$
which may be solved for α, β, γ and δ to give

$$\alpha = C,\ \beta = 3C,\ \gamma = 3C,\ \delta = C$$

Hence,

$$\begin{aligned} N_4 &= -\alpha e^{-6t} + \beta e^{-4t} - \gamma e^{-2t} + \delta \\ &= -Ce^{-6t} + 3Ce^{-4t} - 3Ce^{-2t} + C \end{aligned}$$

■ **10(a)**

(i) Characteristic equation of $\mathbf{A}$ is $\lambda^2 - 3\lambda + 2 = 0$ so by the Cayley–Hamilton theorem

$$\mathbf{A}^2 = 3\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{A}^3 = 3(3\mathbf{A} - 2\mathbf{I}) - 2\mathbf{A} = 7\mathbf{A} - 6\mathbf{I} = \begin{bmatrix} 8 & 0 \\ 7 & 1 \end{bmatrix}$$

$$\mathbf{A}^4 = 7(3\mathbf{A} - 2\mathbf{I}) - 6\mathbf{A} = 15\mathbf{A} - 14\mathbf{I} = \begin{bmatrix} 16 & 0 \\ 15 & 1 \end{bmatrix}$$

$$\mathbf{A}^5 = 15(3\mathbf{A} - 2\mathbf{I}|) - 14\mathbf{A} = 31\mathbf{A} - 30\mathbf{I} = \begin{bmatrix} 32 & 0 \\ 31 & 1 \end{bmatrix}$$

$$\mathbf{A}^6 = 31(3\mathbf{A} - 2\mathbf{I}) - 30\mathbf{A} = 63\mathbf{A} - 62\mathbf{I} = \begin{bmatrix} 64 & 0 \\ 63 & 1 \end{bmatrix}$$

$$\mathbf{A}^7 = 63(3\mathbf{A} - 2\mathbf{I}) - 62\mathbf{A} = 127\mathbf{A} - 126\mathbf{I} = \begin{bmatrix} 128 & 0 \\ 127 & 1 \end{bmatrix}$$

Thus, $\mathbf{A}^7 - 3\mathbf{A}^6 + \mathbf{A}^4 + 3\mathbf{A}^3 - 2\mathbf{A}^2 + 3\mathbf{I} = \begin{bmatrix} -29 & 0 \\ -32 & 3 \end{bmatrix}$

(ii) Eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2, \lambda_2 = 1$. Thus,

$$\mathbf{A}^k = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \text{ where } \alpha_0 \text{ and } \alpha_1 \text{ satisfy}$$
$$2^k = \alpha_0 + 2\alpha_1, 1 = \alpha_0 + \alpha_1$$
$$\alpha_1 = 2^k - 1,\ \alpha_0 = 2 - 2^k$$

Thus, $\mathbf{A}^k = \begin{bmatrix} \alpha_0 + 2\alpha_1 & 0 \\ \alpha_1 & \alpha_0 + \alpha_1 \end{bmatrix} = \begin{bmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{bmatrix}$

10(b) Eigenvalues of $\mathbf{A}$ are $\lambda_1 = -2, \lambda_2 = 0$. Thus,

$$e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \text{ where } \alpha_0 \text{ and } \alpha_1 \text{ satisfy}$$
$$e^{-2t} = \alpha_0 - 2\alpha_1,\ 1 = \alpha_0 \ \Rightarrow \alpha_0 = 1, \alpha_1 = \frac{1}{2}(1 - e^{-2t})$$

Thus, $e^{\mathbf{A}t} = \begin{bmatrix} \alpha_0 & \alpha_1 \\ 0 & \alpha_0 - 2\alpha_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$

■ **11** The matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ has the single eigenvalue $\lambda = 1$ (multiplicity 3)

$(\mathbf{A} - \mathbf{I}) = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is of rank 2 so has nullity $3 - 2 = 1$ indicating that there is only one eigenvector corresponding to $\lambda = 1$.

This is readily determined as

$$\mathbf{e}_1 = [1\ 0\ 0]^T$$

The corresponding Jordan canonical form comprises a single block so

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Taking $\mathbf{T} = \mathbf{A} - \mathbf{I}$ the triad of vectors (including generalized eigenvectors) has the form $\{\mathbf{T}^2\omega,\ \mathbf{T}\ \omega, \omega\}$ with $\mathbf{T}^2\omega = \mathbf{e}_1$. Since $\mathbf{T}^2 = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we may take $\omega = [0\ 0\ \frac{1}{8}]^T$. Then, $\mathbf{T}\ \omega = [\frac{2}{8}\ \frac{1}{8}\ 0]^T$. Thus, the triad of vectors is

$$\mathbf{e}_1 = [1\ 0\ 0]^T,\ \mathbf{e}_1^* = [\tfrac{3}{8}\ \tfrac{1}{2}\ 0]^T,\ \mathbf{e}_1^{**} = [0\ 0\ \tfrac{1}{8}]^T$$

The corresponding modal matrix is

$$\mathbf{M} = \begin{bmatrix} 1 & \frac{3}{8} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}$$

$$\mathbf{M}^{-1} = 16 \begin{bmatrix} \frac{1}{16} & -\frac{3}{64} & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \text{ and by matrix multiplication}$$

$$\mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M} = 16 \begin{bmatrix} \frac{1}{16} & -\frac{3}{64} & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{8} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{J}$$

■ **12** Substituting $x = X\cos\omega t,\ y = Y\cos\omega t,\ z = Z\cos\omega t$ gives

$$\begin{aligned} -\omega^2 X &= -2X + Y \\ -\omega^2 Y &= X - 2Y + Z \\ -\omega^2 Z &= Y - 2Z \end{aligned}$$

or taking $\lambda = \omega^2$

$$\begin{aligned} (\lambda - 2)X + Y &= 0 \\ X + (\lambda - 2)Y + Z &= 0 \\ Y + (\lambda - 2)Z &= 0 \end{aligned}$$

For non-trivial solution

$$\begin{vmatrix} \lambda - 2 & 1 & 0 \\ 1 & \lambda - 2 & 1 \\ 0 & 1 & \lambda - 2 \end{vmatrix} = 0$$

$$\text{that is, } (\lambda - 2)[(\lambda - 2)^2 - 1] - (\lambda - 2) = 0$$

$$(\lambda - 2)(\lambda^2 - 4\lambda + 2) = 0$$

$$\text{so } \lambda = 2 \text{ or } \lambda = 2 \pm \sqrt{2}$$

When $\lambda = 2$, $Y = 0$ and $X = -Z$ so $X : Y : Z = 1 : 0 : -1$
When $\lambda = 2 + \sqrt{2}$, $X = Z$ and $Y = -\sqrt{2}X$ so $X : Y : Z = 1 : -\sqrt{2} : 1$
When $\lambda = 2 - \sqrt{2}$, $X = Z$ and $Y = \sqrt{2}X$ so $X : Y : Z = 1 : \sqrt{2} : 1$

■ **13** In each section $\mathbf{A}$ denotes the matrix of the quadratic form.

13(a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ has principal minors of 2, $\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1$ and $\det \mathbf{A} = 0$

so by Sylvester's condition (c) the quadratic form is positive-semidefinite.

13(b) $\mathbf{A} = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 7 & 0 \\ -2 & 0 & 2 \end{bmatrix}$ has principal minors of 3, $\begin{vmatrix} 3 & -2 \\ -2 & 7 \end{vmatrix} = 17$ and $\det \mathbf{A} = 6$

so by Sylvester's condition (a) the quadratic form is positive-definite.

13(c) $\mathbf{A} = \begin{bmatrix} 16 & 16 & 16 \\ 16 & 36 & 8 \\ 16 & 8 & 17 \end{bmatrix}$ has principal minors of 16, $\begin{vmatrix} 16 & 16 \\ 16 & 36 \end{vmatrix} = 320$ and $\det \mathbf{A} = -704$

so none of Sylvester's conditions are satisfied and the quadratic form is indefinite.

13(d) $\mathbf{A} = \begin{bmatrix} -21 & 15 & -6 \\ 15 & -11 & 4 \\ -6 & 4 & -2 \end{bmatrix}$ has principal minors of -21, $\begin{vmatrix} -21 & 15 \\ 15 & -11 \end{vmatrix} = 6$ and $\det \mathbf{A} = 0$

so by Sylvester's condition (d) the quadratic form is negative-semidefinite.

13(e) $\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -5 \end{bmatrix}$ has principal minors of -1, $\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 2$ and $\det \mathbf{A} = -4$ so by Sylvester's condition (b) the quadratic form is negative-definite.

■ **14** $\mathbf{A}\,\mathbf{e}_1 = \begin{bmatrix} \frac{7}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 4 & -1 & 0 \\ -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Hence, $\mathbf{e}_1 = [1\ 2\ 3]^T$ is an eigenvector with $\lambda_1 = 1$ the corresponding eigenvalue. Eigenvalues are given by

$$0 = \begin{vmatrix} -\frac{7}{2} - \lambda & -\frac{1}{2} & -\frac{1}{2} \\ 4 & -1 - \lambda & 0 \\ -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} - \lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 + \lambda - 3$$

$$= (\lambda - 1)(\lambda^2 + 2\lambda + 3)$$

$$= -(\lambda - 1)(\lambda - 3)(\lambda + 1)$$

so the other two eigenvalues are $\lambda_2 = 3, \lambda_3 = -1$.

Corresponding eigenvectors are the solutions of

$$(-\tfrac{7}{2} - \lambda_i)e_{i1} - \tfrac{1}{2}e_{i2} - \tfrac{1}{2}e_{i3} = 0$$

$$4e_{i1} - (1 + \lambda_i)e_{i2} + 0e_{i3} = 0$$

$$-\tfrac{3}{2}e_{i1} + \tfrac{3}{2}e_{i2} + (\tfrac{1}{2} - \lambda_i)e_{i3} = 0$$

Taking $i = 2, 3$ gives the eigenvectors as

$$\mathbf{e}_2 = [1\ 1\ 0]^T,\ \mathbf{e}_3 = [0\ -1\ 1]^T$$

The differential equations can be written in the vector–matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x}\ ,\ \mathbf{x} = [x\ y\ z]^T$$

so, in special form, the general solution is

$$\begin{aligned}\mathbf{x} &= \alpha e^{\lambda_1 t}\mathbf{e}_1 + \beta e^{\lambda_2 t}\mathbf{e}_2 + \gamma e_{\lambda_3 t}\mathbf{e}_3 \\ &= \alpha e^t \begin{bmatrix}1\\2\\3\end{bmatrix} + \beta e^{3t} \begin{bmatrix}1\\1\\0\end{bmatrix} + \gamma e^{-t} \begin{bmatrix}0\\-1\\1\end{bmatrix}\end{aligned}$$

With $x(0) = 2,\ y(0) = 4,\ z(0) = 6$ we have

$$\alpha = 2,\ \beta = 0,\ \gamma = 0$$

so

$$\mathbf{x} = 2e^t \begin{bmatrix}1\\2\\3\end{bmatrix}$$

that is, $x = 2e^t,\ y = 4e^t,\ z = 6e^t$.

- **15(a)**

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix}1.2 & 0.9 & -4\\1.6 & 1.2 & 3\end{bmatrix}\begin{bmatrix}1.2 & 1.6\\0.9 & 1.2\\-4 & 3\end{bmatrix} = \begin{bmatrix}18.25 & -9\\-9 & 13\end{bmatrix}$$

Eigenvalues λ_{i} given by

$$\begin{aligned}(18.25 - \lambda)(13 - \lambda) - 81 = 0 &\Rightarrow (\lambda - 25)(\lambda - 6.25) = 0\\ &\Rightarrow \lambda_1 = 25, \lambda_2 = 6.25\end{aligned}$$

having corresponding eigenvectors

$$\begin{aligned}\mathbf{u}_1 &= [\,-4 \quad 3\,]^T \Rightarrow \hat{\mathbf{u}}_1 = [\,-\tfrac{4}{5} \quad \tfrac{3}{5}\,]^T\\ \mathbf{u}_2 &= [\,3 \quad 4\,]^T \Rightarrow \hat{\mathbf{u}}_2 = [\,\tfrac{3}{5} \quad \tfrac{4}{5}\,]\end{aligned}$$

leading to the orthogonal matrix

$$\hat{\mathbf{U}} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1.2 & 1.6 \\ 0.9 & 1.2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 1.2 & 0.9 & -4 \\ 1.6 & 1.2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2.25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

Eigenvalues μ_{i} given by

$$(25-\mu)\left[(4-\mu)(2.25-\mu)-9\right] = 0 \Rightarrow (25-\mu)\mu(\mu-6.25) = 0$$

$$\Rightarrow \mu_1 = 25, \mu_2 = 6.25, \mu_3 = 0$$

with corresponding eigenvalues

$$\begin{aligned} \mathbf{v}_1 &= \hat{\mathbf{v}}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \\ \mathbf{v}_2 &= \begin{bmatrix} 4 & 3 & 0 \end{bmatrix}^T \Rightarrow \hat{\mathbf{v}}_2 = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 0 \end{bmatrix}^T \\ \mathbf{v}_3 &= \begin{bmatrix} -3 & 4 & 0 \end{bmatrix}^T \Rightarrow \hat{\mathbf{v}}_3 = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \end{bmatrix}^T \end{aligned}$$

leading to the orthogonal matrix

$$\hat{\mathbf{V}} = \begin{bmatrix} 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 1 & 0 & 0 \end{bmatrix}$$

The singular values of $\boldsymbol{A}$ are $\sigma_1 = \sqrt{25} = 5$ and $\sigma_2 = \sqrt{6.25} = 2.5$ so that $\mathbf{\Sigma} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2.5 & 0 \end{bmatrix}$ giving the SVD form of $\boldsymbol{A}$ as

$$\mathbf{A} = \hat{\mathbf{U}}\mathbf{\Sigma}\hat{\mathbf{V}}^T = \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2.5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \end{bmatrix}$$

(Direct multiplication confirms $\mathbf{A} = \begin{bmatrix} 1.2 & 0.9 & -4 \\ 1.6 & 1.2 & 3 \end{bmatrix}$)

(b) $\mathbf{A}^\dagger = \hat{\mathbf{V}}\boldsymbol{\Sigma}^*\hat{\mathbf{U}}^T = \begin{bmatrix} 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{2}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \frac{1}{125} \begin{bmatrix} 24 & 32 \\ 18 & 24 \\ -20 & 15 \end{bmatrix}$

$$= \begin{bmatrix} 0.192 & 0.256 \\ 0.144 & 0.192 \\ -0.16 & 0.12 \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^\dagger = \mathbf{I}$$

CHECK

$$LHS = \frac{1}{125} \begin{bmatrix} 1.2 & 0.9 & -4 \\ 1.6 & 1.2 & 1 \end{bmatrix} \begin{bmatrix} 24 & 32 \\ 18 & 24 \\ -24 & 15 \end{bmatrix} = \frac{1}{125} \begin{bmatrix} 125 & 0 \\ 0 & 125 \end{bmatrix} = \mathbf{I} = RHS$$

(c) Since $\boldsymbol{A}$ is of full rank 2 and there are more columns than rows

$$\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1} = \begin{bmatrix} 1.2 & 1.6 \\ 0.9 & 1.2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 18.25 & -9 \\ -9 & 13 \end{bmatrix}^{-1} = \frac{1}{156.25} \begin{bmatrix} 1.2 & 1.6 \\ 0.9 & 1.2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 13 & 9 \\ 9 & 18.25 \end{bmatrix}$$

$$= \frac{1}{156.25} \begin{bmatrix} 30 & 40 \\ 22.5 & 30 \\ -25 & 18.25 \end{bmatrix} = \begin{bmatrix} 0.192 & 0.256 \\ 0.144 & 0.192 \\ -0.16 & 0.12 \end{bmatrix}$$

which checks with the answer in (b).

■ **16** (a) Using partitioned matrix multiplication the SVD form of $\boldsymbol{A}$ may be expressed in the form

$$\mathbf{A} = \hat{\mathbf{U}}\boldsymbol{\Sigma}\hat{\mathbf{V}}^T = [\,\hat{\mathbf{U}}_r \quad \hat{\mathbf{U}}_{m-r}\,] \begin{bmatrix} \mathbf{S} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{V}}_r^T \\ \hat{\mathbf{V}}_{n-r}^T \end{bmatrix} = \hat{\mathbf{U}}_r\mathbf{S}\hat{\mathbf{V}}_r^T$$

(b) Since the diagonal elements in $\boldsymbol{S}$ are non-zero the pseudo inverse may be expressed in the form

$$\mathbf{A}^\dagger = \hat{\mathbf{V}}\boldsymbol{\Sigma}^*\hat{\mathbf{U}}^T = \hat{\mathbf{V}}_r\mathbf{S}^{-1}\hat{\mathbf{U}}_r^T$$

(c) From the solution to Q46, exercises 1.8.4, the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$ has a single singularity $\sigma_1 = \sqrt{18}$ so $r = 1$ and $\mathbf{S}$ is a scalar $\sqrt{18}$; $\hat{\mathbf{U}}_r = \hat{\mathbf{U}}_1 = \hat{\mathbf{u}}_1 = [\,\frac{1}{3} \;\; -\frac{2}{3} \;\; \frac{2}{3}\,]^T$ and

$$\hat{\mathbf{V}}_r = \hat{\mathbf{V}}_1 = \hat{\mathbf{v}}_1 = \left[\,\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}\,\right]^T$$

The SVD form of $\boldsymbol{A}$ is

$$\mathbf{A} = \hat{\mathbf{u}}_1 S \hat{\mathbf{v}}_1^T = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \sqrt{18} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

with direct multiplication confirming $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

Thus, the pseudo inverse is

$$\mathbf{A}^\dagger = \hat{\mathbf{v}}_1 S^{-1} \hat{\mathbf{u}}_1^T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \frac{1}{\sqrt{18}} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$
$$= \frac{1}{18} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

which agrees with the answer obtained in Q46, Exercises 1.8.4

■ **17** $\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x} + \mathbf{b}u\ ,\ y = \mathbf{c}^T\mathbf{x}$

Let $\lambda_i, e_i, i = 1, 2, \ldots, n,$ be the eigenvalues and corresponding eigenvectors of $\mathbf{A}$. Let $\mathbf{M} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n]$ then since λ_i's are distinct the $\mathbf{e}_i$'s are linearly independent and $\mathbf{M}^{-1}$ exists. Substituting $\mathbf{x} = \mathbf{M}\ \boldsymbol{\xi}$ gives

$$\mathbf{M}\ \dot{\boldsymbol{\xi}} = \mathbf{A}\ \mathbf{M}\ \boldsymbol{\xi} + \mathbf{b}u$$

Premultiplying by M^{-1} gives

$$\dot{\boldsymbol{\xi}} = \mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M}\ \boldsymbol{\xi} + \mathbf{M}^{-1}\ \mathbf{b}u = \boldsymbol{\Lambda}\ \boldsymbol{\xi} + \mathbf{b}_1 u$$

where $\boldsymbol{\Lambda} = \mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M} = (\lambda_i \delta_{ij}), i, j = 1, 2, \ldots, n,$ and $\mathbf{b}_1 = \mathbf{M}^{-1}\mathbf{b}$

Also, $y = \mathbf{c}^T\mathbf{x}\ \Rightarrow\ \mathbf{y} = \mathbf{c}^T\mathbf{M}\boldsymbol{\xi} = \mathbf{c}_1^T\boldsymbol{\xi},\ \mathbf{c}_1^T = \mathbf{c}^T\mathbf{M}$. Thus, we have the desired canonical form.

If the vector $\mathbf{b}_1$ contains a zero element then the corresponding mode is uncontrollable and consequently $(\mathbf{A}_1\ \mathbf{b}_1\ \mathbf{c})$ is uncontrollable. If the matrix $\mathbf{c}^T$ has a zero element then the system is unobservable.

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$ having corresponding eigenvectors $\mathbf{e}_1 = [1\ 3\ 1]^T,\ \mathbf{e}_2 = [3\ 2\ 1]^T$ and $\mathbf{e}_3 = [1\ 0\ 1]^T$.

The modal matrix

$$\mathbf{M} = [\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3] = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ with } \mathbf{M}^{-1} = -\frac{1}{6}\begin{bmatrix} 2 & -2 & -2 \\ -3 & 0 & 3 \\ 1 & 2 & -7 \end{bmatrix}$$

so canonical form is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{4}{3} \end{bmatrix} u$$

$$y = [1\ \ -4\ \ -2][\xi_1\ \xi_2\ \xi_3]^T$$

We observe that the system is uncontrollable but observable. Since the system matrix $\mathbf{A}$ has positive eigenvalues the system is unstable. Using Kelman matrices

(i) $\mathbf{A}^2 = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 4 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, $\mathbf{A}\ \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{A}^2\ \mathbf{b} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$

Thus, $[\mathbf{b}\ \ \mathbf{A}\ \mathbf{b}\ \ \mathbf{A}^2\ \mathbf{b}] = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 2 & 4 \\ -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and is of rank 2
so the system is uncontrollable.

(ii) $[\mathbf{c}\ \ \mathbf{A}^T\ \mathbf{c}\ \ (A^T)^2\mathbf{c}] = \begin{bmatrix} -2 & -3 & -3 \\ 1 & 0 & 2 \\ 0 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and is of full rank 3
so the system is observable.

■ **18** Model is of form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and making the transformation $\mathbf{x} = \mathbf{M}\mathbf{z}$ gives

$$\mathbf{M}\dot{\mathbf{z}} = \mathbf{A}\mathbf{M}\mathbf{z} + \mathbf{B}\mathbf{u} \Rightarrow \dot{\mathbf{z}} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\mathbf{z} + \mathbf{M}^{-1}\mathbf{B}\mathbf{u} \Rightarrow \dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mathbf{M}^{-1}\mathbf{B}\mathbf{u}$$

where $\mathbf{M}$ and $\mathbf{\Lambda}$ are respectively the modal and spectral matrices of $\mathbf{A}$.
The eigenvalues of $\boldsymbol{A}$ are given by

$$\begin{vmatrix} -2-\lambda & -2 & 0 \\ 0 & -\lambda & 1 \\ 0 & -3 & -4-\lambda \end{vmatrix} = 0 \Rightarrow -(2-\lambda)(4\lambda + \lambda^2 + 3) = 0$$

$$\Rightarrow (\lambda+2)(\lambda+1)(\lambda+3) = 0$$

$$\Rightarrow \lambda_1 - 1, \lambda_2 = -2, \lambda_3 = -3$$

with corresponding eigenvectors

$$\mathbf{e}_1 = [\,-2 \quad 1 \quad -2\,]^T, \mathbf{e}_2 = [\,1 \quad 0 \quad 0\,]^T \text{ and } \mathbf{e}_3 = [\,-2 \quad -1 \quad 3\,]^T$$

Thus, the modal and spectral matrices are

$$\mathbf{M} = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & -4 \\ -1 & 0 & -1 \end{bmatrix} \text{and} \mathbf{\Lambda} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\text{and } \det \mathbf{M} = -2 \Rightarrow \mathbf{M}^{-1} = \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 1 & 4 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \mathbf{M}^{-1}\mathbf{B} = \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 1 & 4 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 2 \\ 3 & 6 \\ \frac{1}{2} & 1 \end{bmatrix} \text{ leading to the canonical form}$$

$$\dot{\mathbf{z}} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 2 \\ 3 & 6 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

From (1.99a) the solution is given by

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & 0 & 0 \\ 0 & e^{-2(t-\tau)} & 0 \\ 0 & 0 & e^{-3(t-\tau)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & 2 \\ 3 & 6 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \tau \\ 1 \end{bmatrix} d\tau$$

$$\text{with } \mathbf{z}(0) = \mathbf{M}^{-1}\mathbf{x}(0) = \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 1 & 4 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix} = [\,\tfrac{17}{2} \quad 34 \quad \tfrac{7}{2}\,]^T. \text{ Thus,}$$

$$\mathbf{z} = \begin{bmatrix} \frac{17}{2}e^{t} \\ 34e^{-2t} \\ \frac{7}{2}e^{-3t} \end{bmatrix} + \int_0^t \begin{bmatrix} (2+\frac{1}{2}\tau)e^{-(t-\tau)} \\ (6+3\tau)e^{-2(t-\tau)} \\ (1+\frac{1}{2}\tau)e^{-3(t-\tau)} \end{bmatrix} d\tau \Rightarrow \mathbf{z} = \begin{bmatrix} \frac{17}{2}e^{t} \\ 34e^{-2t} \\ \frac{7}{2}e^{-3t} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1}{2}t + \frac{3}{2} - \frac{3}{2}e^{-t} \\ \frac{3}{2}t + \frac{9}{4} - \frac{9}{4}e^{-2t} \\ \frac{1}{6}t + \frac{5}{18} - \frac{5}{18}e^{-3t} \end{bmatrix} \Rightarrow \mathbf{z} = \begin{bmatrix} \frac{1}{2}t + \frac{3}{2} + 7e^{-t} \\ \frac{3}{2}t + \frac{9}{4} - \frac{127}{4}e^{-2t} \\ \frac{1}{6}t + \frac{5}{18} + \frac{29}{9}e^{-3t} \end{bmatrix}$$

$$\text{giving } \mathbf{x} = \mathbf{M}\mathbf{z} = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & -1 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2}t + \frac{3}{2} + 7e^{-t} \\ \frac{3}{2}t + \frac{9}{4} - \frac{127}{4}e^{-2t} \\ \frac{1}{6}t + \frac{5}{18} + \frac{29}{9}e^{-3t} \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} -14e^{-t} + \frac{127}{4}e^{-2t} - \frac{58}{9}e^{-3t} + \frac{1}{6}t - \frac{47}{36} \\ 7e^{-t} - \frac{29}{9}e^{-3t} + \frac{1}{3}t + \frac{11}{9} \\ -7e^{-t} + \frac{29}{3}e^{-3t} - \frac{2}{3} \end{bmatrix}$$

■ **19(a)** Eigenvalues of the matrix given by

$$0 = \begin{vmatrix} 5-\lambda & 2 & -1 \\ 3 & 6-\lambda & -9 \\ 1 & 1 & 1-\lambda \end{vmatrix} \underset{=}{C_1 - C_2} \begin{vmatrix} 3-\lambda & 2 & -1 \\ -3+\lambda & 6-\lambda & -9 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} 1 & 2 & -1 \\ 0 & 8-\lambda & -10 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(\lambda^2 - 9\lambda + 18) = (3-\lambda)(\lambda-3)(\lambda-6)$$

so the eigenvalues are $\lambda_1 = 6, \lambda_2 = \lambda_3 = 3$

When $\lambda = 3, \mathbf{A} - 3\mathbf{I} = \begin{bmatrix} 2 & 2 & -1 \\ 3 & 3 & -9 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is of rank 2

so there is only $3 - 2 = 1$ corresponding eigenvectors.

The eigenvector corresponding to $\lambda_1 = 6$ is readily determined as $\mathbf{e}_1 = [3\ 2\ 1]^T$.

Likewise the single eigenvector corresponding to $\lambda_2 = 6$ is determined as

$$\mathbf{e}_2 = [1\ \ -1\ 0]^T$$

The generalized eigenvector $\mathbf{e}_2^*$ determined by

$$(\mathbf{A} - 2\mathbf{I})\mathbf{e}_2^* = \mathbf{e}_2$$

$$\text{or} \quad 3e_{21}^* + 2e_{22}^* - e_{23}^* = 1$$

$$3e_{21}^* + 3e_{22}^* - 9e_{23}^* = -1$$

$$e_{21}^* + e_{22}^* - 2e_{23}^* = 0$$

giving $\mathbf{e}_2^* = [\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]^T$.

For convenience, we can take the two eigenvectors corresponding to $\lambda = 3$ as

$$\mathbf{e}_2 = [3\ \ -3\ 0]^T,\ \mathbf{e}_2^* = [1\ 1\ 1]^T$$

The corresponding Jordan canonical form being $\mathbf{J} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

19(b) The generalised modal matrix is then

$$\mathbf{M} = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}\ \mathbf{M} = \begin{bmatrix} 5 & 2 & -1 \\ 3 & 6 & -9 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 1 \\ 2 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 9 & 6 \\ 12 & -9 & 0 \\ 6 & 0 & 3 \end{bmatrix}$$

$$\mathbf{M}\ \mathbf{J} = \begin{bmatrix} 3 & 3 & 1 \\ 2 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 9 & 6 \\ 12 & -9 & 0 \\ 6 & 0 & 3 \end{bmatrix}$$

so $\mathbf{A}\ \mathbf{M} = \mathbf{M}\ \mathbf{J}$

19(c) $\mathbf{M}^{-1} = -\frac{1}{9} \begin{bmatrix} -3 & -3 & 6 \\ -1 & 2 & -1 \\ 3 & 3 & -15 \end{bmatrix}$, $e^{\mathbf{J}t} = \begin{bmatrix} e^{6t} & 0 & 0 \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}$

so

$$\mathbf{x}(t) = -\frac{1}{9} \begin{bmatrix} 3 & 3 & 1 \\ 2 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{6t} & 0 & 0 \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -3 & -3 & 6 \\ -1 & 2 & -1 \\ 3 & 3 & -15 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 9e^{6t} - 9(1+t)e^{3t} \\ 6e^{6t} + (3+9t)e^{3t} \\ 3e^{6t} - 3e^{3t} \end{bmatrix}$$

■ **20** Substituting $\mathbf{x} = e^{\lambda t}\mathbf{u}$, where $\mathbf{u}$ is a constant vector, in $\mathbf{x} = \mathbf{A}\ \mathbf{x}$ gives

$$\lambda^2\mathbf{u} = \mathbf{A}\ \mathbf{u} \text{ or } (\mathbf{A} - \lambda^2\mathbf{I})\mathbf{u} = 0 \tag{1}$$

so that there is a non-trivial solution provided

$$|\ \mathbf{A} - \lambda^2\mathbf{I}\ | = 0 \tag{2}$$

If $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ are the solutions of (2) and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ the corresponding solutions of (1) define

$$\mathbf{M} = [\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_n] \text{ and } \mathbf{S} = \text{diag } (\lambda_1^2\ \lambda_2^2\ \ldots\ \lambda_n^2)$$

Applying the transformation $\mathbf{x} = \mathbf{M}\ \mathbf{q}$, $\mathbf{q} = [q_1\ q_2\ \ldots\ q_n]$ gives

$$\mathbf{M}\ \ddot{\mathbf{q}} = \mathbf{A}\ \mathbf{M}\ \mathbf{q}$$

$$\text{giving} \quad \ddot{\mathbf{q}} = \mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M}\ \mathbf{q} \text{ provided } \mathbf{u}_1,\ \mathbf{u}_2, \ldots, u_n \text{ are linearly independent}$$

$$\text{so that } \ddot{\mathbf{q}} = \mathbf{S}\ \mathbf{q} \text{ since } \mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M} = \mathbf{S}$$

This represents n differential equations of the form

$$\ddot{q}_i = \lambda_i^2 q_i\ ,\ i = 1, 2, \ldots, n$$

When $\lambda_i^2 < 0$ this has the solution of the form

$$q_i = C_i \sin(\omega_i t + \alpha_i)$$

where C_i and α_i are arbitrary constants and $\lambda_i = j\omega_i$
The given differential equations may be written in the vector–matrix form

$$\dot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is of the above form

$$\ddot{\mathbf{x}} = \mathbf{A}\ \mathbf{x}$$

$0 = |\ \mathbf{A} - \lambda^2 \mathbf{I}\ |$ gives $(\lambda^2)^2 + 5(\lambda^2) + 4 = 0$ or $\lambda_1^2 = -1, \lambda_2^2 = -4$.
Solving the corresponding equation

$$(\mathbf{A} - \lambda_i^2 \mathbf{I})\ \mathbf{u}_i = 0$$

we have that $\mathbf{u}_1 = [1\ 1]^T$ and $\mathbf{u}_2 = [2\ -1]^T$. Thus, we take

$$\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$$

The normal modes of the system are given by

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

giving

$$q_1(t) = C_1 \sin(t + \alpha_1) \equiv \gamma_1 \sin t + \beta_1 \cos t$$
$$q_2(t) = C_2 \sin(2t + \alpha_2) \equiv \gamma_2 \sin 2t + \beta_2 \cos 2t$$

Since $\mathbf{x} = \mathbf{M}\,\mathbf{q}$ we have that $\mathbf{q}(0) = \mathbf{M}^{-1}\mathbf{x}(0) = -\frac{1}{3}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix}$
also $\dot{\mathbf{q}}(0) = \mathbf{M}^{-1}\dot{\mathbf{x}}(0)$ so that $\dot{q}_1(0) = 2$ and $\dot{q}_2(0) = 0$
Using these initial conditions we can determine $\gamma_1, \beta_1, \gamma_2$ and β_2 to give

$$q_1(t) = \frac{5}{3}\cos t + 2\sin t$$
$$q_2(t) = -\frac{1}{3}\cos 2t$$

The general displacements $x_1(t)$ and $x_2(t)$ are then given by $\mathbf{x} = \mathbf{M}\,\mathbf{q}$ so

$$x_1 = q_1 + 2q_2 = \frac{5}{3}\cos t + 2\sin t - \frac{2}{3}\cos 2t$$
$$x_2 = q_1 - q_2 = \frac{5}{3}\cos t + 2\sin t - \frac{1}{3}\cos 2t$$

2

Numerical Solution of Ordinary Differential Equations

Exercises 2.3.4

■ **1** Euler's method for the solution of the differential equation $\dfrac{dx}{dt} = f(t, x)$ is

$$X_{n+1} = X_n + hF_n = X_n + hf(t_n, X_n)$$

Applying this to the equation $\dfrac{dx}{dt} = -\frac{1}{2}xt$ with $x(0) = 1$ and a step size of $h = 0.1$ yields

$$\begin{aligned}
x_0 &= x(0) = 1 \\
X_1 &= x_0 + hf(t_0, x_0) = x_0 + h\left(-\tfrac{1}{2}x_0t_0\right) \\
&= 1 - 0.1 \times \tfrac{1}{2} \times 1 \times 0 = 1.0000 \\
X_2 &= X_1 + hf(t_1, X_1) = X_1 + h\left(-\tfrac{1}{2}X_1t_1\right) \\
&= 1.0000 - 0.1 \times \tfrac{1}{2} \times 1.0000 \times 0.1 = 0.9950 \\
X_3 &= X_2 + hf(t_2, X_2) = X_2 + h\left(-\tfrac{1}{2}X_2t_2\right) \\
&= 0.9950 - 0.1 \times \tfrac{1}{2} \times 0.9950 \times 0.2 \times 0.98505
\end{aligned}$$

Hence Euler's method with step size $h = 0.1$ gives the estimate $X(0.3) = 0.98505$.

■ **2** Euler's method for the solution of the differential equation $\dfrac{dx}{dt} = f(t, x)$ is

$$X_{n+1} = X_n + hF_n = X_n + hf(t_n, X_n)$$

Applying this to the equation $\dfrac{dx}{dt} = -\frac{1}{2}xt$ with $x(1) = 0.1$ and a step size of $h = 0.025$ yields

$$\begin{aligned}
x_0 &= x(1) = 0.1 \\
X_1 &= x_0 + hf(t_0, x_0) = x_0 + h\left(-\tfrac{1}{2}x_0 t_0\right) \\
&= 0.1 - 0.025 \times \tfrac{1}{2} \times 0.1 \times 1 = 0.09875 \\
X_2 &= X_1 + hf(t_1, X_1) = X_1 + h\left(-\tfrac{1}{2}X_1 t_1\right) \\
&= 0.09875 - 0.025 \times \tfrac{1}{2} \times 0.09875 \times 1.025 = 0.09748 \\
X_3 &= X_2 + hf(t_2, X_2) = X_2 + h\left(-\tfrac{1}{2}X_2 t_2\right) \\
&= 0.09748 - 0.025 \times \frac{1}{2} \times 0.09748 \times 1.050 = 0.09621 \\
X_4 &= X_3 + hf(t_3, X_3) = X_3 + h\left(-\tfrac{1}{2}X_3 t_3\right) \\
&= 0.09621 - 0.025 \times \tfrac{1}{2} \times 0.09621 \times 1.075 = 0.09491
\end{aligned}$$

Hence Euler's method with step size $h = 0.1$ gives the estimate $X(1.1) = 0.09491$.

■ **3** Euler's method for the solution of the differential equation $\dfrac{dx}{dt} = f(t, x)$ is

$$X_{n+1} = X_n + hF_n = X_n + hf(t_n, X_n)$$

Applying this to the equation $\dfrac{dx}{dt} = \dfrac{x}{2(t+1)}$ with $x(0.5) = 1$ and a step of $h = 0.1$ yields

$$\begin{aligned}
x_0 &= x(0.5) = 1 \\
X_1 &= x_0 + hf(t_0, x_0) = x_0 + h\frac{x_0}{2(t_0+1)} = 1 + 0.1\frac{1}{2(0.5+1)} = 1.0333 \\
X_2 &= X_1 + hf(t_1, X_1) = X_1 + h\frac{X_1}{2(t_1+1)} = 1.0333 + 0.1\frac{1.0333}{2(0.6+1)} = 1.0656
\end{aligned}$$

(Note that $t_n = t_0 + nh = 0.5 + 0.1n$.) X_3, X_4 and X_5 may be computed in similar fashion. It is usually easier to set out numerical solutions in a systematic tabular form such as the following:

n	t_n	X_n	$f(t_n, X_n)$	$X_n + hf(t_n, X_n)$
0	0.5	1.0000	0.3333	1.0333
1	0.6	1.0333	0.3229	1.0656

2	0.7	1.0656	0.3134	1.0969
3	0.8	1.0969	0.3047	1.1274
4	0.9	1.1274	0.2967	1.1571
5	1.0	1.1571		

Hence Euler's method with step size $h = 0.1$ gives the estimate $X(1) = 1.1571$.

■ **4** Euler's method for the solution of the differential equation $\dfrac{dx}{dt} = f(t, x)$ is

$$X_{n+1} = X_n + hF_n = X_n + hf(t_n, X_n)$$

Applying this to the equation $\dfrac{dx}{dt} = \dfrac{4 - t}{t + x}$ with $x(0) = 1$ and a step size of $h = 0.05$ yields

$$x_0 = x(0.0) = 1$$

$$X_1 = x_0 + hf(t_0, x_0) = x_0 + h\frac{4 - t_0}{t_0 + x_0} = 1 + 0.05\frac{4 - 0}{0 + 1} = 1.2000$$

$$X_2 = X_1 + hf(t_1, X_1) = X_1 + h\frac{4 - t_1}{t_1 + X_1} = 1.2000 + 0.05\frac{4 - 0.05}{0.05 + 1.2000} = 1.3580$$

(Note that $t_n = t_0 + nh = 0.0 + 0.05n$.) $X_3, X_4, \ldots, X_{10}$ may be computed in similar fashion.

It is usually easier to set out numerical solutions in a systematic tabular form such as the following:

t_n	X_n	$f(t_n, X_n)$	$X_n + hf(t_n, X_n)$
0.00	1.0000	4.0000	1.2000
0.05	1.2000	3.1600	1.3580
0.10	1.3580	2.6749	1.4917
0.15	1.4917	2.3451	1.6090
0.20	1.6090	2.1006	1.7140
0.25	1.7140	1.9093	1.8095
0.30	1.8095	1.7540	1.8972
0.35	1.8972	1.6242	1.9784
0.40	1.9784	1.5136	2.0541
0.45	2.0541	1.4177	2.1250
0.50	2.1250		

Hence Euler's method with step size $h = 0.05$ gives the estimate $X(0.5) = 2.1250$.

■ **5** Figure 2.1 shows a suitable pseudocode program for computing the estimates $X_a(2)$ and $X_b(2)$. Figure 2.2 shows a Pascal implementation of the pseudocode program.

```
procedure deriv (t, x → f)
    f ← x*t/(t*t + 2)
endprocedure

t_start ← 1
x_start ← 2
t_end ← 2
write (vdu, "Enter step size")
read (keyboard, h)
write (printer, t_start, x_start)
t ← t_start
x ← x_start
repeat
    deriv (t, x → f)
    t ← t + h
    x ← x + h*f
    write (printer, t,x)
until t >= t_end
```

Figure 2.1: Pseudocode algorithm for Exercise 5

Using this program the results $X_a(2) = 2.811489$ and $X_b(2) = 2.819944$ were obtained. Using the method described in Section 2.3.6, the error in $X_b(2)$ will be approximately equal to $X_b(2) - X_a(2) = -0.008455$ and so the best estimate of $X(2)$ is $2.819944 + 0.008455 = 2.828399$. The desired error bound is 0.1% of this value, 0.0028 approximately. Since Euler's method is a first-order method, the error in the estimate of $X(2)$ varies like h; so, to achieve an error of 0.0028, a step size of no more than $(0.0028/0.008455) \times 0.05 = 0.0166$ is required. We will choose a sensible step size which is less than this, say $h = 0.0125$. This yields an estimate $X(2) = 2.826304$.

The exact solution of the differential equation may be obtained by separation:

$$\frac{dx}{dt} = \frac{xt}{t^2+2} \Rightarrow \int \frac{dx}{x} = \int \frac{t\ dt}{t^2+2} \Rightarrow \ln x = \tfrac{1}{2}\ln(t^2+2) + C \Rightarrow x = \pm D\sqrt{t^2+2}$$

$$x(1) = 2 \Rightarrow 2 = \pm\sqrt{3}D \Rightarrow x = 2\sqrt{\frac{t^2+2}{3}}$$

Hence $x(2) = 2\sqrt{2} = 2.828427$ and the true errors in $X_a(2)$, $X_b(2)$ and the final estimate of $X(2)$ are 0.016938, 0.008483 and 0.002123 respectively. The estimate, $X(2)$, derived using the step size $h = 0.0125$ is comfortably within the 0.1% error requirement.

```
var t_start, x_start, t_end, h,x,t,f:real;

procedure deriv (t,x:real;var f:real);
begin
    f := x*t/(t*t + 2)
end;

begin
    t_start := 1;
    x_start := 2;
    t_end := 2;
    write ('Enter step size ==> ');
    readln (h);
    writeln (t_start :5 :2, x_start :10 :6);
    t := t_start;
    x := x_start;
    repeat
       deriv(t,x,f);
       t := t + h;
       x := x + h*f;
       writeln (t:5:2, x:10:6);
    until t >= t_end;
end.
```

Figure 2.2: Pascal program for Exercise 5

■ **6** The programs shown in Figures 2.1 and 2.2 may readily be modified to solve this problem. Estimates $X_a(2) = 1.573065$ and $X_b(2) = 1.558541$ should be obtained. Using the method described in Section 2.3.6, the error in $X_b(2)$ will be approximately equal to $X_b(2) - X_a(2) = -0.014524$ and so the best estimate of $X(2)$ is $1.558541 - 0.014524 = 1.544017$. The desired error bound is 0.2% of this value, 0.0031 approximately. Since Euler's method is a first-order method, the error in the estimate of $X(2)$ varies like h so, to achieve an error of 0.0031, a step size of no more than $(0.0031/0.014524) \times 0.05 = 0.0107$ is required. We will choose a sensible step size which is less than this, say $h = 0.01$. This yields an estimate $X(2) = 1.547462$.

The exact solution of the differential equation may be obtained by separation:

$$\frac{dx}{dt} = \frac{1}{xt} \Rightarrow \int x dx = \int \frac{dt}{t} \Rightarrow \tfrac{1}{2}x^2 = \ln t + C \Rightarrow x = \pm\sqrt{2(\ln t + C)}$$

$$x(1) = 1 \Rightarrow 1 = 2C \Rightarrow x(t) = \sqrt{2\ln t + 1}$$

Hence $x(2) = \sqrt{2\ln 2 + 1} = 1.544764$ and the true errors in $X_a(2), X_b(2)$ and the final estimate of $X(2)$ are -0.028301, -0.013777 and -0.002698 respectively. The estimate, $X(2)$, derived using the step size $h = 0.01$ is comfortably within the 0.2% error requirement.

■ **7** The programs shown in Figures 2.1 and 2.2 may readily be modified to solve this problem. Estimates $X_a(1.5) = 2.241257$ and $X_b(1.5) = 2.206232$ should be obtained. Using the method described in Section 2.3.6, the error in $X_b(1.5)$ will be approximately equal to $X_b(1.5) - X_a(1.5) = -0.035025$ and so the best estimate of $X(1.5)$ is $2.206232 - 0.035025 = 2.171207$. The desired error bound is 0.25% of this value, 0.0054 approximately. Since Euler's method is a first-order method, the error in the estimate of $X(1.5)$ varies like h; so, to achieve an error of 0.0054, a step size of no more than $(0.0054/0.035025) \times 0.025 = 0.0039$ is required. If we choose $h = 0.04$, this yields an estimate $X(1.5) = 2.183610$.

The exact solution of the differential equation may be obtained by separation:

$$\frac{dx}{ct} = \frac{1}{\ln x} \Rightarrow \int \ln x dx = \int dt \Rightarrow x\ln x - x = t + C$$
$$x(1) = 1.2 \Rightarrow 1.2\ln 1.2 - 1.2 = 1 + C \Rightarrow$$
$$C = -1.981214 \Rightarrow x\ln x - x = t - 1.981214$$

Hence, by any non-linear equation solving method (e.g. Newton–Raphson), we may obtain $x(1.5) = 2.179817$ and the true errors in $X_a(1.5), X_b(1.5)$ and the final estimate of $X(1.5)$ are 0.061440, 0.026415 and 0.003793 respectively. The estimate, $X(1.5)$, derived using the step size $h = 0.04$ is comfortably within the 0.25% error requirement.

Exercises 2.3.9

■ **8** The starting process, using the second-order predictor–corrector method, is

$$\hat{X}_1 = x_0 + hf(t_0, x_0)$$
$$X_1 = x_0 + \tfrac{1}{2}h\left(f(t_0, x_0) + f(t_1, \hat{X}_1)\right)$$

and the second-order Adams–Bashforth method is

$$X_{n+1} = X_n + \tfrac{1}{2}h\left(3f(t_n, X_n) - f(t_{n-1}, X_{n-1})\right)$$

8(a) Applying this method to the problem $\dfrac{dx}{dt} = x^2 \sin t - x, x(0) = 0.2$ with $h = 0.1$, we have

$$\hat{X}_1 = x_0 + hf(t_0, x_0) = 0.2 + 0.1 \times (0.2^2 \sin 0 - 0.2) = 0.1800$$
$$X_1 = x_0 + \tfrac{1}{2}h\left(f(t_0, x_0) + f(t_1, \hat{X}_1)\right)$$
$$= 0.2 + \tfrac{1}{2}0.1 \times (0.2^2 \sin 0 - 0.2 + 0.18^2 \sin 0.1 - 0.18) = 0.1812$$
$$X_2 = X_1 + \tfrac{1}{2}h\left(3f(t_1, X_1) - f(t_0, x_0)\right)$$
$$= 0.1812 + \tfrac{1}{2}0.1 \times \left(3(0.1812^2 \sin 0.1 - 0.1812) - (0.2^2 \sin 0 - 0.2)\right) = 0.1645$$

X_3, X_4 and X_5 are obtained as X_2. The computation is most efficiently set out as a table.

n	t_n	X_n	$f(t_n, X_n)$	$\frac{1}{2}h(3f(t_n, X_n) - f(t_{n-1}, X_{n-1}))$	X_{n+1}
0	0.0	0.2000	−0.2000	(use predictor–corrector)	0.1812
1	0.1	0.1812	−0.1779	−0.016685	0.1645
2	0.2	0.1645	−0.1591	−0.014970	0.1495
3	0.3	0.1495	−0.1429	−0.013480	0.1360
4	0.4	0.1360	−0.1288	−0.012175	0.1238
5	0.5	0.1238			

Hence $X(0.5) = 0.1238$.

8(b) Applying this method to the problem $\dfrac{dx}{dt} = x^2e^{tx}, x(0.5) = 0.5$ with $h = 0.1$, we have

$$\begin{aligned}
\hat{X}_1 &= x_0 + hf(t_0, x_0) = 0.5 + 0.1 \times 0.5^2 e^{0.5\times 0.5} = 0.5321 \\
X_1 &= x_0 + \tfrac{1}{2}h\left(f(t_0, x_0) + f(t_1, \hat{X}_1)\right) \\
&= 0.5 + \tfrac{1}{2}0.1 \times (0.5^2 e^{0.5\times 0.5} + 0.5321^2 e^{0.6\times 0.5321}) = 0.5355 \\
X_2 &= X_1 + \tfrac{1}{2}h\left(3f(t_1, X_1) - f(t_0, x_0)\right) \\
&= 0.5355 + \tfrac{1}{2}0.1 \times \left(3 \times 0.5355^2 e^{0.6\times 0.5355} - 0.5^2 e^{0.5\times 0.5}\right) = 0.5788
\end{aligned}$$

X_3, X_4, X_5, X_6 and X_7 are obtained as X_2. The computation is most efficiently set out as a table.

n	t_n	X_n	$f(t_n, X_n)$	$\frac{1}{2}h(3f(t_n, X_n) - f(t_{n-1}, X_{n-1}))$	X_{n+1}
0	0.5	0.5000	0.3210	(use predictor–corrector)	0.5355
1	0.6	0.5355	0.3955	0.043275	0.5788
2	0.7	0.5788	0.5024	0.055585	0.6344
3	0.8	0.6344	0.6685	0.075155	0.7095
4	0.9	0.7095	0.9534	0.109585	0.8191
5	1.0	0.8191	1.5221	0.180645	0.9998
6	1.1	0.9998	3.0021	0.374210	1.3740
7	1.2	1.3740			

Hence $X(1.2) = 1.3740$.

■ **9** The starting process, using the second-order predictor–corrector method, is

$$\begin{aligned}
\hat{X}_1 &= x_0 + hf(t_0, x_0) \\
X_1 &= x_0 + \tfrac{1}{2}h\left(f(t_0, x_0) + f(t_1, \hat{X}_1)\right) \\
\hat{X}_2 &= X_1 + hf(t_1, X_1) \\
X_2 &= X_1 + \tfrac{1}{2}h\left(f(t_1, X_1) + f(t_2, \hat{X}_2)\right)
\end{aligned}$$

and the third-order Adams–Bashforth method is

$$X_{n+1} = X_n + \tfrac{1}{12}h\left(23f(t_n, X_n) - 16f(t_{n-1}, X_{n-1}) + 5f(t_{n-2}, X_{n-2})\right)$$

Applying this method to the problem $\dfrac{dx}{dt} = \sqrt{x^2 + 2t}, x(0) = 1$ with $h = 0.1$, we have

$$\begin{aligned}
\hat{X}_1 &= x_0 + hf(t_0, x_0) = 1.0 + 0.1 \times \sqrt{1^2 + 2 \times 0} = 1.100 \\
X_1 &= x_0 + \tfrac{1}{2}h\left(f(t_0, x_0) + f(t_1, \hat{X}_1)\right) \\
&= 1.0 + \tfrac{1}{2}0.1 \times \left(\sqrt{1^2 + 2 \times 0} + \sqrt{1.1^2 + 2 \times 0.1}\right) = 1.1094 \\
\hat{X}_2 &= X_1 + hf(t_1, X_1) = 1.1094 + 0.1 \times \sqrt{1.1094^2 + 2 \times 0.1} = 1.2290 \\
X_2 &= X_1 + \tfrac{1}{2}h\left(f(t_1, X_1) + f(t_2, \hat{X}_2)\right) \\
&= 1.1094 + \tfrac{1}{2}0.1 \times \left(\sqrt{1.1094^2 + 2 \times 0.1} + \sqrt{1.2290^2 + 2 \times 0.2}\right) = 1.2383 \\
X_3 &= X_2 + \tfrac{1}{12}h\left(23f(t_2, X_2) - 16f(t_1, X_1) + 5f(t_0, x_0)\right) \\
&= 1.2383 + \tfrac{1}{12}0.1 \times \Big(23\sqrt{1.2383^2 + 2 \times 0.2} - 16\sqrt{1.1094^2 + 2 \times 0.1} \\
&\quad + 5\sqrt{1.0^2 + 2 \times 0}\Big) = 1.3870
\end{aligned}$$

X_4 and X_5 are obtained as X_3. The computation is most efficiently set out as a table.

n	t_n	X_n	$f(t_n, X_n)$	$h(23f(t_n, X_n) - 16f(t_{n-1}, X_{n-1}) + 5f(t_{n-2}, X_{n-2}))/12$	X_{n+1}
0	0.0	1.0000	1.0000	(use predictor–corrector)	1.1094
0	0.1	1.1094	1.1961	(use predictor–corrector)	1.2383
2	0.2	1.2383	1.3905	0.1487	1.3870
3	0.3	1.3870	1.5886	0.1689	1.5559
4	0.4	1.5559	1.7947	0.1901	1.7460
5	0.5	1.7460			

Hence $X(0.5) = 1.7460$.

■ **10** The second-order predictor–corrector method is

$$\hat{X}_{n+1} = X_n + hf(t_n, X_n)$$
$$X_{n+1} = X_n + \tfrac{1}{2}h(f(t_n, X_n) + f(t_{n+1}, \hat{X}_{n+1}))$$

10(a) Applying this method to the problem $\dfrac{dx}{dt} = (2t + x)\sin 2t, x(0) = 0.5$ with $h = 0.05$, we have

$$\hat{X}_1 = x_0 + hf(t_0, x_0) = 0.5 + 0.05 \times (2 \times 0 + 0.5)\sin 0 = 0.5$$
$$X_1 = x_0 + \tfrac{1}{2}h(f(t_0, x_0) + f(t_1, X_1))$$
$$= 0.5 + \tfrac{1}{2}0.05 \times ((2 \times 0 + 0.5)\sin 0 + (2 \times 0.05 + 0.5)\sin(2 \times 0.05)) = 0.5015$$

X_2 to X_{10} are obtained as X_1. The computation is most efficiently set out as a table.

n	t_n	X_n	$f(t_n, X_n)$	$\hat{X}_{n+1}$	$f(t_{n+1}, \hat{X}_{n+1})$	X_{n+1}
0	0.00	0.5000	0.0000	0.5000	0.0599	0.5015
1	0.05	0.5015	0.0150	0.5045	0.1400	0.5065
2	0.10	0.5065	0.0497	0.5135	0.2404	0.5160
3	0.15	0.5160	0.1034	0.5281	0.3614	0.5311
4	0.20	0.5311	0.1752	0.5492	0.5030	0.5527
5	0.25	0.5527	0.2637	0.5780	0.6651	0.5820
6	0.30	0.5820	0.3670	0.6153	0.8474	0.6198
7	0.35	0.6198	0.4832	0.6623	1.0490	0.6673
8	0.40	0.6673	0.6098	0.7199	1.2689	0.7254
9	0.45	0.7254	0.7442	0.7890	1.5054	0.7948
10	0.50	0.7948				

Hence $X(0.5) = 0.7948$.

10(b) Applying this method to the problem $\dfrac{dx}{dt} = -\dfrac{1 + x}{\sin(t + 1)}, x(0) = -2$ with $h = 0.1$, we have

$$\hat{X}_1 = x_0 + hf(t_0, x_0) = -2 + 0.1 \times -\frac{1 - 2}{\sin(0 + 1)} = -1.8812$$
$$X_1 = x_0 + \tfrac{1}{2}h\left(f(t_0, x_0) + f(t_1, \hat{X}_1)\right)$$
$$= -2 + \tfrac{1}{2}0.1 \times \left(-\frac{1 - 2}{\sin(0 + 1)} - \frac{1 - 1.8812}{\sin(0.1 + 1)}\right) = -1.8911$$

X_2 to X_{10} are obtained as X_1. The computation is most efficiently set out as a table.

n	t_n	X_n	$f(t_n, X_n)$	$\hat{X}_{n+1}$	$f(t_{n+1}, \hat{X}_{n+1})$	X_{n+1}
0	0.0	−2.0000	−1.3072	−1.8812	0.9887	−1.8911
1	0.1	−1.8911	−1.2343	−1.7912	0.8488	−1.7987
2	0.2	−1.7987	−1.1802	−1.7130	0.7400	−1.7189
3	0.3	−1.7189	−1.1416	−1.6443	0.6538	−1.6489
4	0.4	−1.6489	−1.1162	−1.5830	0.5845	−1.5867
5	0.5	−1.5867	−1.1028	−1.5279	0.5281	−1.5309
6	0.6	−1.5309	−1.1005	−1.4778	0.4818	−1.4803
7	0.7	−1.4803	−1.1092	−1.4318	0.4434	−1.4339
8	0.8	−1.4339	−1.1295	−1.3893	0.4114	−1.3910
9	0.9	−1.3910	−1.1624	1.3497	0.3846	1.3511
10	1.0	−1.3511				

Hence $X(1.0) = -1.3511$.

■ **11** Taylor's theorem states that

$$f(t+h) = f(t) + h\frac{df}{dt}(t) + \frac{h^2}{2!}\frac{d^2f}{dt^2}(t) + \frac{h^3}{3!}\frac{d^3f}{dt^3}(t) + \frac{h^4}{4!}\frac{d^4f}{dt^4}(t) + K$$

Applying this to $\dfrac{dx}{dt}(t-h)$ and $\dfrac{dx}{dt}(t-2h)$ yields

$$\frac{dx}{dt}(t-h) = \frac{dx}{dt}(t) - h\frac{d^2x}{dt^2}(t) + \frac{h^2}{2!}\frac{d^3x}{dt^3}(t) + O(h^3)$$
$$\frac{dx}{dt}(t-2h) = \frac{dx}{dt}(t) - 2h\frac{d^2x}{dt^2}(t) + \frac{4h^2}{2!}\frac{d^3x}{dt^3}(t) + O(h^3)$$

Multiplying the first equation by 2 and subtracting the second yields

$$2\frac{dx}{dt}(t-h) - \frac{dx}{dt}(t-2h) = \frac{dx}{dt}(t) - h^2\frac{d^3x}{dt^3}(t) + O(h^3)$$
$$\text{that is, } h^2\frac{d^3x}{dt^3}(t) = -2\frac{dx}{dt}(t-h) + \frac{dx}{dt}(t-2h) + \frac{dx}{dt}(t) + O(h^3)$$

Multiplying the first equation by 4 and subtracting the second yields

$$4\frac{dx}{dt}(t-h) - \frac{dx}{dt}(t-2h) = 3\frac{dx}{dt}(t) - 2h\frac{d^2x}{dt^2}(t) + O(h^3)$$
$$\text{that is, } 2h\frac{d^2x}{dt^2}(t) = -4\frac{dx}{dt}(t-h) + \frac{dx}{dt}(t-2h) + 3\frac{dx}{dt}(t) + O(h^3)$$

Now Taylor's theorem yields

$$x(t+h) = x(t) + h\frac{dx}{dt}(t) + \frac{h^2}{2!}\frac{d^2x}{dt^2} + \frac{h^3}{3!}\frac{d^3x}{dt^3}(t) + O(h^3)$$

Hence, substituting for $h\dfrac{d^2x}{dt^2}(t)$ and $h^2\dfrac{d^3x}{dt^3}(t)$ yields

$$\begin{aligned}x(t+h) &= x(t) + h\frac{dx}{dt}(t) + \frac{h}{4}\left(-4\frac{dx}{dt}(t-h) + \frac{dx}{dt}(t-2h) + 3\frac{dx}{dt}(t) + O(h^3)\right)\\ &\quad + \frac{h}{6}\left(-2\frac{dx}{dt}(t-h) + \frac{dx}{dt}(t-2h) + \frac{dx}{dt}(t) + O(h^3)\right) + O(h^3)\\ &= x(t) + \frac{h}{12}\left(23\frac{dx}{dt}(t) - 16\frac{dx}{dt}(t-h) + 5\frac{dx}{dt}(t-2h)\right) + O(h^3)\end{aligned}$$

■ **12** Taylor's theorem states that

$$f(t+h) = f(t) + h\frac{df}{dt}(t) + \frac{h^2}{2!}\frac{d^2f}{dt^2}(t) + \frac{h^3}{3!}\frac{d^3f}{dt^3}(t) + \frac{h^4}{4!}\frac{d^4f}{dt^4}(t) + K$$

Applying this to $\dfrac{dx}{dt}(t+h)$ and $\dfrac{dx}{dt}(t-h)$ yields

$$\begin{aligned}\frac{dx}{dt}(t+h) &= \frac{dx}{dt}(t) + h\frac{d^2x}{dt^2}(t) + \frac{h^2}{2!}\frac{d^3x}{dt^3}(t) + O(h^3)\\ \frac{dx}{dt}(t-h) &= \frac{dx}{dt}(t) - h\frac{d^2x}{dt^2}(t) + \frac{h^2}{2!}\frac{d^3x}{dt^3}(t) + O(h^3)\end{aligned}$$

Summing the two equations yields

$$\begin{aligned}\frac{dx}{dt}(t+h) + \frac{dx}{dt}(t-h) &= 2\frac{dx}{dt}(t) + h^2\frac{d^3x}{dt^3}(t) + O(h^3)\\ \text{that is, } h^2\frac{d^3x}{dt^3}(t) &= \frac{dx}{dt}(t+h) + \frac{dx}{dt}(t-h) - 2\frac{dx}{dt}(t) + O(h^3)\end{aligned}$$

Subtracting the second from the first yields

$$\begin{aligned}\frac{dx}{dt}(t+h) - \frac{dx}{dt}(t-h) &= 2h\frac{d^2x}{dt^2}(t) + O(h^3)\\ \text{that is, } 2h\frac{d^2x}{dt^2}(t) &= \frac{dx}{dt}(t+h) - \frac{dx}{dt}(t-h) + O(h^3)\end{aligned}$$

Now Taylor's theorem gives

$$x(t+h) = x(t) + h\frac{dx}{dt}(t) + \frac{h^2}{2!}\frac{d^2x}{dt^2} + \frac{h^3}{3!}\frac{d^3x}{dt^3}(t) + O(h^4)$$

Hence, substituting for $h\dfrac{d^2x}{dt^2}(t)$ and $h^2\dfrac{d^3x}{dt^3}(t)$, we have

$$\begin{aligned} x(t+h) &= x(t) + h\frac{dx}{dt}(t) + \frac{h}{4}\left(\frac{dx}{dt}(t+h) + \frac{dx}{dt}(t-h) + O(h^3)\right) \\ &\quad + \frac{h}{6}\left(\frac{dx}{dt}(t+h) + \frac{dx}{dt}(t-h) - 2\frac{dx}{dt}(t) + O(h^3)\right) + O(h^4) \\ &= x(t) + \frac{h}{12}\left(5\frac{dx}{dt}(t+h) + 8\frac{dx}{dt}(t) - \frac{dx}{dt}(t-h)\right) + O(h^4) \end{aligned}$$

$$\text{that is, } x_{n+1} = x_n + \frac{h}{12}\left(5\frac{dx}{dt_{n+1}} + 8\frac{dx}{dt_n} - \frac{dx}{dt_{n-1}}\right) + O(h^4)$$

■ **13** Taylor's theorem states that

$$f(t+h) = f(t) + h\frac{df}{dt}(t) + \frac{h^2}{2!}\frac{d^2f}{dt^2}(t) + \frac{h^3}{3!}\frac{d^3f}{dt^3}(t) + \frac{h^4}{4!}\frac{d^4f}{dt^4}(t) + K$$

Applying this to $x(t-h)$ and $\dfrac{dx}{dt}(t-h)$ yields

$$\begin{aligned} x(t-h) &= x(t) - h\frac{dx}{dt}(t) + \frac{h^2}{2!}\frac{d^2x}{dt^2}(t) - \frac{h^3}{3!}\frac{d^3x}{dt^3}(t) + O(h^4) \\ \frac{dx}{dt}(t-h) &= \frac{dx}{dt}(t) - h\frac{d^2x}{dt^2}(t) + \frac{h^2}{2!}\frac{d^3x}{dt^3}(t) + O(h^3) \end{aligned}$$

Multiplying the first equation by 2, the second equation by h and adding yields

$$\begin{aligned} 2x(t-h) + h\frac{dx}{dt}(t-h) &= 2x(t) - h\frac{dx}{dt}(t) + \frac{h^3}{6}\frac{d^3x}{dt^3}(t) + O(h^4) \\ \text{that is, } \frac{h^3}{6}\frac{d^3x}{dt^3}(t) &= 2x(t-h) + h\frac{dx}{dt}(t-h) - 2x(t) + h\frac{dx}{dt}(t) + O(h^4) \end{aligned}$$

Multiplying the first equation by 3, the second equation by h and adding yields

$$\begin{aligned} 3x(t-h) + h\frac{dx}{dt}(t-h) &= 3x(t) - 2h\frac{dx}{dt}(t) + \frac{h^2}{2}\frac{d^2x}{dt^2}(t) + O(h^4) \\ \text{that is, } \frac{h^2}{2}\frac{d^2x}{dt^2}(t) &= 3x(t-h) + h\frac{dx}{dt}(t-h) - 3x(t) + 2h\frac{dx}{dt}(t) + O(h^4) \end{aligned}$$

Now Taylor's theorem gives

$$x(t+h) = x(t) + h\frac{dx}{dt}(t) + \frac{h^2}{2!}\frac{d^2x}{dt^2} + \frac{h^3}{3!}\frac{d^3x}{dt^3}(t) + O(h^4)$$

Hence, substituting for $h^2\dfrac{d^2x}{dt^2}(t)$ and $h^3\dfrac{d^3x}{dt^3}(t)$, we have

$$\begin{aligned} x(t+h) &= x(t) + h\frac{dx}{dt}(t) + \left(3x(t-h) + h\frac{dx}{dt}(t-h) - 3x(t) + 2h\frac{dx}{dt}(t) + O(h^4)\right) \\ &\quad + \left(2x(t-h) + h\frac{dx}{dt}(t-h) - 2x(t) + h\frac{dx}{dt}(t) + O(h^4)\right) + O(h^4) \\ &= -4x(t) + 5x(t-h) + 4h\frac{dx}{dt}(t) + 2h\frac{dx}{dt}(t-h) + O(h^4) \end{aligned}$$

$$\text{that is, } x_{n+1} = -4x_n + 5x_{n-1} + 2h\left(2\frac{dx}{dt_n} + \frac{dx}{dt_{n-1}}\right) + O(h^4)$$

This gives rise to the approximate scheme $X_{n+1} = -4X_n + 5X_{n-1} + 2h(2F_n + F_{n-1})$ which may equally be written as $X_{n+1} = 5[X_{n-1} + 2hF_n] - 4[X_n + \frac{1}{2}h(3F_n - F_{n-1})]$; in other words, this scheme gives an approximation for X_{n+1} which is 5 times the central difference approximation minus 4 times the second-order Adams–Bashforth approximation. Because of the inclusion of the central difference approximation, this scheme will be unstable whenever the central difference scheme is.

■ **14** The predictor–corrector scheme specified is

$$\begin{aligned} \hat{X}_{n+1} &= X_n + \tfrac{1}{2}h(3F_n - F_{n-1}) = X_n + \tfrac{1}{2}h(3f(t_n, X_n) - f(t_{n-1}, X_{n-1})) \\ X_{n+1} &= X_n + \tfrac{1}{12}h(5F_{n+1} + 8F_n - F_{n-1}) = X_n + \tfrac{1}{12}h(5f(t_{n+1}, \hat{X}_{n+1}) \\ &\quad + 8f(t_n, X_n) - f(t_{n-1}, X_{n-1})) \end{aligned}$$

This is obviously not self-starting and requires that one initial step is taken using a self-starting method. For the problem $\dfrac{dx}{dt} = x^2 + t^2, x(0.3) = 0.1$ with a step size

of $h = 0.05$, using the fourth-order Runge–Kutta method for the initial step we obtain

$$\begin{aligned}
c_1 &= hf(t_0, x_0) = h(x_0^2 + t_0^2) = 0.05 \times (0.1^2 + 0.3^2) = 0.0050 \\
c_2 &= hf(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}c_1) = 0.05 \times \left((0.1 + \tfrac{1}{2}0.0050)^2 + (0.3 + \tfrac{1}{2}0.05)^2\right) \\
&= 0.0058 \\
c_3 &= hf(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}c_2) = 0.05 \times \left((0.1 + \tfrac{1}{2}0.0058)^2 + (0.3 + \tfrac{1}{2}0.05)^2\right) \\
&= 0.0058 \\
c_4 &= hf(t_0 + h, x_0 + c_3) = 0.05 \times \left((0.1 + 0.0058)^2 + (0.3 + 0.05)^2\right) \\
&= 0.0067 \\
X_1 &= x_0 + \tfrac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4) = 0.1 + \tfrac{1}{6}(0.0050 + 2 \times 0.0058 \\
&\quad + 2 \times 0.0058 + 0.0067) = 0.1058
\end{aligned}$$

Now we can say

$$\begin{aligned}
\hat{X}_2 &= X_1 + \tfrac{1}{2}h\,(3f(t_1, X_1) - f(t_0, x_0)) \\
&= 0.1058 + \tfrac{1}{2}0.05 \times \left(3(0.1058^2 + 0.35^2) - (0.1^2 + 0.3^2)\right) = 0.1133 \\
X_2 &= X_1 + \tfrac{1}{12}h\left(5f(t_2, \hat{X}_2) + 8f(t_1, X_1) - f(t_0, x_0)\right) \\
&= 0.1058 + \tfrac{1}{12}0.05 \times \left(5(0.1133^2 + 0.4^2) + 8(0.1058^2 + 0.35^2) - (0.1^2 + 0.3^2)\right) \\
&= 0.1134
\end{aligned}$$

Computing X_3 and X_4 in a similar manner and setting the computation out in tabular fashion we obtain:

n	t_n	X_n	$f(t_n, X_n)$	$\hat{X}_{n+1}$	$f(t_{n+1}, \hat{X}_{n+1})$	X_{n+1}
0	0.30	0.1000	0.1000	(done by Runge–Kutta method)		0.1058
1	0.35	0.1058	0.1337	0.1133	0.1728	0.1134
2	0.40	0.1134	0.1729	0.1230	0.2176	0.1231
3	0.45	0.1231	0.2177	0.1351	0.2683	0.1352
4	0.50	0.1352				

Hence $X(0.5) = 0.1352$.

■ **15** The fourth-order Runge–Kutta method for the solution of the differential equation $\dfrac{dx}{dt} = f(t,x)$ using a stepsize of h is given by

$$
\begin{aligned}
c_1 &= hf(t_n, X_n)\\
c_2 &= hf(t_n + \tfrac{1}{2}h, X_n + \tfrac{1}{2}c_1)\\
c_3 &= hf(t_n + \tfrac{1}{2}h, X_n + \tfrac{1}{2}c_2)\\
c_4 &= hf(t_n + h, X_n + c_3)\\
X_{n+1} &= X_n + \tfrac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4)
\end{aligned}
$$

15(a) To solve the equation $\dfrac{dx}{dt} = x + t + xt, x(0) = 1,$ using a stepsize of $h = 0.15$, we write

$$
\begin{aligned}
c_1 &= hf(t_0, x_0) = 0.15 \times (1 + 0 + 1 \times 0) = 0.1500\\
c_2 &= hf(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}c_1)\\
&= 0.15 \times ((1 + \tfrac{1}{2}0.1500) + (0 + \tfrac{1}{2}0.15) + (1 + \tfrac{1}{2}0.1500) \times (0 + \tfrac{1}{2}0.15))\\
&= 0.1846\\
c_3 &= hf(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}c_2)\\
&= 0.15 \times ((1 + \tfrac{1}{2}0.1846) + (0 + \tfrac{1}{2}0.15) + (1 + \tfrac{1}{2}0.1846) \times (0 + \tfrac{1}{2}0.15))\\
&= 0.1874\\
c_4 &= hf(t_0 + h, x_0 + c_3) = 0.15 \times ((1 + 0.1874) + (0 + 0.15) + (1 + 0.1874)\\
&\quad \times (0 + 0.15)) = 0.2273\\
X_1 &= x_0 + \tfrac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4) = 1 + \tfrac{1}{6}(0.1500 + 2 \times 0.1846\\
&\quad + 2 \times 0.1874 + 0.2273) = 1.1869
\end{aligned}
$$

X_2, X_3, X_4 and X_5 may be computed in a similar manner. Setting the computation out in tabular fashion we obtain:

n	t_n	X_n	c_1	c_2	c_3	c_4	X_{n+1}
0	0.00	1.0000	0.1500	0.1846	0.1874	0.2273	1.1869
1	0.15	1.1869	0.2272	0.2727	0.2769	0.3304	1.4630
2	0.30	1.4630	0.3303	0.3921	0.3984	0.4724	1.8603
3	0.45	1.8603	0.4721	0.5583	0.5681	0.6728	2.4266
4	0.60	2.4266	0.6724	0.7954	0.8109	0.9623	3.2345
5	0.75	3.2345					

Hence $X(0.75) = 3.2345$.

15(b) To solve the equation $\dfrac{dx}{dt} = \dfrac{1}{x+t}, x(1) = 2,$ using a stepsize of $h = 0.1$ we write

$$c_1 = hf(t_0, x_0) = 0.1 \times \frac{1}{2+1} = 0.0333$$

$$c_2 = hf(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}c_1) = 0.1 \times \frac{1}{(2 + \frac{1}{2}0.0333) + (1 + \frac{1}{2}0.1)} = 0.0326$$

$$c_3 = hf(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}c_2) = 0.1 \times \frac{1}{(2 + \frac{1}{2}0.0326) + (1 + \frac{1}{2}0.1)} = 0.0326$$

$$c_4 = hf(t_0 + h, x_0 + c_3) = 0.1 \times \frac{1}{(2 + 0.0326) + (1 + 0.1)} = 0.0319$$

$$X_1 = x_0 + \tfrac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4) = 2 + \tfrac{1}{6}(0.0333 + 2 \times 0.0326 + 2 \times 0.0326 + 0.0319) = 2.0326$$

$X_2, X_3, \ldots, X_{10}$ may be computed in a similar manner. Setting the computation out in tabular fashion we obtain:

n	t_n	X_n	c_1	c_2	c_3	c_4	X_{n+1}
0	1.0	2.0000	0.0333	0.0326	0.0326	0.0319	2.0326
1	1.1	2.0326	0.0319	0.0313	0.0313	0.0306	2.0639
2	1.2	2.0639	0.0306	0.0300	0.0300	0.0295	2.0939
3	1.3	2.0939	0.0295	0.0289	0.0289	0.0284	2.1228
4	1.4	2.1228	0.0284	0.0279	0.0279	0.0274	2.1507
5	1.5	2.1507	0.0274	0.0269	0.0269	0.0265	2.1777
6	1.6	2.1777	0.0265	0.0260	0.0260	0.0256	2.2037
7	1.7	2.2037	0.0256	0.0252	0.0252	0.0248	2.2289
8	1.8	2.2289	0.0248	0.0244	0.0244	0.0241	2.2534
9	1.9	2.2534	0.0241	0.0237	0.0237	0.0234	2.2771
10	2.0	2.2771					

Hence $X(2) = 2.2771$.

■ **16** In this exercise the differential equation problem $\frac{dx}{dt} = x^2 + t^{\frac{3}{2}}, x(0) = -1$, is solved using a variety of methods.

■ **16(a)** Figure 2.3 shows a pseudocode algorithm for solving the equation using the second-order Adams–Bashforth method with a second-order predictor–corrector starting step and Figure 2.4 shows a Pascal program derived from it.

```
procedure deriv (t, x → f)
    f ← x*x+sqrt (t)*t
endprocedure

t_start ← 0
x_start ← −1
t_end ← 2
write (vdu, "Enter step size")
read (keyboard, h)
write (printer, t_start , x_start)
deriv (t_start, x_start → f)
x_hat ← x_start + h*f
deriv(t_start + h, x_hat → f_hat)
t ← t_start + h
x ← x_start + h*(f + f_hat)/2
write (printer, t,x)
f_n_minus_one ← f
repeat
    deriv (t, x → f)
    t ← t + h
    x ← x + h*(3*f − f_n_minus_one)/2
    f_n_minus_one ← f
    write (printer, t,x)
until t >= t_end
```

Figure 2.3: Pseudocode algorithm for Exercise 16(a)

```
var x_start, x_hat, x,t_start, t_end; t:real;
    h,f,f_hat, f_n_minus_one:real;

procedure deriv (t,x:real;var f:real);
begin
   f := x*x+sqrt(t)*t
end;

begin
   t_start := 0;
x_start := -1;
   t_end := 2;
   write('Enter step size ==>');
   readln(h);
   writeln(t_start:10:3, x_start:10:6);
   deriv(t_start, x_start, f);
   x_hat := x_start + h*f;
   deriv(t_start + h, x_hat, f_hat);
   t :=t_start + h;
   x :=x_start + h *(f+f_hat)/2;
   writeln(t:10:3, x:10:6);
   f_n_minus_one := f;
   repeat
      deriv(t,x,f);
      t := t + h;
      x := x + h*(3*f-f_n_minus_one)/2;
      f_n_minus_one := f;
      writeln(t:10:3, x:10:6);
   until t >= t_end;
end.
```

Figure 2.4: Pascal program for Exercise 16(a)

Using this program with $h = 0.2$ gives $X(2) = 2.242408$ and, with $h = 0.1, X(2) = 2.613104$. The method of Richardson extrapolation given in Section 2.3.6 gives the estimated error in the second of these as $(2.613104 - 2.242408)/3 = 0.123565$. For 3 decimal place accuracy in the final estimate we need error ≤ 0.0005; in other words, the error must be reduced by a factor of $0.123565/0.0005 = 247.13$. Since Adams–Bashforth is a second-order method, the required step length will be $0.1/\sqrt{247.13} = 0.0064$. Rounding this down to a suitable size suggests that a step size of $h = 0.005$ will give a solution accurate to 3 decimal places. In fact the program of Figure 2.4 yields, with $h = 0.005, X(2) = 2.897195$. With

$h = 0.0025$ it gives $X(2) = 2.898175$. Richardson extrapolation predicts the error in the $h = 0.0025$ solution as 0.000327 and therefore that in the $h = 0.005$ as 0.001308. The required accuracy was therefore not achieved using $h = 0.005$ but was achieved with $h = 0.0025$.

16(b) Figure 2.5 shows a pseudocode algorithm for solving the equation using the second-order predictor–corrector method and Figure 2.6 shows a Pascal program derived from it.

```
procedure deriv (t, x → f)
    f ← x*x+sqrt (t)*t
endprocedure

t_start ← 0
x_start ← −1
t_end ← 2
write (vdu, "Enter step size")
read (keyboard, h)
write (printer, t_start , x_start)
t ← t_start
x ← x_start
repeat
    deriv (t, x → f)
    x_hat ← x + h*f
    derive (t + h, x_hat → f_hat)
    t ← t + h
    x ← x + h*(f + f_hat)/2
    write (printer, t,x)
until t >= t_end
```

Figure 2.5: Pseudcode algorithm for Exercise 2.16(b)

Using this program with $h = 0.2$ gives $X(2) = 2.788158$ and, with $h = 0.1, X(2) = 2.863456$. The method of Richardson extrapolation given in Section 2.3.6 gives the estimated error in the second of these as $(2.863456 - 2.788158)/3 = 0.025099$. For 3 decimal place accuracy in the final estimate we need error ≤ 0.0005; in other words, the error must be reduced by a factor of $0.025099/0.0005 = 50.20$. Since the second-order predictor–corrector method is being used, the required step size will be $0.1/\sqrt{50.20} = 0.014$. Rounding this down to a suitable size suggests that a step size of $h = 0.0125$ will give a solution accurate to 3 decimal places. In fact the program of Figure 2.6 yields, with $h = 0.0125, X(2) = 2.897876$. With

$h = 0.00625$ it gives $X(2) = 2.898349$. Richardson extrapolation predicts the error in the $h = 0.00625$ solution as 0.000158 and therefore that in the $h = 0.0125$ as 0.000632. The required accuracy was therefore not quite achieved using $h = 0.0125$ but was achieved with $h = 0.00625$.

```
var x_start,x_hat,x,t_start,t_end, t:real;
      h,f,f_hat:real;

procedure deriv(t,x:real;var f:real);
    begin
      f := x*x+sqrt(t)*t
    end;

begin
    t_start := 0;
    x_start := -1;
    t_end := 2;
    write('Enter step size ==> ');
    readln(h);
    writeln(t_start:10:3,x_start:10:6);
    t := t_start;
    x := x_start;
    repeat
       deriv(t,x,f);
       x_hat := x + h*f;
       deriv(t + h,x_hat,f_hat);
       t := t + h;
       x := x + h*(f + f_hat)/2;
       writeln (t:10:3,x:10:6);
    until t >= t_end;
end.
```

Figure 2.6: Pascal program for Exercise 16(b)

16(c) Figure 2.7 shows a pseudocode algorithm for solving the equation using the fourth-order Runge–Kutta method and Figure 2.8 shows a Pascal program derived from it. Using this program with $h = 0.4$ gives $X(2) = 2.884046$ and, with $h = 0.2, X(2) = 2.897402$. The method of Richardson extrapolation given

in Section 2.3.6 gives the estimated error in the second of these as $(2.897402 - 2.884046)/15 = 0.000890$. For 5 decimal place accuracy in the final estimate we need error ≤ 0.000005; in other words, the error must be reduced by a factor of $0.000890/0.000005 = 178$. Since Runge–Kutta is a fourth-order method the required step size will be $0.2/(178)^{1/4} = 0.0547$. Rounding this down to a suitable size suggests that a step size of $h = 0.05$ will give a solution accurate to 5 decimal places. In fact the program of Figure 2.8 yields, with $h = 0.05, X(2) = 2.89850975$. With $h = 0.025$ it gives $X(2) = 2.89850824$. Richardson extrapolation predicts the error

```
procedure deriv (t,x → f)
    f → x*x + sqrt(t)*t
endprocedure

t_start ← 0
x_start ← −1
t_end ← 2
write(vdu, "Enter step size")
read(keyboard, h)
write (printer,t_start,x_start)
t ← t_start
x ← x_start
repeat
    deriv(t, x, → f)
    c1 ← h*f
    deriv(t + h/2, x + c1/2 → f)
    c2 ← h*f
    deriv(t + h/2, x + c2/2 → f)
    c3 ← h*f
    deriv(t + h, x + c3 → f)
    c4 ← h*f
    t ← t + h
    x ← x + (c1 + 2*c2 + 2*c3 + c4)/6
    write(printer,t,x)
until t >= t_end
```

Figure 2.7: Pseudocode algorithm for Exercise 16(c)

```
{Program for exercise 2.16c}
var x_start,x,t_start,t_end,t:real;
      h,f,c1,c2,c3,c4:real;

procedure deriv (t,x:real;var f:real);
    begin
       f := x*x+sqrt(t)*t
    end;

begin
   t_start := 0;
   x_start := -1;
   t_end := 2;
   write('Enter step size ==> ');
   readln(h);
   writeln(t_start:10:3,x_start:10:6);
   t := t_start;
   x := x_start;
   repeat
      deriv(t,x,f);
      c1 := h*f;
      deriv(t + h/2,x + c1/2, f);
      c2 := h*f;
      deriv(t + h/2,x + c2/2, f);
      c3 := h*f;
      deriv(t + h,x + c3, f);
      c4 := h*f;
      t := t + h;
      x := x + (c1 + 2*c2 + 2*c3 + c4)/6;
      writeln(t:10:3,x:10:6);
   until t >= t_end;
end.
```

Figure 2.8: Pascal program for Exercise 16(c)

in the $h = 0.025$ solution as 0.000000101 and therefore that in the $h = 0.05$ as 0.00000161. The required accuracy was therefore achieved using $h = 0.05$.

■ **17** The pseudocode algorithm shown in Figure 2.7 and the Pascal program in Figure 2.8 may easily be modified to solve this problem. With a step size of $h = 0.5$ the estimate $X(3) = 1.466489$ is obtained, whilst with a step size of $h = 0.25$, the estimate is $X(3) = 1.466476$. Richardson extrapolation suggests that the step

size of $h = 0.25$ gives an error of 0.00000087 which is comfortably within the 0.000005 range, which 5 decimal place accuracy requires. Hence, $X(3) = 1.46648$ to 5 decimal places. In fact, of course, the analytic solution of the problem is $\dfrac{e}{e^{1-t} + e - 1}$ and so $x(3) = 1.466474$.

Exercises 2.4.3

■ **18** In each part of this question, the technique is to introduce new variables to represent each derivative of the dependent variable up to one less than the order of the equation. This can be done by inspection.

18(a)

$$\frac{dx}{dt} = v, \qquad x(0) = 1$$
$$\frac{dv}{dt} = 4xt - 6(x^2 - t)v, \qquad v(0) = 2$$

18(b)

$$\frac{dx}{dt} = v, \qquad x(1) = 2$$
$$\frac{dv}{dt} = -4(x^2 - t^2), \qquad v(1) = 0.5$$

18(c)

$$\frac{dx}{dt} = v, \qquad x(0) = 0$$
$$\frac{dv}{dt} = -\sin v - 4x, \qquad v(0) = 0$$

18(d)

$$\frac{dx}{dt} = v, \qquad x(0) = 1$$
$$\frac{dv}{dt} = w, \qquad v(0) = 2$$
$$\frac{dw}{dt} = e^{2t} + x^2 t - 6e^t v - tw, \qquad w(0) = 0$$

18(e)

$$\frac{dx}{dt} = v, \qquad x(1) = 1$$
$$\frac{dv}{dt} = w, \qquad v(1) = 0$$
$$\frac{dw}{dt} = \sin t - tw - x^2, \qquad w(1) = -2$$

18(f)

$$\begin{aligned} \frac{dx}{dt} &= v, & x(2) &= 0 \\ \frac{dv}{dt} &= w, & v(2) &= 0 \\ \frac{dw}{dt} &= (tw + t^2x^2)^2, & w(2) &= 2 \end{aligned}$$

18(g)

$$\begin{aligned} \frac{dx}{dt} &= v, & x(0) &= 0 \\ \frac{dv}{dt} &= w, & v(0) &= 0 \\ \frac{dw}{dt} &= u, & w(0) &= 4 \\ \frac{du}{dt} &= \ln t - x^2 - xw, & u(0) &= -3 \end{aligned}$$

18(h)

$$\begin{aligned} \frac{dx}{dt} &= v, & x(0) &= a \\ \frac{dv}{dt} &= w, & v(0) &= 0 \\ \frac{dw}{dt} &= u, & w(0) &= b \\ \frac{du}{dt} &= t^2 + 4t - 5 + \sqrt{xt} - v - (v-1)tu, & u(0) &= 0 \end{aligned}$$

■ **19** First, we recast the equation as a pair of coupled first-order equations in the form

$$\begin{aligned} \frac{dx}{dt} &= f_1(t, x, v), & x(0) &= x_0 \\ \frac{dv}{dt} &= f_2(t, x, v), & v(0) &= v_0 \end{aligned}$$

This yields

$$\begin{aligned} \frac{dx}{dt} &= v, & x(0) &= 0 \\ \frac{dv}{dt} &= \sin t - x - x^2 v, & v(0) &= 1 \end{aligned}$$

Now, applying Euler's method to the two equations we have

$$\begin{aligned}
& X_1 = x_0 + h_1 f(t_0, x_0, v_0) && V_1 = v_0 + h f_2(t_0, x_0, v_0) \\
\text{that is, } & X_1 = 0 + 0.1 \times 1 = 0.10000 && V_1 = 1.00000 = 1 + 0.1 \times (\sin 0 - 0 - 0^2 \times 1) \\
& X_2 = X_1 + h f_1(t_1, X_1, V_1) && V_2 = V_1 + h f_2(t_1, X_1, V_1) \\
\text{that is, } & X_2 = 0.10000 + 0.1 \times 1.00000 && V_2 = 1 + 0.1 \times (\sin 0.10000 \\
& \quad = 0.20000 && \qquad - 0.10000 - 0.10000^2 \times 1.00000) \\
& && \quad = 0.99898 \\
& X_3 = X_2 + h f_1(t_2, X_2, V_2) \\
\text{that is, } & X_2 = 0.20000 + 0.1 \times 0.99898 \\
& \quad = 0.29990
\end{aligned}$$

■ **20** The second-order Adams–Bashforth method applied to a pair of coupled equations is

$$\begin{aligned}
X_{n+1} &= X_n + \tfrac{1}{2}h\,(3f_1(t_n, X_n, Y_n) - f_1(t_{n-1}, X_{n-1}, Y_{n-1})) \\
Y_{n+1} &= Y_n + \tfrac{1}{2}h\,(3f_2(t_n, X_n, Y_n) - f_2(t_{n-1}, X_{n-1}, Y_{n-1}))
\end{aligned}$$

First, we recast the differential equation as a pair of coupled first-order equations.

$$\begin{aligned}
\frac{dx}{dt} &= v, && x(0) = 0 \\
\frac{dv}{dt} &= \sin t - x - x^2 v, && v(0) = 1
\end{aligned}$$

Now, since the Adams–Bashforth method is a two-step process, we need to start the computation with another method. We use the second-order predictor–corrector. This has the form

$$\begin{aligned}
\hat{X}_{n+1} &= X_n + h f_1(t_n, X_n, Y_n) \hat{Y}_{n+1} = Y_n + h f_2(t_n, X_n, Y_n) \\
X_{n+1} &= X_n + \frac{1}{2}h\left(f_1(t_{n+1}, \hat{X}_{n+1}, \hat{Y}_{n+1}) + f_1(t_n, X_n, Y_n)\right) \\
Y_{n+1} &= Y_n + \frac{1}{2}h\left(f_2(t_{n+1}, \hat{X}_{n+1}, \hat{Y}_{n+1}) + f_2(t_n, X_n, Y_n)\right)
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\hat{X}_1 = 0 + 0.1 \times 1 = 0.10000 \hat{V}_1 = 1 + 0.1 \times (0 - 0 - 0^2 \times 1)1 = 1.00000 \\
&X_1 = 0 + 0.05 \times (1 + 1) = 0.10000 V_1 = 1 + 0.05 \times (-0.01017 + 0) = 0.99949
\end{aligned}$$

Now, continuing using the Adams–Bashforth method we have

$$X_2 = X_1 + \tfrac{1}{2}h\,(3f_1(t_1, X_1, Y_1) - f_1(t_0, x_0, v_0))$$
$$V_2 = V_1 + \tfrac{1}{2}h\,(3f_2(t_1, X_1, V_1) - f_2(t_0, x_0, v_0))$$

that is,
$$X_2 = 0.10000 + 0.05 \times (3 \times 0.99949 - 1) = 0.19992$$
$$V_2 = 0.99949 + 0.05 \times (3 \times -0.01016 - 0) = 0.99797$$
$$X_3 = X_2 + \tfrac{1}{2}h\,(3f_1(t_2, X_2, V_2) - f_1(t_1, X_1, V_1))$$

that is,
$$X_3 = 0.19992 + 0.05 \times (3 \times 0.99797 - 0.99949) = 0.29964$$

■ **21** First, we formulate the problem as a set of 3 coupled first-order differential equations

$$\frac{dx}{dt} = u, \qquad x(0.5) = -1$$
$$\frac{du}{dt} = v, \qquad u(0.5) = 1$$
$$\frac{dv}{dt} = x^2 - v(x - t) - u^2, \qquad v(0.5) = 2$$

We can then solve these by the predictor–corrector method. Notice that we need to compute the predicted values for all three variables before computing the corrected values for any of them.

$$\hat{X}_1 = x_0 + hf_1(t_0, x_0, u_0, v_0) = -1 + 0.05(1) = -0.95000$$
$$\hat{U}_1 = u_0 + hf_2(t_0, x_0, u_0, v_0) = 1 + 0.05(2) = 1.10000$$
$$\hat{V}_1 = v_0 + hf_3(t_0, x_0, u_0, v_0) = 2 + 0.05((-1)^2 - 2(-1 - 0.5) - 1) = 2.15000$$

$$X_1 = x_0 + \tfrac{1}{2}h\left(f_1\left(t_1, \hat{X}_1, \hat{U}_1, \hat{V}_1\right) + f_1(t_0, x_0, u_0, v_0)\right)$$
$$= -1 + 0.025 \times (1.10000 + 1.00000) = -0.94750$$
$$U_1 = u_0 + \tfrac{1}{2}h\left(f_2\left(t_1, \hat{X}_1, \hat{U}_1, \hat{V}_1\right) + f_2(t_0, x_0, u_0, v_0)\right)$$
$$= 1 + 0.025 \times (2.15000 + 2.00000) = 1.10375$$
$$V_1 = v_0 + \tfrac{1}{2}h\left(f_3\left(t_1, \hat{X}_1, \hat{U}_1, \hat{V}_1\right) + f_3(t_0, x_0, u_0, v_0)\right)$$
$$= 2 + 0.025 \times (2.91750 + 3.00000) = 2.14794$$

Continuing the process we obtain

$$\hat{X}_2 = -0.94750 + 0.05 \times 1.10375 = -0.89231$$
$$\hat{U}_2 = 1.10375 + 0.05 \times 2.14794 = 1.21115$$
$$\hat{V}_2 = 2.14794 + 0.05 \times 2.89603 = 2.29274$$

$$X_2 = -0.94750 + 0.025 \times (1.21115 + 1.10375) = -0.88963$$
$$U_2 = 1.10375 + 0.025 \times (2.29274 + 2.14794) = 1.21477$$
$$V_2 = 2.14794 + 0.025 \times (2.75083 + 2.89603) = 2.28911$$

$$\hat{X}_3 = -0.88963 + 0.05 \times 1.21477 = -0.82889$$
$$\hat{U}_3 = 1.21477 + 0.05 \times 2.28911 = 1.32923$$
$$\hat{V}_3 = 2.28911 + 0.05 \times 2.72570 = 2.42539$$
$$X_3 = -0.88963 + 0.025 \times (1.32923 + 1.21477) = -0.82603$$

■ **22** The first step in solving this problem is to convert the problem to a pair of coupled first-order differential equations

$$\frac{dx}{dt} = v, \qquad x(0) = 0$$
$$\frac{dv}{dt} = \sin t - x^2 v - x, \qquad v(0) = 1$$

A pseudocode algorithm to compute the value of $X(1.6)$ is shown in Figure 2.9. Using a program derived from this algorithm with a step size $h = 0.4$ gives $X(1.6) = 1.220254$ and, with a step size $h = 0.2$, gives $X(1.6) = 1.220055$. The method of Richardson extrapolation, given in Section 2.3.6, is equally applicable to problems such as this one involving coupled equations. Since the Runge–Kutta method has a local error of $O(h)^5$ the global error will be $O(h^4)$. The method therefore gives the estimated error in the second value of $X(1.6)$ as $(1.220254 - 1.220055)/15 = 0.000013$. For 6 decimal place accuracy in the final estimate we need error ≤ 0.0000005; in other words, the error must be reduced by a factor of $0.000013/0.0000005 = 26$. Since Runge–Kutta is a fourth-order method the required step length will be $0.2/\sqrt[4]{26} = 0.088$. Rounding this down to a suitable size suggests that a step size of $h = 0.08$ will give a solution accurate to 6 decimal places. In fact the program yields, with $h = 0.08, X(1.6) = 1.2200394$. With $h = 0.04$ it gives $X(1.6) = 1.2200390$. Richardson extrapolation predicts the

error in the $h = 0.04$ solution as 0.00000003 and therefore that in the $h = 0.08$ as 0.0000005. The required accuracy was therefore just achieved using $h = 0.08$.

```
{program solves a pair of ordinary differential equations by the 4th
order Runge-Kutta Method}
procedure f1(t, x, v → f1)
    f1 ← v
endprocedure

procedure f2(t, x, v → f2)
    f2 ← sin(t) − x*x*v − x
endprocedure

{procedure computes values of x and v at the next time step}
procedure rk4(t, x, v, h → xn, vn)
    c11 ← h*f1(t, x, v)
    c21 ← h*f2(t, x, v)
    c12 ← h*f1(t + h/2, x + c11/2, v + c21/2)
    c22 ← h*f2(t + h/2, x + c11/2, v + c21/2)
    c13 ← h*f1(t + h/2, x + c12/2, v + c22/2)
    c23 ← h*f2(t + h/2, x + c12/2, v + c22/2)
    c14 ← h*f1(t + h, x + c13, v + c23)
    c24 ← h*f2(t + h, x + c13, v + c23)
    xn ← x + (c11 + 2*(c12 + c13) + c14)/6
    vn ← v + (c21 + 2*(c22 + c23) + c24)/6
endprocedure

t_start ← 0
t_end ← 1.6
x0 ← 0
v0 ← 1
write(vdu, "Enter step size")
read(keyboard, h)
write(printer,t_start,x0)
t ← t_start
x ← x0
v ← v0
repeat
    rk4(t, x, v, h → xn, vn)
    x ← xn
    v ← vn
    t ← t + h
    write(printer,t,x)
until t >= t_end
```

Figure 2.9: Pseudocode algorithm for Exercise 22

■ **23** The first step in solving this problem is to convert the problem to a set of coupled first-order differential equations

$$\begin{aligned} \frac{dx}{dt} &= v, & x(0.5) &= -1 \\ \frac{dv}{dt} &= w, & v(0.5) &= 1 \\ \frac{dw}{dt} &= x^2 - v^2 - (x-t)w, & w(0.5) &= 2 \end{aligned}$$

A pseudocode algorithm to compute the value of $X(2.2)$ is shown in Figure 2.11. The procedures used in the algorithm are defined in Figure 2.10.

```
{procedures for pseudocode algorithm in figure 2.11}
procedure f1(t, x, v, w → f1)
    f1 ← v
endprocedure

procedure f2(t, x, v, w → f2)
    f2 ← w
endprocedure

procedure f3(t, x, v, w → f3)
    f3 ← x*x − v*v − (x − t)*w
endprocedure

{procedure computes values of x, v and w at the next time step using
the 4th order Runge-Kutta procedure}
procedure rk4(t, x, v, w, h → xn, vn, wn)
    c11 ← h*f1(t, x, v, w)
    c21 ← h*f2(t, x, v, w)
    c31 ← h*f3(t, x, v, w)
    c12 ← h*f1(t + h/2, x + c11/2, v + c21/2, w + c31/2)
    c22 ← h*f2(t + h/2, x + c11/2, v + c21/2, w + c31/2)
    c32 ← h*f3(t + h/2, x + c11/2, v + c21/2, w + c31/2)
    c13 ← h*f1(t + h/2, x + c12/2, v + c22/2, w + c32/2)
    c23 ← h*f2(t + h/2, x + c12/2, v + c22/2, w + c32/2)
    c33 ← h*f3(t + h/2, x + c12/2, v + c22/2, w + c32/2)
    c14 ← h*f1(t + h, x + c13, v + c23, w + c33)
    c24 ← h*f2(t + h, x + c13, v + c23, w + c33)
    c34 ← h*f3(t + h, x + c13, v + c23, w + c33)
```

(Continued)

```
    xn ← x + (c11 + 2*(c12 + c13) + c14)/6
    vn ← v + (c21 + 2*(c22 + c23) + c24)/6
    wn ← w + (c31 + 2*(c32 + c33) + c34)/6
endprocedure

{procedure computes values of x, v and w at the next time step using
the 3rd order predictor-corrector procedure}
procedure pc3(t, xo, vo, wo, x, v, w, h → xn, vn, wn)
    xp ← x + h*(3*f1(t, x, v, w) − f1(t − h, xo, vo, wo))/2
    vp ← v + h*(3*f2(t, x, v, w) − f2(t − h, xo, vo, wo))/2
    wp ← w + h*(3*f3(t, x, v, w) − f3(t − h, xo, vo, wo))/2
    xn ← x + h*(5*f1(t + h, xp, vp, wp) + 8*f1(t, x, v, w)
                        -f1(t-h,xo,vo,wo))/12
    vn ← v + h*(5*f2(t + h, xp, vp, wp) + 8*f2(t, x, v, w)
                        -f2(t-h,xo,vo,wo))/12
    wn ← w + h*(5*f3(t + h, xp, vp, wp) + 8*f3(t, x, v, w)
                        -f3(t-h,xo,vo,wo))/12
endprocedure
```

Figure 2.10: Pseudocode procedures for algorithm for Exercise 23

Using a program derived from this algorithm with a step size $h = 0.1$ gives $X(2.2) = 2.923350$ and, with a step size $h = 0.05$, gives $X(2.2) = 2.925418$. The method of Richardson extrapolation given in Section 2.3.6, is equally applicable to problems such as this one involving coupled equations. Since the third-order predictor–corrector method used has a local error of $O(h^4)$, the global error will be $O(h^3)$. The method therefore gives the estimated error in the second value of $X(2.2)$ as $2.923350-2.925418)/7 = -0.000295$. For 6 decimal place accuracy in the final estimate we need error ≤ 0.0000005; in other words, the error must be reduced by a factor of $0.000295/0.0000005 = 590$. Since we are using a third-order method, the required step length will be $0.05/\sqrt[3]{590} = 0.00596$. Rounding this down to a suitable size suggests that a step size of $h = 0.005$ will give a solution accurate to 6 decimal places. In fact the program yields, with $h = 0.005, X(2.2) = 2.92575057$. With $h = 0.0025$ it gives $X(2.2) = 2.92575089$. Richardson extrapolation predicts the error in the $h = 0.0025$ solution as 0.000000046 and therefore that in the

$h = 0.005$ as 0.00000037. The required accuracy was therefore comfortably achieved using $h = 0.005$.

```
{program solves three ordinary differential equations by the
3rd order predictor-corrector method}
t_start ← 0.5
t_end ← 2.2
x_start ← −1
v_start ← 1
w_start ← 2
write(vdu, "Enter step size")
read(keyboard, h)
write(printer, t_start, x_start)
t ← t_start
xo ← x_start
vo ← v_start
wo ← w_start
rk4(t, xo, vo, wo, h → x, v, w)
t ← t + h
write(printer,t,x)
repeat
    pc3(t, xo, vo, wo, x, v, w, h → xn, vn, wn)
    xo ← x
    vo ← v
    wo ← w
    x ← xn
    v ← vn
    w ← wn
    t ← t + h
    write (printer, t, x)
until t >= t_end
```

Figure 2.11: Pseudocode algorithm for Exercise 23

Review exercises 2.7

■ **1** Euler's method for the solution of the differential equation $\dfrac{dx}{dt} = f(t, x)$ is

$$X_{n+1} = X_n + hF_n = X_n + hf(t_n, X_n)$$

Applying this to the equation $\dfrac{dx}{dt} = \sqrt{x}$ with $x(0) = 1$ and a step size of $h = 0.1$ yields

$$
\begin{aligned}
x_0 &= x(0) = 1 \\
X_1 &= x_0 + hf(t_0, x_0) = x_0 + h\sqrt{x_0} = 1 + 0.1 \times \sqrt{1} = 1.1000 \\
X_2 &= X_1 + hf(t_1, X_1) = X_1 + h\sqrt{X_1} = 1.1000 + 0.1\sqrt{1.1000} = 1.2049 \\
X_3 &= X_2 + hf(t_2, X_2) = X_2 + h\sqrt{X_2} = 1.2049 + 0.1\sqrt{1.2049} = 1.3146 \\
X_4 &= X_3 + hf(t_3, X_3) = X_3 + h\sqrt{X_3} = 1.3146 + 0.1\sqrt{1.3146} = 1.4293 \\
X_5 &= X_4 + hf(t_4, X_4) = X_4 + h\sqrt{X_4} = 1.4293 + 0.1\sqrt{1.4293} = 1.5489
\end{aligned}
$$

Hence Euler's method with step size $h = 0.1$ gives the estimate $X(0.5) = 1.5489$.

■ **2** Euler's method for the solution of the differential equation $\dfrac{dx}{dt} = f(t, x)$ is

$$X_{n+1} = X_n + hF_n = X_n + hf(t_n, X_n)$$

Applying this to the equation $\dfrac{dx}{dt} = -e^{xt}$ with $x(1) = 1$ and a step size of $h = 0.05$ yields

$$
\begin{aligned}
x_0 &= x(1) = 1 \\
X_1 &= x_0 + hf(t_0, x_0) = x_0 + h(-e^{x_0 t_0}) = 1 - 0.05\exp(1 \times 1) = 0.86409 \\
X_2 &= X_1 + hf(t_1, X_1) = X_1 + h(-e^{X_1 t_1}) \\
&= 0.86409 - 0.05\exp(0.86409 \times 1.05) = 0.74021 \\
X_3 &= X_2 + hf(t_2, X_2) = X_2 + h(-e^{X_2 t_2}) \\
&= 0.74021 - 0.05\exp(0.74021 \times 1.10) = 0.62733 \\
X_4 &= X_3 + hf(t_3, X_3) = X_3 + h(-e^{X_3 t_3}) \\
&= 0.62733 - 0.05\exp(0.62733 \times 1.15) = 0.52447
\end{aligned}
$$

Hence Euler's method with step size $h = 0.05$ gives the estimate $X(1.1) = 0.52447$.

■ **3** This question could be solved by hand computation or using a computer program based on a simple modification of the pseudocode algorithm given in Figure 2.1. With a step size of $h = 0.1$ it is found that $X(0.4) = 1.125584$ and, with a step size of $h = 0.05, X(0.4) = 1.142763$. Using the Richardson

extrapolation method, because Euler's method is a first-order method and the global error is therefore of *O(h)*, the error in the estimate of $X(0.4)$ is approximately $1.142763 - 1.125584 = 0.017179$. To obtain $X(0.4)$ accurate to 2 decimal places we need error ≤ 0.005. To achieve this we would need to reduced step size by a factor of $0.017179/0.005 = 3.44$. This suggests a step size of $0.05/3.44 = 0.0145$. Rounding this down to a sensible figure suggests trying a step size of 0.0125 or 0.01.

■ **4** This question could be solved by hand computation or using a computer program based on a simple modification of the pseudocode algorithm given in Figure 2.1. With a step size of $h = 0.05$ it is found that $X(0.25) = 2.003749$ and, with a step size of $h = 0.025, X(0.25) = 2.004452$. Using the Richardson extrapolation method, because Euler's method is a first-order method and the global error is therefore of $O(h)$, the error in the estimate of $X(0.25)$ is approximately $2.004452 - 2.003749 = 0.000703$. To obtain $X(0.25)$ accurate to 3 decimal places we need error ≤ 0.0005. To achieve this we would need to reduce step size by a factor of $0.0007/0.0005 = 1.4$. This suggests a step size of $0.025/1.4 = 0.0179$. Rounding this down to a sensible figure suggests trying a step size of 0.0166667 or 0.0125.

■ **5** This question could be solved by hand computation or using a computer program based on a simple modification of the pseudocode algorithm given in Figure 2.5. By either method it is found that $X_1(1.2) = 2.374037, X_2(1.2) = 2.374148$ and $X_3(1.2) = 2.374176$. The local error of the second-order predictor–corrector is $O(h^3)$ so the global error is $O(h^2)$. Hence it is expected

$$|X - x| \propto h^2 \quad \text{that is,} \quad x \approx X + \alpha h^2$$

$$\text{therefore, } |X_1 - X_2| \approx \left| x - \alpha h^2 - \left(x - \alpha \left(\frac{h}{2} \right)^2 \right) \right| = \frac{3}{4} \alpha h^2$$

$$\text{and } |X_2 - X_3| \approx \left| x - \alpha \left(\frac{h}{2} \right)^2 - \left(x - \alpha \left(\frac{h}{4} \right)^2 \right) \right| = \frac{3}{16} \alpha h^2$$

$$\text{Hence } \frac{|X_1 - X_2|}{|X_2 - X_3|} \approx \frac{\frac{3}{4} a h^2}{\frac{3}{16} a h^2} \approx 4$$

In fact we find that $\dfrac{|X_1 - X_2|}{|X_2 - X_3|} = \dfrac{|2.374037 - 2.374148|}{|2.374148 - 2.374176|} = \dfrac{|-0.000111|}{|-0.000028|} = 3.97 \approx 4$

■ **6** This question is best solved using a computer program based on a simple modification of the pseudocode given Figure 2.7. Let X_1 denote the solution using a step size of $h = 0.2$, X_2 that using $h = 0.1$ and X_3 that using $h = 0.05$. By either method it is found that $X_1(2) = 5.19436687, X_2(2) = 5.19432575$ and $X_3(2) = 5.19432313$. The local error of the fourth-order Runge–Kutta method is $O(h^5)$ so the global error is $O(h^4)$. Hence, by Richardson extrapolation, we may expect

$$x(2) = X_2(2) + \alpha h^4 = X_3(2) + \alpha\left(\frac{h}{2}\right)^4$$

$$\text{therefore } \alpha h^4 \approx 16(x(2) - X_3(2))$$

$$\text{therefore } x(2) = X_2(2) + 16(x(2) - X_3(2))$$

Hence $x(2) = \dfrac{16X_3(2) - X_2(2)}{15} = \dfrac{16 \times 5.19432313 - 5.19432575}{15} = 5.19432296$ and the most accurate estimate of $x(2)$ is 5.19432296.

■ **7** The boundary conditions for this problem are $p(r_0) = p_0$ and $p(r_1) = 0$. Hence we have

$$p + r\frac{dp}{dr} = 2a - p \quad \Rightarrow \quad \frac{dp}{dr} + \frac{2p}{r} = \frac{2a}{r}$$

This is a linear differential equation, so we first find the integrating factor.
$\int \frac{2}{r} dr = 2\ln r$ so the integrating factor is $e^{2\ln r} = r^2$

$$\text{therefore, } r^2\frac{dp}{dr} + 2rp = 2ar$$

$$\text{that is, } r^2 p = ar^2 + C$$

$$\text{Now } p(r_0) = p_0 \quad \Rightarrow \quad r_0^2 p_0 = a r_0^2 + C \quad \Rightarrow \quad C = r_0^2(p_0 - a)$$

$$p(r_1) = 0 \quad \Rightarrow \quad 0 = a r_1^2 + C = a r_1^2 + r_0^2(p_0 - a) \quad \Rightarrow \quad a = \frac{r_0^2 p_0}{r_0^2 - r_1^2}$$

Hence $p(r) = \dfrac{r_0^2 p_0}{r_0^2 - r_1^2} + \left(p_0 - \dfrac{r_0^2 p_0}{r_0^2 - r_1^2}\right)\dfrac{r_0^2}{r^2} = \dfrac{r_0^2 p_0}{r_1^2 - r_0^2}\left(\dfrac{r_1^2}{r^2} - 1\right)$

and $p(1.5) = \dfrac{1^2 1}{2^2 - 1^2}\left(\dfrac{2^2}{1.5^2} - 1\right) = \frac{1}{3}\left(\frac{16}{9} - 1\right) = \frac{7}{27}$

The problem to solve numerically is $\dfrac{dp}{dr} = -\dfrac{2(3p+1)}{3r}$, $p(1) = 1$. This may easily be solved using a modification of the pseudocode algorithm of Figure 2.7 or of the program of Figure 2.8. We find that, using a step size of $h = 0.05$, $p(1.5) = 0.25925946$.

■ **8** The first step is to recast the problem as a set of three coupled (linked) first-order ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} &= v, & x(1) &= 0.2 \\ \frac{dv}{dt} &= w, & v(1) &= 1 \\ \frac{dw}{dt} &= \sin(t) + xt - 4v^2 - w^2, & w(1) &= 0 \end{aligned}$$

Figure 2.12 shows a pseudocode algorithm for the solution of these three equations by Euler's Method.

```
{program solves three ordinary differential equations by the Euler
method}
procedure f1(t, x, v, w → f1)
    f1 ← v
endprocedure

procedure f2(t, x, v, w → f2)
    f2 ← w
endprocedure

procedure f3(t, x, v, w → f3)
    f3 ← sin(t) + x*t − 4*v*v − w*w
endprocedure
```

(Continued)

```
{procedure computes values of x, v and w at the next time step using
the Euler procedure}
procedure euler(t, x, v, w, h → xn, vn, wn)
    xn ← x + h*f1(t, x, v, w)
    vn ← v + h*f2(t, x, v, w)
    wn ← w + h*f3(t, x, v, w)
endprocedure

t_start ← 1.0
t_end ← 2.0
x_start ← 0.2
v_start ← 1
w_start ← 0
write(vdu, "Enter step size")
read(keyboard, h)
write(printer, t_start, x_start)
t ← t_start
x ← x_start
v ← v_start
w ← w_start
repeat
    euler(t, x, v, w, h → xn, vn, wn)
    x ← xn
    v ← vn
    w ← wn
    t ← t + h
    write(printer, t,x)
until t >= t_end
```

Figure 2.12: Pseudocode algorithm for Review Exercise 8

Using a program derived from this algorithm with a step size $h = 0.025$ gives $X(2) = 0.847035$ and, with a step size $h = 0.0125$, gives $X(2) = 0.844067$. The method of Richardson extrapolation, given in Section 2.3.6, is equally applicable to problems such as this one involving coupled equations. Since Euler's method has a local error of $O(h^2)$ the global error will be $O(h)$. The method therefore gives the estimated error in the second value of $X(2)$ as $(0.847035 - 0.844067)/1 = 0.002968$. This is less than 5 in the third decimal place, so we have two significant figures of accuracy. The best estimate we can make is that $x(2) = 0.84$ to 2dp.

■ **9** The first step is to recast the problem as a set of two coupled (linked) first-order ordinary differential equations

$$\frac{dx}{dt} = v, \qquad x(0) = 0.02$$
$$\frac{dv}{dt} = (1 - x^2)v - 40x, \qquad v(0) = 0$$

Figure 2.13 shows a pseudocode algorithm for the solution of these equations by the second-order predictor–corrector method. Using a program derived from this algorithm with a stepsize $h = 0.02$ gives $X(4) = 0.147123$ and, with a step size $h = 0.01$, gives $X(4) = 0.146075$. The method of Richardson extrapolation, given in Section 2.3.6, is applicable to problems such as this one involving coupled equations. Since the second-order predictor–corrector method has a local error of $O(h^3)$, the global error will be $O(h^2)$. The method therefore gives the estimated error in the second value of $X(4)$ as $(0.147123 - 146075)/3 = 0.001048$. For 4 decimal place accuracy in the final estimate we need error ≤ 0.00005; in other words, the error must be reduced by a factor of $0.001048/0.00005 = 20.96$. Since this predictor–corrector is a second-order method the required step length will be $0.01/\sqrt{20.96} = 0.0022$. Rounding this down to a suitable size suggests that a step size of $h = 0.002$ will give a solution accurate to 4 decimal places. In fact the program yields, with $h = 0.002, X(4) = 0.145813$. With $h = 0.001$ it gives $X(4) = 0.145807$. Richardson extrapolation predicts the error in the $h = 0.001$ solution as 0.000002 and therefore that in the $h = 0.002$ as 0.000008. The required accuracy was therefore comfortably achieved using $h = 0.002$. We can be confident that $x(4) = 0.1458$ to 4dp.

```
{program solves two ordinary differential equations by the second
order predictor-corrector method}
procedure f1(t, x, v → f1)
    f1 ← v
endprocedure

procedure f2(t, x, v → f2)
    f2 ← (1 − x*x)*v − 40*x
endprocedure
```

(Continued)

```
{procedure computes values of x, v and w at the next time step using
the second order predictor-corrector procedure}
procedure pc2(t, x, v, h → xn, vn)
    xp ← x + h*f1(t, x, v)
    vp ← v + h*f2(t, x, v)
    xn ← x + h*(f1(t + h, xp, vp) + f1(t, x, v))/2
    vn ← v + h*(f2(t + h, xp, vp) + f2(t, x, v))/2
endprocedure

t_start ← 0.0
t_end ← 4.0
x_start ← 0.02
v_start ← 0
write(vdu, "Enter step size")
read(keyboard, h)
write printer, t_start, x_start
t ← t_start
x ← x_start
v ← v_start
repeat
    pc2(t, x, v, h → xn, vn)
    x ← xn
    v ← vn
    t ← t + h
    write(printer,t,x)
until t >= t_end
```

Figure 2.13: Pseudcode algorithm for Review Exercise 9

■ **10** The first step is to recast the problem as a set of three coupled (linked) first-order ordinary differential equations.

$$\begin{aligned} \frac{dx}{dt} &= v, & x(1) &= -1 \\ \frac{dv}{dt} &= w, & v(1) &= 1 \\ \frac{dw}{dt} &= \sin(t) + xt - 4v^3 - \sqrt{|w|}, & w(1) &= 2 \end{aligned}$$

A minor modification of the pseudocode algorithm shown in Figure 2.9 provides an algorithm for the solution of this problem. Using a program derived from this algorithm with a step size $h = 0.1$ gives $X(2.5) = -0.651076$ and,

with a step size $h = 0.05$, gives $X(2.5) = -0.653798$. We use the method of Richardson extrapolation, given in Section 2.3.6. Since the Runge–Kutta method has a local error of $O(h^5)$ the global error will be $O(h^4)$. The method therefore gives the estimated error in the second value of $X(2.5)$ as $(0.651076 - 0.653798)/15 = -0.000181$. For 4 decimal place accuracy in the final estimate we need error ≤ 0.00005; in other words, the error must be reduced by a factor of $0.000181/0.00005 = 3.63$. Since the Runge–Kutta is a fourth-order method the required step length will be $0.05\sqrt[4]{3.63} = 0.036$. Rounding this down to a suitable size suggests that a stepsize of $h = 0.025$ will give a solution accurate to 4 decimal places. In fact the program yields, with $h = 0.025, X(2.5) = -0.653232$. With $h = 0.0125$ it gives $X(2.5) = -0.653217$. Richardson extrapolation predicts the error in the $h = 0.0125$ solution as 0.0000009 and therefore that in the $h = 0.025$ as 0.000015. The required accuracy is therefore easily achieved using $h = 0.025$. We can be confident that $x(2.5) = -0.6532$ to 4dp.

3

Vector Calculus

Exercises 3.1.2

- **1(a)** $f(x, y) = c \Rightarrow x^2 + y^2 = 1 + \mathrm{e}^c$

  Contours are a family of concentric circles, centre (0,0) and radius > 1.

1(b) $f(x, y) = c \Rightarrow y = (1 + x)\tan c$

Contours are a family of straight lines whose y intercept equals their slope and pass through (-1,0).

- **2(a)** Flow lines are given by $\frac{\mathrm{d}x}{\mathrm{d}t} = y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 6x^2 - 4x$.

  Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x^2 - 4x}{y}$$
$$\int y \, \mathrm{d}y = \int (6x^2 - 4x) \, \mathrm{d}x + c$$
$$\frac{1}{2}y^2 = 2x^3 - 2x^2 + c$$
$$y^2 = 4x^2(x - 1) + C$$

2(b) Flow lines are given by $\frac{\mathrm{d}x}{\mathrm{d}t} = y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{6}x^3 - x$.

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{1}{6}x^3 - x}{y}$$
$$\int y \, \mathrm{d}y = \int (\frac{1}{6}x^3 - x) \, \mathrm{d}x + c$$
$$\frac{1}{2}y^2 = \frac{1}{24}x^4 - \frac{1}{2}x^2 + c$$
$$y^2 = \frac{1}{12}x^2(x^2 - 12) + C$$

■ **3(a)** Level surfaces are given by $f(\mathbf{r}) = c \Rightarrow z = c + xy$.

3(b) Level surfaces are given by $f(\mathbf{r}) = c \Rightarrow z = c - \mathrm{e}^{-xy}$.

■ **4(a)** Field lines are given by $\frac{\mathrm{d}x}{\mathrm{d}t} = xy$, $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 + 1$, $\frac{\mathrm{d}z}{\mathrm{d}t} = z$.

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z \Rightarrow z = A\mathrm{e}^t$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + y^2 \Rightarrow y = \tan(t + \alpha)$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = xy \Rightarrow \ln x = C - \ln(\cos(t + \alpha))$$
$$x = \frac{B}{\cos(t + \alpha)}$$

Since $1 + \tan^2\theta = \sec^2\theta \Rightarrow 1 + y^2 = \left(\frac{x}{B}\right)^2$ is a hyperbola, the curve is on a hyperbolic cylinder.

4(b) Field lines are given by $\frac{\mathrm{d}x}{\mathrm{d}t} = yz$, $\frac{\mathrm{d}y}{\mathrm{d}t} = zx$, $\frac{\mathrm{d}z}{\mathrm{d}t} = xy$.
Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} \Rightarrow y^2 = x^2 - c$$
$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{x}{z} \Rightarrow z^2 = x^2 + k$$

The curve is the intersection of these hyperbolic cylinders.

■ **5(a)**

$$\frac{\partial f}{\partial x} = yz - 2x \qquad \frac{\partial f}{\partial y} = xz + 1 \qquad \frac{\partial f}{\partial z} = xy - 1$$
$$\frac{\partial^2 f}{\partial x^2} = -2 \qquad \frac{\partial^2 f}{\partial y^2} = 0 \qquad \frac{\partial^2 f}{\partial z^2} = 0$$
$$\frac{\partial^2 f}{\partial x \partial y} = z \qquad \frac{\partial^2 f}{\partial y \partial x} = z \qquad \frac{\partial^2 f}{\partial x \partial z} = y$$
$$\frac{\partial^2 f}{\partial z \partial x} = y \qquad \frac{\partial^2 f}{\partial y \partial z} = x \qquad \frac{\partial^2 f}{\partial z \partial y} = x$$

5(b)

$$\frac{\partial f}{\partial x} = 2xyz^3 \qquad \frac{\partial f}{\partial y} = x^2z^3 \qquad \frac{\partial f}{\partial z} = 3x^2yz^2$$
$$\frac{\partial^2 f}{\partial x^2} = 2yz^3 \qquad \frac{\partial^2 f}{\partial y^2} = 0 \qquad \frac{\partial^2 f}{\partial z^2} = 6x^2yz$$
$$\frac{\partial^2 f}{\partial x \partial y} = 2xz^3 \qquad \frac{\partial^2 f}{\partial y \partial x} = 2xz^3 \qquad \frac{\partial^2 f}{\partial x \partial z} = 6xyz^2$$
$$\frac{\partial^2 f}{\partial z \partial x} = 6xyz^2 \qquad \frac{\partial^2 f}{\partial y \partial z} = 3x^2z^2 \qquad \frac{\partial^2 f}{\partial z \partial y} = 3x^2z^2$$

5(c)

$$\frac{\partial f}{\partial x} = \frac{z}{1+(\frac{y}{x})^2}\left(\frac{-y}{x^2}\right) = \frac{-yz}{x^2+y^2} \qquad \frac{\partial f}{\partial y} = \frac{z}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{xz}{x^2+y^2}$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{2xyz}{(x^2+y^2)^2} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{-2xyz}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial z} = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\frac{\partial^2 f}{\partial z^2} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{z}{x^2+y^2} - \frac{2x^2z}{(x^2+y^2)^2} = \frac{z(x^2+y^2)-2x^2z}{(x^2+y^2)^2} = \frac{(y^2-x^2)z}{(x^2+y^2)^2}$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{-z}{x^2+y^2} + \frac{2y^2z}{(x^2+y^2)^2} = \frac{2y^2z-z(x^2+y^2)}{(x^2+y^2)^2} = \frac{(y^2-x^2)z}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \qquad \frac{\partial^2 f}{\partial z \partial x} = \frac{-y}{x^2+y^2}$$
$$\frac{\partial^2 f}{\partial y \partial z} = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \qquad \frac{\partial^2 f}{\partial z \partial y} = \frac{x}{x^2+y^2}$$

■ **6(a)** $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = -1$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 2$, $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{-1}{(t-1)^2}$

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= 2(t^3-1)(3t^2) + 2(2t)(2) + (-1)\left(\frac{-1}{(t-1)^2}\right) \\ &= \{2t(3t^4-3t+4)(t-1)^2+1\}/(t-1)^2 \\ &= \{2t(3t^6-6t^5+3t^4-3t^3+10t^2-11t+4)+1\}/(t-1)^2\end{aligned}$$

6(b)

$$\frac{\partial f}{\partial x} = yz,\ \frac{\partial f}{\partial y} = xz,\ \frac{\partial f}{\partial z} = xy,\ \frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^{-t}(\cos t - \sin t),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\mathrm{e}^{-t}(\cos t + \sin t),\ \frac{\mathrm{d}z}{\mathrm{d}t} = 1$$

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= t\mathrm{e}^{-t}\cos t.\mathrm{e}^{-t}(\cos t - \sin t) - t\mathrm{e}^{-t}\sin t.\mathrm{e}^{-t}(\cos t + \sin t) + \mathrm{e}^{-2t}\sin t\cos t.(1)\\ &= t\mathrm{e}^{-2t}(\cos^2 t - \sin^2 t - 2\sin t\cos t) + \mathrm{e}^{-2t}\sin t\cos t\\ &= t\mathrm{e}^{-2t}(\cos 2t - \sin 2t) + \frac{1}{2}\mathrm{e}^{-2t}\sin 2t\end{aligned}$$

■ **7** $r^2 = x^2 + y^2 + z^2,\ \tan\phi = \frac{y}{x},\ \tan\theta = \frac{(x^2+y^2)^{1/2}}{z}$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\phi,\ \frac{\partial \phi}{\partial y} = \frac{1}{1+(\frac{y}{x})^2}\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{\cos\phi}{r\sin\theta}$$

$$\frac{\partial\theta}{\partial y} = \frac{\partial}{\partial y}\{\tan^{-1}\left[\frac{(x^2+y^2)^{1/2}}{z}\right]\} = \frac{yz}{(x^2+y^2+z^2)(x^2+y^2)^{1/2}} = \frac{\sin\phi\cos\theta}{r}$$

$$\frac{\partial f}{\partial y} = \sin\theta\sin\phi\frac{\partial f}{\partial r} + \frac{\cos\phi}{r\sin\theta}\frac{\partial f}{\partial\phi} + \frac{\sin\phi\cos\theta}{r}\frac{\partial f}{\partial\theta}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta,\ \frac{\partial\phi}{\partial z} = 0$$

$$\frac{\partial\theta}{\partial z} = \frac{1}{1+\frac{x^2+y^2}{z^2}}\left[-\frac{(x^2+y^2)^{1/2}}{z^2}\right] = \frac{-(x^2+y^2)^{1/2}}{x^2+y^2+z^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial f}{\partial z} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}$$

■ **8** $\frac{\partial u}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial x},\ \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \frac{\partial u}{\partial x} = \frac{x}{r}\frac{\mathrm{d}f}{\mathrm{d}r}$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{x}{r}\right)\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{x}{r}\frac{\partial}{\partial x}\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)\\ &= \frac{r - x(\frac{x}{r})}{r^2}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{x}{r}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}\frac{x}{r}\\ &= \frac{y^2+z^2}{r.r^2}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{x^2}{r^2}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}\end{aligned}$$

Similarly (by symmetry), $\frac{\partial^2 u}{\partial y^2} = \frac{x^2+y^2}{r.r^2}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{y^2}{r^2}\frac{\partial^2 f}{\partial r^2}$, $\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = \frac{x^2+z^2}{r.r^2}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{z^2}{r^2}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2(x^2+y^2+z^2)}{r.r^2}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{x^2+y^2+z^2}{r^2}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}$$
$$= \frac{2}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{\mathrm{d}^2 f}{\mathrm{d}r^2}$$

Hence, the result.

■ **9** $V(x,y,z) = \frac{1}{z}\exp\left\{-\frac{x^2+y^2}{4z}\right\}$

$$\frac{\partial V}{\partial x} = \frac{1}{z}\exp\left\{-\frac{x^2+y^2}{4z}\right\}\left[\frac{-x}{2z}\right]$$
$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{z}\exp\left\{-\frac{x^2+y^2}{4z}\right\}\left[\frac{-x}{2z}\right]^2 - \frac{1}{2z^2}\exp\left\{-\frac{x^2+y^2}{4z}\right\}$$
$$\frac{\partial V}{\partial y} = \frac{1}{z}\exp\left\{-\frac{x^2+y^2}{4z}\right\}\left[\frac{-y}{2z}\right]$$
$$\frac{\partial^2 V}{\partial y^2} = \frac{1}{z}\exp\left\{-\frac{x^2+y^2}{4z}\right\}\left[\frac{-y}{2z}\right]^2 - \frac{1}{2z^2}\exp\left\{-\frac{x^2+y^2}{4z}\right\}$$
$$\frac{\partial V}{\partial z} = -\frac{1}{z^2}\exp\left\{-\frac{x^2+y^2}{4z}\right\} + \frac{1}{z}\exp\left\{-\frac{x^2+y^2}{4z}\right\}\left[\frac{x^2+y^2}{4z^2}\right]$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial V}{\partial z}$$

■ **10** $V = \sin 3x \cos 4y \cosh 5z$

$$\frac{\partial^2 V}{\partial x^2} = -9\sin 3x\cos 4y\cosh 5z$$
$$\frac{\partial^2 V}{\partial y^2} = -16\sin 3x\cos 4y\cosh 5z$$
$$\frac{\partial^2 V}{\partial z^2} = 25\sin 3x\cos 4y\cosh 5x$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Exercises 3.1.4

■ **11** $x + y = u, \quad y = uv$

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} = 1, \quad \frac{\partial y}{\partial u} = v \Rightarrow \frac{\partial x}{\partial u} = 1 - v$$

$$\frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} = 0, \quad \frac{\partial y}{\partial v} = u \Rightarrow \frac{\partial x}{\partial v} = -u$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 - v & v \\ -u & u \end{vmatrix} = u - uv - (-uv) = u$$

■ **12** $x + y + z = u,\ y + z = uv,\ z = uvw$

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} + \frac{\partial z}{\partial u} = 1,\ \frac{\partial y}{\partial u} + \frac{\partial z}{\partial u} = v,\ \frac{\partial z}{\partial u} = vw$$

$$\Rightarrow \frac{\partial z}{\partial u} = vw,\ \frac{\partial y}{\partial u} = v(1 - w),\ \frac{\partial x}{\partial u} = 1 - v$$

$$\frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} + \frac{\partial z}{\partial v} = 0,\ \frac{\partial y}{\partial v} + \frac{\partial z}{\partial v} = u,\ \frac{\partial z}{\partial v} = uw$$

$$\frac{\partial z}{\partial v} = uw,\ \frac{\partial y}{\partial v} = u - uw,\ \frac{\partial x}{\partial v} = -u$$

$$\frac{\partial x}{\partial w} + \frac{\partial y}{\partial w} + \frac{\partial z}{\partial w} = 0,\ \frac{\partial y}{\partial w} + \frac{\partial z}{\partial w} = 0,\ \frac{\partial z}{\partial w} = uv$$

$$\Rightarrow \frac{\partial z}{\partial w} = uv,\ \frac{\partial y}{\partial w} = -uv,\ \frac{\partial x}{\partial w} = 0$$

$$\Rightarrow \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 - v & v - vw & vw \\ -u & u - uw & uw \\ 0 & -uv & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 - v & v & vw \\ -u & u & uw \\ 0 & 0 & uv \end{vmatrix} = uv \begin{vmatrix} 1 - v & v \\ -u & u \end{vmatrix}$$

$$= u^2 v \begin{vmatrix} 1 - v & v \\ -1 & 1 \end{vmatrix} = u^2 v$$

■ **13** $x = \mathrm{e}^u \cos v,\ y = \mathrm{e}^u \sin v$

$$\frac{\partial x}{\partial u} = \mathrm{e}^u \cos v, \quad \frac{\partial y}{\partial u} = \mathrm{e}^u \sin v$$

$$\frac{\partial x}{\partial v} = -\mathrm{e}^u \sin v, \quad \frac{\partial y}{\partial v} = \mathrm{e}^u \cos v$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \mathrm{e}^u \cos v & \mathrm{e}^u \sin v \\ -\mathrm{e}^u \sin v & \mathrm{e}^u \cos v \end{vmatrix} = \mathrm{e}^{2u}(\cos^2 v + \sin^2 v) = \mathrm{e}^{2u}$$

$$x^2 + y^2 = \mathrm{e}^{2u} \Rightarrow u = \frac{1}{2}\ln(x^2 + y^2)$$

$$\frac{y}{x} = \tan v \Rightarrow v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2},\ \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}\ \frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2},\ \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{x^2+y^2} & -\frac{y}{x^2+y^2} \\ \frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} = \frac{1}{\mathrm{e}^{2u}}$$

Hence, the result.

■ **14**

$$\begin{aligned}
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} (-\sin x \cos y - \lambda \cos x \sin y) & (\cos x \cos y - \lambda \sin x \sin y) \\ (-\cos x \sin y - \lambda \sin x \cos y) & (-\sin x \sin y + \lambda \cos x \cos y) \end{vmatrix} \\
&= -(\sin x \cos y + \lambda \cos x \sin y)(-\sin x \sin y + \lambda \cos x \cos y) \\
&\quad + (\cos x \sin y + \lambda \sin x \cos y)(\cos x \cos y - \lambda \sin x \sin y) \\
&= -[-\sin^2 x \sin y \cos y + \lambda \sin x \cos x \cos^2 y - \lambda \sin x \cos x \sin^2 y \\
&\quad + \lambda^2 \cos^2 x \sin y \cos y] + [\cos^2 x \sin y \cos y - \lambda \sin x \cos x \sin^2 y \\
&\quad + \lambda \sin x \cos x \cos^2 y - \lambda^2 \sin^2 x \sin y \cos y] \\
&= \sin y \cos y - \lambda^2 \sin y \cos y
\end{aligned}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = -1 \quad \text{or} \quad 1$$

■ **15**

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2Kx & 3 & (3z+6y) \\ 8y & 2 & (2z+6x) \\ 2z & 1 & (2y+3x) \end{vmatrix}$$

$$= 2\begin{vmatrix} Kx-3z & 0 & (3z-9x) \\ 4y-2z & 0 & (2z-4y) \\ z & 1 & (2y+3x) \end{vmatrix}$$

$$= -2\begin{vmatrix} Kx-3z & 3z-9x \\ 4y-2z & 2z-4y \end{vmatrix} = 4(z-2y)(-Kx+9x)$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0 \Rightarrow K = 9$$

$$u = 9x^2 + 4y^2 + z^2$$

$$v^2 = 9x^2 + 4y^2 + z^2 + 12xy + 6xz + 4yz$$

$$2w = 12xy + 6xz + 4yz$$

$$u = v^2 - 2w$$

■ **16**

$$1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} \text{ (differentiating } u = g(x,y) \text{ with respect to } u)$$

$$0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} \text{ (differentiating } v = h(x,y) \text{ with respect to } u)$$

$$\Rightarrow \frac{\partial x}{\partial u} = \frac{\begin{vmatrix} 1 & \frac{\partial u}{\partial y} \\ 0 & \frac{\partial v}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{\partial v}{\partial y}/J$$

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x}/J$$

Similarly, differentiating $u = g(x,y)$ and $v = h(x,y)$ with respect to v obtains the other two expressions.

■ **17**

$$u = \mathrm{e}^x \cos y, \ v = \mathrm{e}^{-x} \sin y$$

$$\frac{\partial u}{\partial x} = \mathrm{e}^x \cos y = u, \ \frac{\partial v}{\partial x} = -\mathrm{e}^{-x} \sin y = -v$$

$$\frac{\partial u}{\partial y} = -\mathrm{e}^x \sin y, \ \frac{\partial v}{\partial y} = \mathrm{e}^{-x} \cos y$$

$$\frac{\partial x}{\partial u} = \frac{\mathrm{e}^{-x}\cos y}{\cos^2 y - \sin^2 y}$$
$$\frac{\partial x}{\partial v} = \frac{\mathrm{e}^{x}\sin y}{\cos^2 y - \sin^2 y}$$
$$\frac{\partial y}{\partial u} = \frac{\mathrm{e}^{-x}\sin y}{\cos^2 y - \sin^2 y}$$
$$\frac{\partial y}{\partial v} = \frac{\mathrm{e}^{x}\cos y}{\cos^2 y - \sin^2 y}$$

Since $2uv = 2\sin y\cos y = \sin 2y$, it is possible to express these results in terms of u and v.

$$\sin y = \frac{1}{2}(1 + \sqrt{1 - 4u^2v^2})$$
$$\cos y = \frac{1}{2}(1 - \sqrt{1 - 4u^2v^2})$$
$$\mathrm{e}^{x} = \frac{1}{2u}(1 + \sqrt{1 - 4u^2v^2})$$

Exercises 3.1.6

- **18(a)**

$$\frac{\partial}{\partial y}(y^2 + 2xy + 1) = 2y + 2x$$
$$\frac{\partial}{\partial x}(2xy + x^2) = 2y + 2x$$

Therefore, it is an exact differential.
Let $\frac{\partial f}{\partial x} = y^2 + 2xy + 1$, then $f(x, y) = xy^2 + x^2y + x + c(y)$

$$\text{and} \quad \frac{\partial f}{\partial y} = 2xy + x^2 + \frac{\mathrm{d}c}{\mathrm{d}y}$$

But, $\frac{\partial f}{\partial y} = 2xy + x^2$ from the question. Hence, $\frac{\mathrm{d}c}{\mathrm{d}y} = 0$; so c is independent of x and $y \Rightarrow f(x, y) = x^2y + y^2x + x + c$.

- **18(b)**

$$\frac{\partial}{\partial y}(2xy^2 + 3y\cos 3x) = 4xy + 3\cos 3x$$
$$\frac{\partial}{\partial x}(2x^2y + \sin 3x) = 4xy + 3\cos 3x$$

Therefore, it is an exact differential.

Let $\frac{\partial f}{\partial x} = 2xy^2 + 3y\cos 3x$, then $f(x,y) = x^2y^2 + y\sin 3x + c(y)$

$$\text{and } \frac{\partial f}{\partial y} = 2x^2y + y\sin 3x + \frac{\mathrm{d}c}{\mathrm{d}y}$$

Hence, $\frac{\mathrm{d}c}{\mathrm{d}y} = 0$ and c is a constant with respect to both x and y

$$\Rightarrow f(x,y) = x^2y^2 + y\sin 3x + c$$

18(c)

$$\frac{\partial}{\partial y}(6xy - y^2) = 6x - 2y$$
$$\frac{\partial}{\partial x}(2x\mathrm{e}^y - x^2) = 2\mathrm{e}^y - 2x$$

Not equal, so not an exact differential.

18(d)

$$\frac{\partial}{\partial y}(z^3 - 3y) = -3 \qquad \frac{\partial}{\partial z}(z^3 - 3y) = 3z^2 \qquad \frac{\partial}{\partial z}(12y^2 - 3x) = 0$$
$$\frac{\partial}{\partial x}(12y^2 - 3x) = -3 \qquad \frac{\partial}{\partial x}(3xz^2) = 3z^2 \qquad \frac{\partial}{\partial y}(3xz^2) = 0$$

Hence, exact. Let $\frac{\partial f}{\partial x} = z^3 - 3y$, then $f(x,y,z) = z^3x - 3xy + c(y,z)$ and $\frac{\partial f}{\partial y} = -3x + \frac{\partial c}{\partial y}$. This inturn implies that $\frac{\partial c}{\partial y} = 12y^2$ and $c(y,z) = 4y^3 + k(z)$.

$$\frac{\partial f}{\partial z} = 3z^2x + \frac{\partial c}{\partial z} = 3z^2x + \frac{\mathrm{d}k}{\mathrm{d}z}.$$

This inturn implies that $\frac{\mathrm{d}k}{\mathrm{d}z} = 0$ and so $f(x,y,z) = z^3x - 3xy + 4y^3 + K$.

■ **19**

$$\frac{\partial}{\partial y}(y\cos x + \lambda\cos y) = \cos x - \lambda\sin y$$
$$\frac{\partial}{\partial x}(x\sin y + \sin x + y) = \sin y + \cos x$$

Equal, if $\lambda = -1$.

Let $\frac{\partial f}{\partial x} = y\cos x - \cos y$, then $f(x,y) = y\sin x - x\cos y + c(y)$ and $\frac{\partial f}{\partial y} = \sin x + x\sin y + c'(y)$ so that $c'(y) = y$ and $c(y) = \frac{1}{2}y^2 + k$.

Hence, $f(x,y) = y\sin x - x\cos y + \frac{1}{2}y^2 + k$.

$$f(0,1) = 0 \Rightarrow 0 = 0 + 0 + \frac{1}{2} + k \Rightarrow k = -\frac{1}{2}$$

and $f(x,y) = y\sin x - x\cos y + \frac{1}{2}(y^2 - 1)$.

■ **20**

$$\frac{\partial}{\partial y}(10x^2 + 6xy + 6y^2) = 6x + 12y$$
$$\frac{\partial}{\partial x}(9x^2 + 4xy + 15y^2) = 18x + 4y$$

Hence, not exact.

$$\begin{aligned}\frac{\partial}{\partial y}[(2x+3y)^m(10x^2+6xy+6y^2)] &= 3m(2x+3y)^{m-1}(10x^2+6xy+6y^2)\\ &\quad + (2x+3y)^m(6x+12y)\\ \frac{\partial}{\partial x}[(2x+3y)^m(9x^2+4xy+15y^2)] &= 2m(2x+3y)^{m-1}(9x^2+4xy+15y^2)\\ &\quad + (2x+3y)^m(18x+4y)\end{aligned}$$

Hence, exact if

$$3m(10x^2+6xy+6y^2)+(2x+3y)(6x+12y) = 2m(9x^2+4xy+15y^2)+(2x+3y)(18x+4y)$$

Comparing coefficients of x^2 gives $m = 2$. Let

$$\frac{\partial f}{\partial x} = (2x+3y)^2(10x^2+6xy+6y^2) = 40x^4 + 144x^3y + 186x^2y^2 + 126xy^3 + 54y^4$$

$$\Rightarrow f(x,y) = 8x^5 + 36x^4y + 62x^3y^2 + 63x^2y^3 + 54xy^4 + c(y)$$

$$\frac{\partial f}{\partial y} = 36x^4 + 124x^3y + 99x^2y^2 + 216xy^3 + c'(y)$$

$$\Rightarrow c'(y) = 9y^2 \times 15y^2 \Rightarrow c(y) = 27y^5 + k$$

Hence, $f(x,y) = 8x^5 + 36x^4y + 62x^3y^2 + 63x^2y^3 + 5xy^4 + 27y^5 + k$.

Exercises 3.2.2

■ **21** grad $f = (2xyz^2, x^2z^2, 2x^2yz)$.
At (1,2,3), grad $f = (36, 9, 12) = 3(12, 3, 4)$.

■ **21(a)** Unit vector in direction of $(-2, 3, -6)$ is $\frac{(-2,3,-6)}{\sqrt{(4+9+36)}} = \frac{1}{7}(-2, 3, -6)$.
Directional derivative of f in direction of $(-2, 3, -6)$ at (1,2,3) is

$$3(12, 3, 4) \cdot (-2, 3, -6)/7 = -117/7$$

21(b) Maximum rate of change is $|\text{grad } f| = 3\sqrt{(144 + 9 + 16)} = 39$ and is in the direction of grad f, that is, (12,3,4)/13.

■ **22(a)** $\nabla(x^2 + y^2 - z) = (2x, 2y, -1)$

22(b)

$$\nabla\left(z\tan^{-1}\left(\frac{y}{x}\right)\right) = \left(\frac{-zy}{x^2+y^2}, \frac{zx}{x^2+y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right)$$

22(c)

$$\begin{aligned}\nabla\left(\frac{\mathrm{e}^{-x-y+z}}{\sqrt{x^3+y^2}}\right) = &\left(\frac{-\mathrm{e}^{-x-y+z}}{\sqrt{x^3+y^2}} - \frac{1}{2}\frac{3x^2\mathrm{e}^{-x-y+z}}{(x^3+y^2)^{3/2}}, \frac{-\mathrm{e}^{-x-y+z}}{\sqrt{x^3+y^2}}\right.\\ &\left.-\frac{1}{2}\frac{2y\mathrm{e}^{-x-y+z}}{(x^3+y^2)^{3/2}}, \frac{\mathrm{e}^{-x-y+z}}{\sqrt{x^3+y^2}}\right)\\ =&\frac{\mathrm{e}^{-x-y+z}}{(x^3+y^2)^{3/2}}\left(-x^3-y^2-\frac{3}{2}x^2, -x^3-y^2-y, x^3+y^2\right)\end{aligned}$$

22(d)

$$\begin{aligned}\nabla(xyz\sin\pi(x+y+z)) =&(yz\sin\pi(x+y+z) + \pi xyz\cos\pi(x+y+z),\\ &xz\sin\pi(x+y+z) + \pi xyz\cos\pi(x+y+z),\\ &xy\sin\pi(x+y+z) + \pi xyz\cos\pi(x+y+z))\end{aligned}$$

■ **23** grad $(x^2+y^2-z) = (2x, 2y, -1)$.
At (1,1,2), grad $f = (2, 2, -1)$.
Unit vector in the direction of $(4, 4, -2)$ is $\frac{1}{3}(2, 2, -1)$.
Directional derivative is $(2, 2, -1) \cdot \frac{1}{3}(2, 2, -1) = \frac{1}{3}(4 + 4 + 1) = 3$.

■ **24** $\nabla(xy^2 - 3xz + 5) = (y^2 - 3z, 2xy, -3x)$.
At $(1, -2, 3)$, grad $f = (-5, -4, -3)$.
Unit vector in the direction of grad f is $(-5, -4, -3)/\sqrt{50}$.
Unit normal to surface $xy^2 - 3xz + 5 = 0$ at $(1, -2, 3)$ is $(5, 4, 3)/\sqrt{50}$.

■ **25(a)** $r = \sqrt{x^2+y^2+z^2}$

$$\begin{aligned}\nabla r &= \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ &= \frac{1}{r}(x, y, z) \\ &= \frac{\mathbf{r}}{r}\end{aligned}$$

25(b)

$$\begin{aligned}\nabla\left(\frac{1}{r}\right) &= \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}\right) \\ &= -\frac{\mathbf{r}}{r^3}\end{aligned}$$

■ **26**

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= 2xy + z^2 \Rightarrow \phi(x, y, z) = x^2y + xz^2 + f(y, z) \\ \frac{\partial\phi}{\partial y} &= x^2 + z \Rightarrow x^2 + z = x^2 + \frac{\partial f}{\partial y} \Rightarrow f(y, z) = zy + g(z) \\ \frac{\partial\phi}{\partial z} &= y + 2xz \Rightarrow y + 2xz = 2xz + y + \frac{\mathrm{d}g}{\mathrm{d}z}\end{aligned}$$

Hence, $\frac{\mathrm{d}g}{\mathrm{d}z} = 0 \Rightarrow g(z) = c$, a constant.
Hence, $\phi(x, y, z) = x^2y + xz^2 + zy + c$.

■ **27** $\phi(x, y, z) = x^2y - 3xyz + z^3$

grad $\phi = (2xy - 3yz, x^2 - 3xz, -3xy + 3z^2)$

At (3,1,2), grad $\phi = (0, -9, 3)$.

Unit vector in direction of $(3, -2, 6) = (3, -2, 6)/\sqrt{49}$.

Directional derivative at (3,1,2) in the direction of $(3, -2, 6)$ is

$$(0, -9, 3) \cdot (3, -2, 6)/7 = 36/7$$

■ **28** $\nabla(x^2 + y^2 + z^2 - 9) = (2x, 2y, 2z)$.

At $(2, -1, 2)$, grad $(x^2 + y^2 + z^2 - 9) = (4, -2, 4)$.

Unit normal to surface at $(2, -1, 2)$ is $(2, -1, 2)/3$.

$\nabla(x^2 + y^2 - z - 3) = (2x, 2y, -1)$.

At $(2, -1, 2)$, grad $(x^2 + y^2 - z - 3) = (4, -2, -1)$.

Unit normal to surface at $(2, -1, 2)$ is $(4, -2, -1)/\sqrt{21}$.

Let angle between normals be θ, then

$$\cos\theta = \frac{(2, -1, 2)}{3} \cdot \frac{(4, -2, -1)}{\sqrt{21}}$$

$$\Rightarrow \cos\theta = \frac{8}{3\sqrt{21}}, \text{ hence}, \theta = 54.41^\circ$$

■ **29(a)** $\nabla(x^2 + 2y^2 + 3z^2 - 6) = (2x, 4y, 6z)$.

At (1,1,1), grad $f = (2, 4, 6)$, so tangent plane at (1,1,1) is

$$(1, 2, 3) \cdot (x - 1, y - 1, z - 1) = 0 \text{ i.e. } x + 2y + 3z = 6$$

and normal line is

$$\frac{x - 1}{1} = \frac{y - 1}{2} = \frac{z - 1}{3}$$

29(b) $\nabla(2x^2 + y^2 - z^2 + 3) = (4x, 2y, -2z)$

At (1,2,3), grad $f = (4, 4, -6)$, so the tangent plane at (1,2,3) is

$$(2, 2, -3) \cdot (x - 1, y - 2, z - 3) = 0 \text{ i.e. } 2x + 2y - 3z = -3$$

and the normal line is

$$\frac{x - 1}{2} = \frac{y - 2}{2} = \frac{3 - z}{3}$$

29(c) $\nabla(x^2+y^2-z-1)=(2x,2y,-1)$
At (1,2,4), grad $f=(2,4,-1)$, so that the tangent plane is

$$(2,4,-1)\cdot(x-1,y-2,z-4)=0 \text{ i.e. } 2x+4y-z=6$$

and the normal line is

$$\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-4}{-1}$$

■ **30** The change $\Delta\mathbf{r}$ in the vector $\mathbf{r}$ can be resolved into the three directions $\mathbf{u}_r$, $\mathbf{u}_\theta$, $\mathbf{u}_\phi$. Thus,

$$\Delta\mathbf{r}=\Delta r\mathbf{u}_r+r\Delta\theta\mathbf{u}_\theta+r\sin\theta\Delta\phi\mathbf{u}_\phi$$

Hence,

$$\begin{aligned}\text{grad } f&=\lim_{\Delta\mathbf{r}\to\mathbf{0}}\frac{f(\mathbf{r}+\Delta\mathbf{r})-f(\mathbf{r})}{|\Delta\mathbf{r}|}\\&=\frac{\partial f}{\partial r}\mathbf{u}_r+\frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{u}_\theta+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{u}_\phi\end{aligned}$$

Exercises 3.3.2

■ **31(a)** div $(3x^2y,z,x^2)=6xy+0+0=6xy$

31(b) div $(3x+y,2z+x,z-2y)=3+0+1=4$

■ **32** div $\mathbf{F}=2y^2-2yz^3+2yz-3xz^2$.
At $(-1,2,3)$, div $\mathbf{F}=-61$.

■ **33**

$$\nabla(\mathbf{a}\cdot\mathbf{r}) = \nabla(a_1x + a_2y + a_3z)$$
$$= (a_1, a_2, a_3) = \mathbf{a}$$

$$(\mathbf{a}\cdot\nabla)\mathbf{r} = \left(a_1\frac{\partial}{\partial x}, a_2\frac{\partial}{\partial y}, a_3\frac{\partial}{\partial z}\right)(x, y, z)$$
$$= (a_1, a_2, a_3) = \mathbf{a}$$

$$\mathbf{a}(\nabla\cdot\mathbf{r}) = \mathbf{a}(1+1+1) = 3\mathbf{a}$$

■ **34**

$$\nabla\cdot\mathbf{v} = \left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \left(\frac{1}{r} - \frac{y^2}{r^3}\right) + \left(\frac{1}{r} - \frac{z^2}{r^3}\right)$$

$$\text{since } \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{1}{2}\frac{x.(2x)}{(x^2+y^2+z^2)^{3/2}}$$

$$\text{Hence, } \nabla.\mathbf{v} = \frac{3}{r} - \frac{x^2+y^2+z^2}{r^3} = \frac{2}{r}$$

$$\nabla\left(\frac{2}{r}\right) = 2\left(\frac{-\frac{1}{2}.(2x)}{(x^2+y^2+z^2)^{3/2}}, \frac{-\frac{1}{2}(2y)}{(x^2+y^2+z^2)^{3/2}}, \frac{-\frac{1}{2}(2z)}{(x^2+y^2+z^2)^{3/2}}\right)$$
$$= -\frac{2}{r^3}(x, y, z) = -\frac{2\mathbf{r}}{r^3}$$

■ **35**

$$\text{div } \mathbf{F} = 4xy^2 + 9xy^2 + \lambda xy^2 = (4+9+\lambda)xy^2$$
$$\text{div } \mathbf{F} = 0 \Rightarrow \lambda = -13$$

■ **36** In spherical polar coordinates, an element of volume has side Δr in the $\mathbf{u}_r$ direction, $r\Delta\theta$ in the $\mathbf{u}_\theta$ direction and $r\sin\theta\Delta\phi$ in the $\mathbf{u}_\phi$ direction.
The total flow out of the elementary volume is

$$\frac{\partial}{\partial r}(\mathbf{v}\cdot\mathbf{u}_r r^2\sin\theta\Delta\theta\Delta\phi)\Delta r + \frac{\partial}{\partial\theta}(\mathbf{v}\cdot\mathbf{u}_\theta r\sin\theta\Delta\phi\Delta r)\Delta\theta + \frac{\partial}{\partial\phi}(\mathbf{v}\cdot\mathbf{u}_\phi r\Delta\theta\Delta r)\Delta\phi$$

$$+ \text{ terms of order } |\Delta\mathbf{r}|^2$$

Dividing by the volume of the element, $r^2 \sin\theta \Delta\theta \Delta\phi \Delta r$, and proceeding to the limit, we obtain

$$\text{div } \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(r\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(v_\phi)$$

■ **37**

$$\begin{aligned}\text{div}\left(\frac{\mathbf{r}}{r^3}\right) &= \text{div}\left[\left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)\right] \\ &= \frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5} \\ &= \frac{3}{r^3} - 3\frac{(x^2+y^2+z^2)}{r^5} = 0\end{aligned}$$

Exercises 3.3.4

■ **38**

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz^2 & -yz & x+2z \end{vmatrix} = (y, 6xz-1, 0)$$

■ **39**

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x-x, y-y, z-z) = \mathbf{0}$$

■ **40**

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+yz & 2y+zx & 2z+xy \end{vmatrix} = (0,0,0) = \mathbf{0}$$

$$\text{grad } f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial z}\right) \Rightarrow \frac{\partial f}{\partial x} = 2x + yz$$

$$\frac{\partial f}{\partial x} = 2x + yz \Rightarrow f(x,y,z) = x^2 + xyz + g(y,z)$$

$$\frac{\partial f}{\partial y} = 2y + zx \text{ and } \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} \Rightarrow g(y,z) = y^2 + h(z)$$

$$\frac{\partial f}{\partial z} = 2z + xy \text{ and } \frac{\partial f}{\partial z} = xy + \frac{\mathrm{d}h}{\mathrm{d}z} \Rightarrow h(z) = z^2 + c$$

Hence, $f(x,y,z) = x^2 + y^2 + z^2 + xyz + C$.

■ **41**

$$\nabla \times (f\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zx^3 - zy & 0 & -x^4 + xy \end{vmatrix} = (x, 5x^3 - 2y, z)$$

$$f(\nabla \times \mathbf{v}) = (x^3 - y)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 0 & -x \end{vmatrix}$$

$$= (x^3 - y)(0, 2, 0) = (0, 2x^3 - 2y, 0)$$

$$(\nabla f) \times \mathbf{v} = (3x^2, -1, 0) \times \mathbf{v}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3x^2 & -1 & 0 \\ z & 0 & -x \end{vmatrix} = (x, 3x^3, z)$$

■ **42**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy + az^3 & bx^2 + 3z & 6xz^2 + cy \end{vmatrix} = (c - 3, 3az^2 - 6z^2, 2bx - 4x)$$

$\nabla \times \mathbf{F} = \mathbf{0} \Rightarrow c = 3, a = 2, b = 2$

$$\text{grad } \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$

$$\frac{\partial \phi}{\partial x} = 4xy + 2z^3 \Rightarrow \phi(x, y, z) = 2x^2y + 2xz^3 + f(y, z)$$

$$\frac{\partial \phi}{\partial y} = 2x^2 + 3z \text{ and } \frac{\partial \phi}{\partial y} = 2x^2 + \frac{\partial f}{\partial y} \Rightarrow \frac{\partial f}{\partial y} = 3z$$

Hence, $f(y, z) = 3yz + g(z)$.

$$\frac{\partial \phi}{\partial z} = 6xz^2 + 3y \text{ and } \frac{\partial \phi}{\partial z} = 6xz^2 + 3y + \frac{\mathrm{d}g}{\mathrm{d}z} \Rightarrow \frac{\mathrm{d}g}{\mathrm{d}z} = 0$$

Hence, $\phi(x, y, z) = 2x^2y + 2xz^3 + 3yz + C$.

■ **43**

$$\boldsymbol{\omega} = \frac{1}{2}\text{curl } \mathbf{u} = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & xyz \end{vmatrix}$$
$$= \frac{1}{2}(xz, -yz, 2)$$

At (1,3,2), $\boldsymbol{\omega} = \frac{1}{2}(2, -6, 2) = (1, -3, 1)$

$$\Rightarrow \quad |\boldsymbol{\omega}| = \sqrt{11}$$

■ **44**

$$\text{div } \mathbf{v} = a + d$$
$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax + by & cx + dy & 0 \end{vmatrix} = (0, 0, c - b)$$
$$\text{div } \mathbf{v} = 0 \;\Rightarrow\; a = -d$$
$$\text{curl } \mathbf{v} = \mathbf{0} \;\Rightarrow\; c = b$$
$$\mathbf{v} = (ax + by)\mathbf{i} + (bx - ay)\mathbf{j}$$
$$= \text{grad } \phi$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = ax + by \text{ and } \frac{\partial \phi}{\partial y} = bx - ay$$
$$\Rightarrow \quad \phi(x, y) = \frac{1}{2}ax^2 + bxy + f(y)$$

$$\frac{\partial \phi}{\partial y} = bx + f'(y) \;\Rightarrow\; f'(y) = -ay \;\Rightarrow\; f(y) = -\frac{1}{2}ay^2 + K$$

Hence, $\phi(x, y) = \frac{1}{2}ax^2 + bxy - \frac{1}{2}ay^2 + K$.

■ **45** In spherical polar coordinates, an element of volume has side Δr in the $\mathbf{u}_r$ direction, $r\Delta\theta$ in the $\mathbf{u}_\theta$ direction and $r\sin\theta\Delta\phi$ in the $\mathbf{u}_\phi$ direction.
Setting $\mathbf{v}\cdot\mathbf{u}_r = v_r$, $\mathbf{v}\cdot\mathbf{u}_\theta = v_\theta$ and $\mathbf{v}\cdot\mathbf{u}_\phi = v_\phi$, we see that the circulation around the $\mathbf{u}_r$ direction is

$$\frac{\partial}{\partial\theta}(v_\phi r\sin\theta\Delta\phi)\Delta\theta - \frac{\partial}{\partial\phi}(vr\Delta\theta)\Delta\phi + \text{terms of order } \Delta\theta^2 \text{ etc.}$$

The area around which this circulation takes place is $r^2 \sin\theta \Delta\theta \Delta\phi$, so, proceeding to the limit we have

$$(\text{curl } \mathbf{v}) \cdot \mathbf{u}_r = \left\{ \frac{\partial}{\partial\theta}(v_\phi r \sin\theta) - \frac{\partial}{\partial\phi}(v_\theta r) \right\} / (r^2 \sin\theta)$$

$$\text{Similarly} \qquad (\text{curl } \mathbf{v}) \cdot \mathbf{u}_\theta = \left\{ \frac{\partial}{\partial\phi}(v_r) - \frac{\partial}{\partial r}(r \sin\theta v_\phi) \right\} / (r^2 \sin\theta)$$

$$\text{and} \qquad (\text{curl } \mathbf{v}) \cdot \mathbf{u}_\phi = \left\{ \frac{\partial}{\partial r}(r v_\theta) - \frac{\partial}{\partial\theta}(v_r) \right\} / r$$

Hence, the result.

Exercises 3.3.6

- **46**

$$\begin{aligned} \text{grad } g &= \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) \\ &= \left(\frac{\mathrm{d}g}{\mathrm{d}r}\frac{\partial r}{\partial x}, \frac{\mathrm{d}g}{\mathrm{d}r}\frac{\partial r}{\partial y}, \frac{\mathrm{d}g}{\mathrm{d}r}\frac{\partial r}{\partial z} \right) \\ &= \frac{\mathrm{d}g}{\mathrm{d}r}\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) \qquad \text{since } r^2 = x^2 + y^2 + z^2 \\ &= \frac{1}{r}\frac{\mathrm{d}g}{\mathrm{d}r}\mathbf{r} \end{aligned}$$

$$\text{div } [(\mathbf{u} \times \mathbf{r})g] = (\mathbf{u} \times \mathbf{r}) \cdot \text{grad } g + g\nabla \cdot (\mathbf{u} \times \mathbf{r}) \qquad (3.19d)$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{r}) = \mathbf{r} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{r}) \qquad (3.19f)$$

$$\nabla \times \mathbf{r} = 0 \quad \Rightarrow \quad \nabla \cdot (\mathbf{u} \times \mathbf{r}) = \mathbf{r} \cdot \text{curl } \mathbf{u}$$

But $(\mathbf{u} \times \mathbf{r})$ is perpendicular to $\text{grad } g = \dfrac{1}{r}\dfrac{\mathrm{d}g}{\mathrm{d}r}\mathbf{r}$, so

$$(\mathbf{u} \times \mathbf{r}) \cdot \text{grad } g = 0$$

Hence, $\text{div } ((\mathbf{u} \times \mathbf{r})g) = \mathbf{r} \cdot \text{curl } \mathbf{u}$.

- **47** $\phi(x, y, z) = x^2 y^2 z^3, \qquad \mathbf{F}(x, y, z) = (x^2 y, x y^2 z, -y z^2)$

47(a) $\nabla^2\phi = 2y^2z^3 + 2x^2z^3 + 6x^2y^2z$

47(b)

$$\begin{aligned}\text{grad div } \mathbf{F} &= \text{grad } (2xy + 2xyz - 2yz)\\ &= (2y + 2yz, 2x + 2xz - 2z, 2xy - 2y)\end{aligned}$$

47(c)

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xy^2z & -yz^2 \end{vmatrix}$$

$$= \mathbf{i}(-z^2 - xy^2) + \mathbf{j}(0) + \mathbf{k}(y^2z - x^2)$$

$$\text{curl (curl } \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z^2 - xy^2 & 0 & y^2z - x^2 \end{vmatrix}$$

$$= \mathbf{i}(2yz) + \mathbf{j}(2x - 2z) + \mathbf{k}(2xy)$$

■ **48**

$$\begin{aligned}\text{grad } [(\mathbf{r}\cdot\mathbf{r})(\mathbf{a}\cdot\mathbf{r})] &= [\text{grad } (\mathbf{r}\cdot\mathbf{r})](\mathbf{a}\cdot\mathbf{r}) + (\mathbf{r}\cdot\mathbf{r})\text{grad } (\mathbf{a}\cdot\mathbf{r}) \qquad (3.19b)\\ &= 2\mathbf{r}(\mathbf{a}\cdot\mathbf{r}) + (\mathbf{r}\cdot\mathbf{r})\mathbf{a}\end{aligned}$$

$$\begin{aligned}\text{div } \{\text{grad } [(\mathbf{r}\cdot\mathbf{r})(\mathbf{a}\cdot\mathbf{r})]\} &= 2\text{div } [\mathbf{r}(\mathbf{a}\cdot\mathbf{r})] + \text{div } [(\mathbf{r}\cdot\mathbf{r})\mathbf{a}]\\ &= 2\{[\text{div } \mathbf{r}](\mathbf{a}\cdot\mathbf{r}) + \mathbf{r}\cdot\text{grad } (\mathbf{a}\cdot\mathbf{r})\}\\ &\quad + [\text{grad } (\mathbf{r}\cdot\mathbf{r})]\cdot\mathbf{a} + (\mathbf{r}\cdot\mathbf{r})\text{div } \mathbf{a} \qquad (3.19d)\\ &= 2\{3(\mathbf{a}\cdot\mathbf{r}) + \mathbf{r}\cdot\mathbf{a}\} + 2\mathbf{r}\cdot\mathbf{a} + 0\\ &= 10(\mathbf{r}\cdot\mathbf{a})\end{aligned}$$

■ **49**

$$\mathbf{v} = x^3y\mathbf{i} + x^2y^2\mathbf{j} + x^2yz\mathbf{k}$$

$$\nabla^2\mathbf{v} = 6xy\mathbf{i} + 2(x^2+y^2)\mathbf{j} + 2yz\mathbf{k}$$

$$\text{grad div } \mathbf{v} = \text{grad } (3x^2y + 2x^2y + x^2y) = \text{grad } (6x^2y) = 12xy\mathbf{i} + 6x^2\mathbf{j}$$

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3y & x^2y^2 & x^2yz \end{vmatrix} = x^2z\mathbf{i} - 2xyz\mathbf{j} + (2xy^2 - x^3)\mathbf{k}$$

$$\text{curl(curl } \mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2xyz & 2xy^2 - x^3 \end{vmatrix}$$

$$= (4xy + 2xy)\mathbf{i} + (x^2 - 2y^2 + 3x^2)\mathbf{j} + (-2yz)\mathbf{k}$$

grad div $\mathbf{v}$ − curl curl $\mathbf{v} = 6xy\mathbf{i} + 2(x^2+y^2)\mathbf{j} + 2yz\mathbf{k}$ as required.

■ **50**

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & xy & xz \\ xy & 0 & yz \end{vmatrix} = xy^2z\mathbf{i} + x^2yz\mathbf{j} - x^2y^2\mathbf{k}$$

$$\text{div } (\mathbf{u} \times \mathbf{v}) = y^2z + x^2z = (x^2+y^2)z$$

$$\text{curl } \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xy & xz \end{vmatrix} = 0\mathbf{i} - z\mathbf{j} + y\mathbf{k}$$

$$\mathbf{v} \cdot \text{curl } \mathbf{u} = y^2z$$

$$\text{curl } \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 0 & yz \end{vmatrix} = z\mathbf{i} + 0\mathbf{j} - x\mathbf{k}$$

$$\mathbf{u} \cdot \text{curl } \mathbf{v} = -x^2z$$

$$\Rightarrow \quad \mathbf{v} \cdot \text{curl } \mathbf{u} - \mathbf{u} \cdot \text{curl } \mathbf{v} = (x^2+y^2)z$$

$$\text{curl } (\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z & x^2yz & -x^2y^2 \end{vmatrix} = -3x^2y\mathbf{i} + 3xy^2\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{u}\text{div } \mathbf{v} = (xy\mathbf{j} + xz\mathbf{k})(y+y) = 2xy^2\mathbf{j} + 2xyz\mathbf{k}$$

$$\mathbf{v}\text{div } \mathbf{u} = (xy\mathbf{i} + yz\mathbf{k})(x+x) = 2x^2y\mathbf{i} + 2xyz\mathbf{k}$$

$$(\mathbf{v}\cdot\nabla)\mathbf{u} = \left(xy\frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}\right)(xy\mathbf{j} + xz\mathbf{k}) = xy^2\mathbf{j} + 2xyz\mathbf{k}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{v} = \left(xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}\right)(xy\mathbf{i} + yz\mathbf{k}) = x^2y\mathbf{i} + 2xyz\mathbf{k}$$

$$\Rightarrow \quad \Big[\mathbf{u}\text{div } \mathbf{v} - \mathbf{v}\text{div } \mathbf{u} + (\mathbf{v}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{v} = -3x^2y\mathbf{i} + 3xy^2\mathbf{j} + 0\mathbf{k}\Big]$$

51(a)

$$\text{grad}\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$$

$$\text{div}\left(\text{grad}\left(\frac{1}{r}\right)\right) = -\text{div } \frac{\mathbf{r}}{r^3} = -\frac{1}{r^3}\text{div } \mathbf{r} - \mathbf{r}\cdot\text{grad}\left(\frac{1}{r^3}\right)$$

$$= -\frac{3}{r^3} - \mathbf{r}\cdot\left(\frac{-3\mathbf{r}}{r^5}\right) = -\frac{3}{r^3} + \frac{3r^2}{r^5} = 0$$

51(b)

$$\text{curl}\left(\mathbf{k}\times\text{grad}\left(\frac{1}{r}\right)\right) = \text{curl}\left(\frac{-\mathbf{k}\times\mathbf{r}}{r^3}\right) = \text{curl}\left((y\mathbf{i} - x\mathbf{j})\left(\frac{1}{r^3}\right)\right)$$

$$= [\text{curl } (y\mathbf{i} - x\mathbf{j})]\frac{1}{r^3} + \text{grad}\left(\frac{1}{r^3}\right)\times(y\mathbf{i} - x\mathbf{j})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}\frac{1}{r^3} - \frac{3\mathbf{r}}{r^5}\times(y\mathbf{i} - x\mathbf{j})$$

$$= (0\mathbf{i} + 0\mathbf{j} - 2\mathbf{k})\frac{1}{r^3} - \frac{3}{r^5}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ y & -x & 0 \end{vmatrix}$$

$$= -\frac{2\mathbf{k}}{r^3} + \frac{3}{r^5}(-xz\mathbf{i} - yz\mathbf{j} + (x^2 + y^2)\mathbf{k})$$

$$\text{grad}\left(\mathbf{k}\cdot\text{grad}\left(\frac{1}{r}\right)\right) = \text{grad}\left(-\frac{\mathbf{k}\cdot\mathbf{r}}{r^3}\right) = \text{grad}\left(\frac{-z}{r^3}\right)$$

$$= -z\text{grad}\left(\frac{1}{r^3}\right) - (\text{grad } z)\frac{1}{r^3}$$

$$= -z\left(-\frac{3\mathbf{r}}{r^5}\right) - \frac{1}{r^3}\mathbf{k}$$

$$\text{curl}\left(\mathbf{k}\times\text{grad}\left(\frac{1}{r}\right)\right) + \text{grad}\left(\mathbf{k}\cdot\text{grad}\left(\frac{1}{r}\right)\right) = -\frac{3\mathbf{k}}{r^3} + \frac{3}{r^5}(-xz\mathbf{i} - yz\mathbf{j} + (x^2 + y^2)\mathbf{k}) + \frac{3}{r^5}(xz\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}) = 0$$

- **52(a)**

$$\text{grad}\left(\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}\right) = \text{grad}\left[\left(\frac{\mathbf{A}}{r^3}\right)\cdot\mathbf{r}\right]$$

$$= \frac{\mathbf{A}}{r^3}\times\text{curl }\mathbf{r} + \mathbf{r}\times\text{curl}\left(\frac{\mathbf{A}}{r^3}\right) + (\mathbf{r}\cdot\nabla)\left(\frac{\mathbf{A}}{r^3}\right) + \left(\frac{\mathbf{A}}{r^3}\cdot\nabla\right)\mathbf{r} \quad (3.19c)$$

$$= \mathbf{0} + \mathbf{r}\times\left[\text{grad}\left(\frac{1}{r^3}\right)\times\mathbf{A}\right] + \mathbf{A}(\mathbf{r}\cdot\nabla)\left(\frac{1}{r^3}\right) + \frac{\mathbf{A}}{r^3}$$

$$= \mathbf{r}\times\left(-\frac{3\mathbf{r}}{r^5}\times\mathbf{A}\right) + \mathbf{A}\left(-\frac{3r^2}{r^5}\right) + \frac{\mathbf{A}}{r^3}$$

Now, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$$\text{so} \qquad \mathbf{r}\times\left(\mathbf{A}\times\frac{3\mathbf{r}}{r^5}\right) = \left(\mathbf{r}\cdot\frac{3\mathbf{r}}{r^5}\right)\mathbf{A} - (\mathbf{A}\cdot\mathbf{r})\frac{3\mathbf{r}}{r^5}$$

$$\text{Hence} \qquad \text{grad}\left(\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}\right) = \frac{\mathbf{A}}{r^3} - 3\mathbf{r}\frac{(\mathbf{A}\cdot\mathbf{r})}{r^5}$$

52(b)

$$\begin{aligned}
\text{curl}\left(\frac{\mathbf{A}\times\mathbf{r}}{r^3}\right) &= (\mathbf{r}\cdot\nabla)\left(\frac{\mathbf{A}}{r^3}\right) - \mathbf{r}\left(\nabla\cdot\frac{\mathbf{A}}{r^3}\right) - \left(\frac{\mathbf{A}}{r^3}\cdot\nabla\right)\mathbf{r} + \frac{\mathbf{A}}{r^3}(\nabla\cdot\mathbf{r}) \\
&= -\frac{3\mathbf{A}}{r^3} - \mathbf{r}\left(-3\frac{\mathbf{A}\cdot\mathbf{r}}{r^5}\right) - \frac{\mathbf{A}}{r^3} + 3\frac{\mathbf{A}}{r^3} \\
&= \frac{3}{r^5}(\mathbf{A}\cdot\mathbf{r})\mathbf{r} - \frac{\mathbf{A}}{r^3} \\
(\mathbf{A}\times\mathbf{r})\times\mathbf{r} &= (\mathbf{A}\cdot\mathbf{r})\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\mathbf{A} \\
(\mathbf{A}\cdot\mathbf{r})\mathbf{r} &= (\mathbf{A}\times\mathbf{r})\times\mathbf{r} + \mathbf{A}r^2 \\
\text{curl}\left(\frac{\mathbf{A}\times\mathbf{r}}{r^3}\right) &= \frac{3}{r^5}[(\mathbf{A}\times\mathbf{r})\times\mathbf{r}] + \frac{3\mathbf{A}r^2}{r^5} - \frac{\mathbf{A}}{r^3} \\
&= 2\frac{\mathbf{A}}{r^3} + \frac{3}{r^5}(\mathbf{A}\times\mathbf{r})\times\mathbf{r}
\end{aligned}$$

■ **53(a)**

$$\Delta\times\mathbf{r} = \text{curl}\,\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

53(b)

$$\begin{aligned}
(\mathbf{a}\cdot\nabla)\mathbf{r} &= \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\
&= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \mathbf{a}
\end{aligned}$$

53(c)

$$\begin{aligned}
\nabla\times[(\mathbf{a}\cdot\mathbf{r})\mathbf{b} - (\mathbf{b}\cdot\mathbf{r})\mathbf{a}] &= \nabla\times[(\mathbf{a}\times\mathbf{b})\times\mathbf{r}] \\
&= (\mathbf{a}\times\mathbf{b})(\nabla\cdot\mathbf{r}) - [(\mathbf{a}\times\mathbf{b})\cdot\nabla]\mathbf{r} \\
&= 3(\mathbf{a}\times\mathbf{b}) - \mathbf{a}\times\mathbf{b} \\
&= 2\mathbf{a}\times\mathbf{b}
\end{aligned}$$

53(d)

$$\begin{aligned}\nabla \cdot [(\mathbf{a}\cdot\mathbf{r})\mathbf{b} - (\mathbf{b}\cdot\mathbf{r})\mathbf{a}] &= \nabla \cdot [(\mathbf{a}\times\mathbf{b})\times\mathbf{r}] \\ &= (\mathbf{a}\times\mathbf{b})\cdot(\nabla\times\mathbf{r}) \\ &= (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{0}) = 0\end{aligned}$$

■ **54**

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{u}_\phi \qquad \text{(Exercise 30)}$$

$$\nabla\cdot(\nabla f) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)$$

(using Exercise 36)

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)$$

■ **55**

$$\text{div } \mathbf{H} = \frac{1}{c}\left(\text{div}\left(\text{curl}\,\frac{\partial \mathbf{Z}}{\partial t}\right)\right) = 0 \qquad (3.22)$$

$$\text{div } \mathbf{E} = \text{div (curl curl } \mathbf{Z}) = 0 \qquad (3.22)$$

$$\text{curl } \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \text{ becomes}$$

$$\text{curl } \mathbf{H} = \frac{1}{c}\text{curl curl}\,\frac{\partial \mathbf{Z}}{\partial t}$$

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}(\text{curl curl } \mathbf{Z}) = \frac{1}{c}\text{curl curl}\,\frac{\partial \mathbf{Z}}{\partial t}$$

$$\text{curl } \mathbf{E} = \text{curl curl curl } \mathbf{Z}$$

$$\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{c}\text{curl}\,\frac{\partial^2 \mathbf{Z}}{\partial t^2}$$

$$\text{curl } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \;\Rightarrow\; \text{curl curl curl } \mathbf{Z} = -\frac{1}{c}\text{curl}\,\frac{\partial^2 \mathbf{Z}}{\partial t^2}$$

$$\Rightarrow \text{ curl curl } \mathbf{Z} = -\frac{1}{c}\frac{\partial^2 \mathbf{Z}}{\partial t^2}$$

$$\Rightarrow \text{ grad (div } \mathbf{Z}) - \nabla^2\mathbf{Z} = -\frac{1}{c}\frac{\partial^2 \mathbf{Z}}{\partial t^2}$$

$$\text{Hence, } \nabla^2\mathbf{Z} = \frac{1}{c}\frac{\partial^2 \mathbf{Z}}{\partial t^2} \text{ when div } \mathbf{Z} = 0$$

Exercises 3.4.2

■ **56**

$$\int_A^B y\,\mathrm{d}s = \int_3^{24} (2\sqrt{x})\sqrt{1+1/x}\,\mathrm{d}x$$

$$= \int_3^{24} 2\sqrt{x+1}\,\mathrm{d}x = \left[\frac{4}{3}(x+1)^{3/2}\right]_3^{24}$$

$$= \frac{4}{3}[125-8] = 156$$

■ **57** $\partial S \displaystyle\int_A^B [2xy\,dx + (x^2-y^2)\,dy] = I$

$$x^2+y^2=1 \;\Rightarrow\; x\,dx = -y\,dy$$

$$I = \int_{y=0}^{y=1} [-2y^2 + (1-2y^2)]\,dy$$

$$= \int_0^1 (1-4y^2)\,dy = \left[y - \frac{4}{3}y^3\right]_0^1 = -\frac{1}{3}$$

■ **58**

$$\mathbf{r} = (t^3, t^2, t)$$

$$\mathrm{d}\mathbf{r} = (3t^2, 2t, 1)\,\mathrm{d}t$$

$$\int_C \mathbf{V}\cdot\,\mathrm{d}\mathbf{r} = \int_0^1 [(2yz+3x^2)(3t^2) + (y^2+4xz)2t + (2z^2+6xy)1]\,\mathrm{d}t$$

$$= \int_0^1 [6t^5 + 9t^8 + 2t^5 + 8t^5 + 2t^2 + 6t^5]\,\mathrm{d}t$$

$$= \int_0^1 (22t^5 + 9t^8 + 2t^2)\,\mathrm{d}t = \frac{11}{3} + 1 + \frac{2}{3} = \frac{16}{3}$$

■ **59** $\mathbf{A} = (2y+3, xz, yz-x)$.

59(a) $\displaystyle\int_C \mathbf{A}\cdot \mathrm{d}\mathbf{r}$ where $\mathbf{r} = (2t^2, t, t^3)$ and $\mathrm{d}r = (4t, 1, 3t^2)\,\mathrm{d}t$

$$\begin{aligned}\int_C \mathbf{A}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 [(2t+3)4t + (2t^5)1 + (t^4-2t^2)3t^2]\,\mathrm{d}t \\ &= \int_0^1 (12t + 8t^2 - 6t^4 + 2t^5 + 3t^6)\,\mathrm{d}t = \left[6 + \frac{8}{3} - \frac{6}{5} + \frac{1}{3} + \frac{3}{7}\right] \\ &= \frac{288}{35}\end{aligned}$$

59(b) $\displaystyle\int_C \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \int_P^Q \mathbf{A}\cdot\ \mathrm{d}\mathbf{r} + \int_Q^R \mathbf{A}\cdot\ \mathrm{d}\mathbf{r} + \int_R^S \mathbf{A}\cdot\ \mathrm{d}\mathbf{r}$

where $P = (0,0,0)$, $Q = (0,0,1)$, $R = (0,1,1)$, $S = (2,1,1)$

(using straight lines)

On PQ	$\mathbf{A} = 3\mathbf{i}$	$(x = y = 0)$	$\mathbf{r} = z\mathbf{k}$
On QR	$\mathbf{A} = (2y+3)\mathbf{i} + y\mathbf{k}$	$(x = 0, z = 1)$	$\mathbf{r} = y\mathbf{j} + \mathbf{k}$
On RS	$\mathbf{A} = 5\mathbf{i} + x\mathbf{j} + (1-x)\mathbf{k}$	$(y = 1, z = 1)$	$\mathbf{r} = x\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\begin{aligned}\int_C \mathbf{A}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 3\mathbf{i}\cdot\mathbf{k}\,\mathrm{d}z + \int_0^1 [(2y+3)\mathbf{i} + y\mathbf{k}]\cdot\mathbf{j}\,\mathrm{d}y + \int_0^2 [5\mathbf{i} + x\mathbf{j} + (1-x)\mathbf{k}]\cdot\mathbf{i}\,\mathrm{d}x \\ &= 10\end{aligned}$$

since $\mathbf{i}\cdot\mathbf{k} = 0$ etc.

59(c) $\displaystyle\int_C \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \int_P^S \mathbf{A}\cdot\ \mathrm{d}\mathbf{r}$

where C is a straight line, $P = (0,0,0)$ and $S = (2,1,1)$.

Parametrically, straight line is $\mathbf{r} = (2,1,1)t$, so

$$\begin{aligned}\int_C \mathbf{A}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 [(2t+3)\mathbf{i} + 2t^2\mathbf{j} + (t^2-2t)\mathbf{k}]\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k})\,\mathrm{d}t \\ &= \int_0^1 [4t + 6 + 2t^2 + t^2 - 2t]\,\mathrm{d}t = \int_0^1 (2t + 6 + 3t^2)\,\mathrm{d}t \\ &= [1 + 6 + 1] = 8\end{aligned}$$

■ **60** F is conservative if there exists a ϕ such that

$$\mathbf{F} = (y^2\cos x + z^3, 2y\sin x - 4, 3xz^2 + z) = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

Such a ϕ is readily determined giving

$$\mathbf{F} = -\text{grad}\left(4y - y^2\sin x - xz^3 - \tfrac{1}{2}z^2\right)$$

Hence, work done in moving an object is

$$\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = -\left[4y - y^2\sin x - xz^3 - \frac{1}{2}z^2\right]_{(0,1,-1)}^{(\pi/2,-1,2)}$$

$$= \left[4 - \frac{1}{2}\right] - [-5 - 4\pi - 2] = 4\pi + 10.5$$

■ **61(a)** Curve is $\mathbf{r} = \left(t, \frac{1}{4}t^2, \frac{3}{8}t^3\right)$, so that $\mathrm{d}\mathbf{r} = \left(1, \frac{1}{2}t, \frac{9}{8}t^2\right)\mathrm{d}t$ and $\mathbf{F} = \left(3t^2, \frac{3}{4}t^4 - \frac{1}{4}t^2, \frac{3}{8}t^3\right)$

$$\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \int_0^2 \left[3t^2 + \frac{3}{8}t^5 - \frac{1}{8}t^3 + \frac{27}{64}t^5\right]\mathrm{d}t$$

$$= \left[t^3 + \frac{3}{48}t^6 - \frac{1}{32}t^4 + \frac{9}{128}t^6\right]_0^2$$

$$= 8 + 4 - \frac{1}{2} + \frac{9}{2} = 16$$

■ **61(b)** Curve is $\mathbf{r} = (2t, t, 3t)$, $0 \le t \le 1$, so that $\mathrm{d}\mathbf{r} = (2, 1, 3)\,\mathrm{d}t$ and $\mathbf{F} = (12t^2, 12t^2 - t, 3t)$.

$$\int_C \mathbf{F}\cdot\,\mathrm{d}r = \int_0^1 (24t^2 + 12t^2 - t + 9t)\,\mathrm{d}t$$

$$= \int_0^1 (36t^2 + 8t)\,\mathrm{d}t = 12 + 4 = 16$$

■ **61(c)** No. If $\mathbf{F}$ is conservative, there is a function $U(x, y, z)$ such that $\mathbf{F} = -\text{grad } U$

Test for existence of U: $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ has to be an exact differential

$$\frac{\partial}{\partial y}(3x^2) \neq \frac{\partial}{\partial x}(2xz - y)$$

Hence, not exact and $\mathbf{F}$ is not conservative.

■ **62** $\mathbf{F} = (3x^2 - y, 2yz^2 - x, 2y^2z)$

$$\frac{\partial}{\partial y}(3x^2 - y) = -1 = \frac{\partial}{\partial x}(2yz^2 - 1)$$

$$\frac{\partial}{\partial z}(3x^2 - y) = 0 = \frac{\partial}{\partial x}(2y^2z)$$

$$\frac{\partial}{\partial y}(2y^2z) = 4yz = \frac{\partial}{\partial z}(2yz^2 - x)$$

Hence, conservative and $\mathbf{F} = -\text{grad } U$ where $U = -x^3 + xy - y^2z^2$.
$\text{div } \mathbf{F} = 6x + 2z^2 + 2y^2 \neq 0$, hence not solenoidal.

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \left[x^3 - xy + y^2z^2\right]_{(0,0,0)}^{(1,2,3)} = 1 - 2 + 36 = 35$$

■ **63** $\mathbf{F} = (2t^3, -t^3, t^4)$, $\mathbf{r} = (t^2, 2t, t^3)$, $\mathrm{d}\mathbf{r} = (2t, 2, 3t^2)\,\mathrm{d}t$

$$\mathbf{F} \times \mathrm{d}\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} \mathrm{d}t$$

$$= [(-3t^5 - 2t^4)\mathbf{i} + (-4t^5)\mathbf{j} + (4t^3 + 2t^4)\mathbf{k}]\,\mathrm{d}t$$

$$\int_C \mathbf{F} \times \mathrm{d}\mathbf{r} = \int_0^1 [(-3t^5 - 2t^4)\mathbf{i} - 4t^5\mathbf{j} + (4t^3 + 2t^4)\mathbf{k}]\,\mathrm{d}t$$

$$= \left[\left(-\frac{1}{2}-\frac{2}{5}\right)\mathbf{i} - \frac{4}{6}\mathbf{j} + \left(1+\frac{2}{5}\right)\mathbf{k}\right]$$

$$= -\frac{9}{10}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{7}{5}\mathbf{k}$$

■ **64**

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3x+y & -x & y-z \\ 2 & -3 & 1 \end{vmatrix} = (3y-3z-x)\mathbf{i} + (y-2z-3x)\mathbf{j} + (-3y-7x)\mathbf{k}$$

$$\mathbf{r} = (2\cos\theta, 2\sin\theta, 0)$$

$$\mathrm{d}\mathbf{r} = (-2\sin\theta, 2\cos\theta, 0)\ \mathrm{d}\theta$$

On circle, $z = 0$ and

$$\mathbf{A}\times\mathbf{B} = (6\sin\theta - 2\cos\theta)\mathbf{i} + (2\sin\theta - 6\cos\theta)\mathbf{j} - (6\sin\theta + 14\cos\theta)\mathbf{k}$$

$$(\mathbf{A}\times\mathbf{B})\times\mathrm{d}\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6\sin\theta - 2\cos\theta & 2\sin\theta - 6\cos\theta & -6\sin\theta - 14\cos\theta \\ -2\sin\theta & 2\cos\theta & 0 \end{vmatrix} \mathrm{d}\theta$$

$$\int_C (\mathbf{A}\times\mathbf{B})\times\mathrm{d}\mathbf{r} = \int_0^{2\pi} \{(6\sin\theta + 14\cos\theta)2\cos\theta\mathbf{i} + (6\sin\theta + 14\cos\theta)2\sin\theta\mathbf{j}$$

$$+ [(6\sin\theta - 2\cos\theta)(2\cos\theta) + (2\sin\theta - 6\cos\theta)(2\sin\theta)]\}\ \mathrm{d}\theta$$

$$\int_0^{2\pi} \sin\theta\cos\theta\ \mathrm{d}\theta = \int_0^{2\pi} \frac{1}{2}\sin 2\theta\ \mathrm{d}\theta = \left[\frac{1}{4}\cos 2\theta\right]_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin^2\theta\ \mathrm{d}\theta = \int_0^{2\pi} \cos^2\theta\ \mathrm{d}\theta = \pi$$

$$\int_C (\mathbf{A}\times\mathbf{B})\times\mathrm{d}\mathbf{r} = 28\pi\mathbf{i} + 12\pi\mathbf{j} + 0\mathbf{k}$$

Exercises 3.4.4

- **65(a)**

$$\begin{aligned}\int_0^3 \int_1^2 xy(x+y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^3 \left[\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\right]_1^2 \mathrm{d}x \\ &= \int_0^3 \left[\frac{1}{2}x^2(4-1) + \frac{1}{3}x(8-1)\right] \mathrm{d}x \\ &= \left[\frac{1}{2}x^3 + \frac{7}{6}x^2\right]_0^3 \\ &= \frac{27}{2} + \frac{21}{2} = 24\end{aligned}$$

- **65(b)**

$$\begin{aligned}\int_2^3 \int_1^5 x^2y\,\mathrm{d}y\,\mathrm{d}x &= \int_2^3 x^2\,\mathrm{d}x \int_1^5 y\,\mathrm{d}y \\ &= \left[\frac{1}{3}x^3\right]_2^3 \left[\frac{1}{2}y^2\right]_1^5 \\ &= \frac{1}{3}(27-8)\frac{1}{2}(25-1) \\ &= 76\end{aligned}$$

- **65(c)**

$$\begin{aligned}\int_{-1}^1 \int_{-2}^2 (2x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x &= \int_{-1}^1 \left[2x^2y + \frac{1}{3}y^3\right]_{-2}^2 \mathrm{d}x \\ &= \int_{-1}^1 \left(8x^2 + \frac{16}{3}\right) \mathrm{d}x = \frac{16}{3} + \frac{32}{3} = 16\end{aligned}$$

- **66**

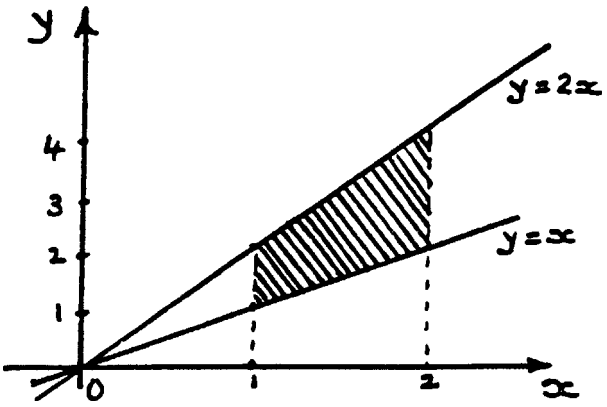
$$\begin{aligned}\int_1^2 \int_0^2 \frac{x^2}{y}\,\mathrm{d}x\,\mathrm{d}y &= \int_1^2 \frac{1}{y}\,\mathrm{d}y \int_0^2 x^2\,\mathrm{d}x \\ &= (\ln 2)\left(\frac{8}{3}\right) = \frac{8}{3}\ln 2\end{aligned}$$

■ **67**

$$\begin{aligned}\int_0^1 \int_0^{1-x} (x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \left[x^2y + \frac{1}{3}y^3\right]_0^{1-x}\,\mathrm{d}x \\ &= \int_0^1 x^2(1-x) + \frac{1}{3}(1-x)^3\,\mathrm{d}x \\ &= \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(1-x)^4\right]_0^1 \\ &= \left[\frac{1}{3} - \frac{1}{4}\right] - \left[-\frac{1}{12}\right] = \frac{1}{6}\end{aligned}$$

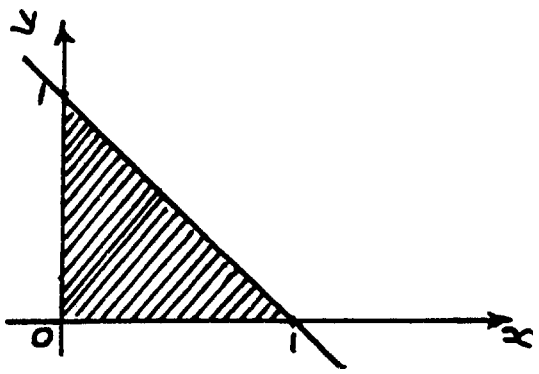
■ **68(a)** $\displaystyle\int_1^2 \mathrm{d}x \int_x^{2x} \frac{\mathrm{d}y}{x^2+y^2}$

$$\begin{aligned}&= \int_1^2 \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_x^{2x}\,\mathrm{d}x \\ &= \int_1^2 \frac{1}{x}(\tan^{-1}2 - \tan^{-1}1)\,\mathrm{d}x \\ &= (\tan^{-1}2 - \tan^{-1}1)[\ln x]_1^2 \\ &= \tan^{-1}\left(\frac{1}{3}\right)\ln 2\end{aligned}$$

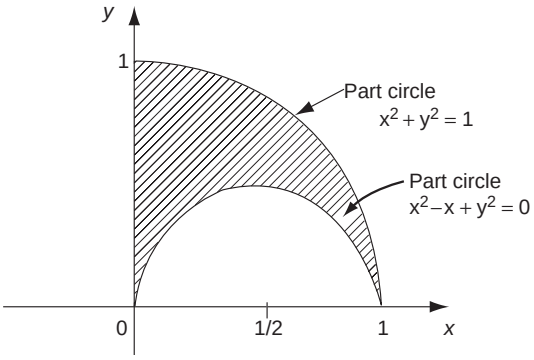


■ **68(b)** $\displaystyle\int_0^1 \mathrm{d}x \int_0^{1-x} (x+y)\,\mathrm{d}y$

$$\begin{aligned}&= \int_0^1 \left[xy + \frac{1}{2}y^2\right]_0^{1-x}\,\mathrm{d}x \\ &= \int_0^1 \left[x(1-x) + \frac{1}{2}(1-x)^2\right]\,\mathrm{d}x \\ &= \left[-\frac{x^2}{2} - \frac{x^3}{3} - \frac{1}{6}(1-x)^3\right]_0^1 \\ &= \left[\frac{1}{2} - \frac{1}{3} + \frac{1}{6}\right] = \frac{1}{3}\end{aligned}$$



■ **68(c)** $\displaystyle\int_0^1 \mathrm{d}x \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2-y^2}}\,\mathrm{d}y$



$$= \int_0^1 \left[\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right]_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \mathrm{d}x$$

$$= \int_0^1 \left[\sin^{-1} 1 - \sin^{-1}\sqrt{\frac{x-x^2}{1-x^2}}\right] \mathrm{d}x$$

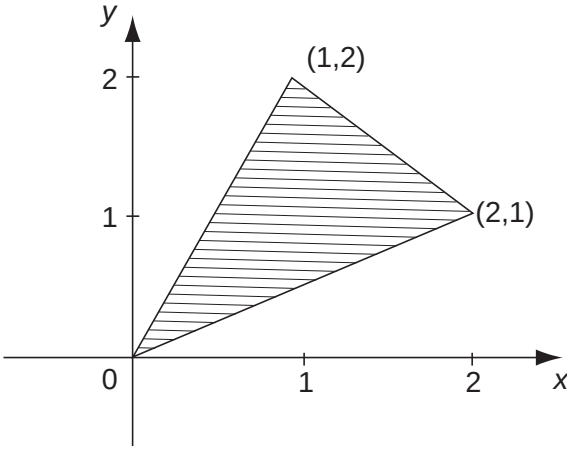
$$= \int_0^1 \left[\sin^{-1} 1 - \sin^{-1}\sqrt{\frac{x}{1+x}}\ \right] \mathrm{d}x$$

$$= \frac{\pi}{2} - \left[x\sin^{-1}\sqrt{\frac{x}{1+x}}\ \right]_0^1 + \frac{1}{2}\int_0^1 \frac{\sqrt{x}}{(1+x)}\,\mathrm{d}x$$

$$= \frac{\pi}{2} - \sin^{-1}\frac{1}{\sqrt{2}} + \left[\sqrt{x} - \tan^{-1}\sqrt{x}\right]_0^1, \text{using substitution } x = \tan^2\theta$$

$$= \frac{\pi}{4} + \left(1 - \frac{\pi}{4}\right) = 1$$

■ **69**



$$\int\int \sin\frac{1}{2}\pi(x+y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \mathrm{d}x \int_{x/2}^{x} \sin\frac{1}{2}\pi(x+y)\,\mathrm{d}y + \int_1^2 \mathrm{d}x \int_{x/2}^{3-x} \sin\frac{1}{2}\pi(x+y)\,\mathrm{d}y$$

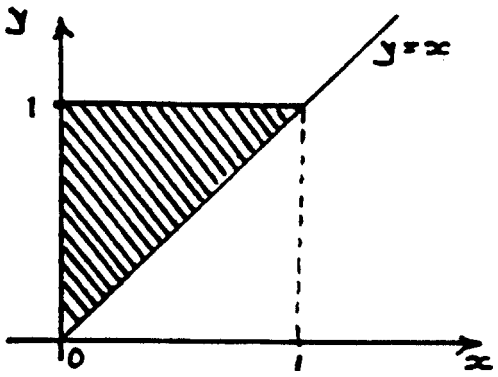
$$= \int_0^1 \left[-\frac{2}{\pi}\cos\frac{1}{2}\pi(x+y)\right]_{x/2}^{x} \mathrm{d}x + \int_1^2 \left[-\frac{2}{\pi}\cos\frac{1}{2}\pi(x+y)\right]_{x/2}^{3-x} \mathrm{d}x$$

$$= \int_0^1 \left[\frac{2}{\pi}\cos\frac{3}{4}\pi x - \frac{2}{\pi}\cos\pi x\right] \mathrm{d}x + \int_1^2 \left[\frac{2}{\pi}\cos\frac{3}{4}\pi x - \frac{2}{\pi}\cos\frac{3\pi}{2}\right] \mathrm{d}x$$

$$= \left[\frac{8}{3\pi^2}\sin\frac{3}{4}\pi x - \frac{2}{\pi^2}\sin\pi x\right]_0^1 + \left[\frac{8}{3\pi^2}\sin\frac{3}{4}\pi x\right]_1^2$$
$$= \frac{8}{3\pi^2}\sin\frac{3}{4}\pi + \frac{8}{3\pi^2}\sin\frac{3\pi}{2} - \frac{8}{3\pi^2}\sin\frac{3\pi}{4}$$
$$= -\frac{8}{3\pi^2}$$

■ **70(a)** $\displaystyle\int_0^1 \mathrm{d}x \int_x^1 \frac{xy}{\sqrt{1+y^4}}\,\mathrm{d}y$

$$= \int_0^1 \mathrm{d}y \int_0^y \frac{xy}{\sqrt{1+y^4}}\,\mathrm{d}x$$
$$= \int_0^1 \left[\frac{\frac{1}{2}x^2y}{\sqrt{1+y^4}}\right]_0^y \mathrm{d}y$$
$$= \int_0^1 \frac{\frac{1}{2}y^3}{\sqrt{1+y^4}}\,\mathrm{d}y = \left[\frac{1}{8}\cdot\frac{2}{1}\sqrt{1+y^4}\right]_0^1$$
$$= \frac{1}{4}(\sqrt{2}-1)$$

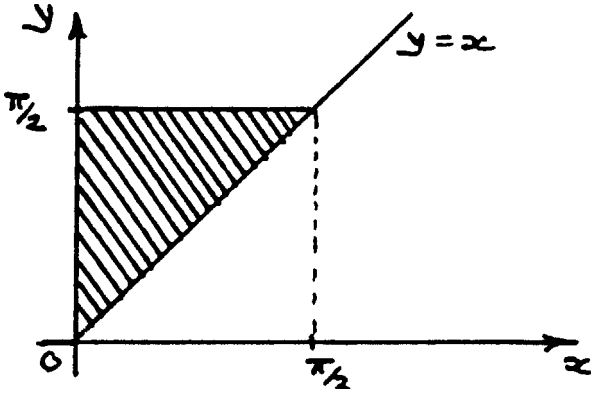


■ **70(b)** $\displaystyle\int_0^{\pi/2} \mathrm{d}y \int_0^y \cos 2y\sqrt{1-k^2\sin^2 x}\,\mathrm{d}x$

$$= \int_0^{\pi/2} \mathrm{d}x \int_x^{\pi/2} \cos 2y\sqrt{1-k^2\sin^2 x}\,\mathrm{d}y$$
$$= \int_0^{\pi/2} \left[\frac{1}{2}\sin 2y\sqrt{1-k^2\sin^2 x}\right]_x^{\pi/2} \mathrm{d}x$$
$$= \int_0^{\pi/2} -\sin x\cos x\sqrt{1-k^2\sin^2 x}\,\mathrm{d}x$$
$$= \int_0^{-1} \frac{1}{2}\sqrt{1+k^2t}\,\mathrm{d}t$$

(Let $t = -\sin^2 x$, then $\mathrm{d}t = -2\sin x\cos x\,\mathrm{d}x$)

$$= \left[\frac{1}{3k^2}(1+k^2t)^{3/2}\right]_0^{-1}$$
$$= \frac{1}{3k^2}((1-k^2)^{3/2}-1)$$



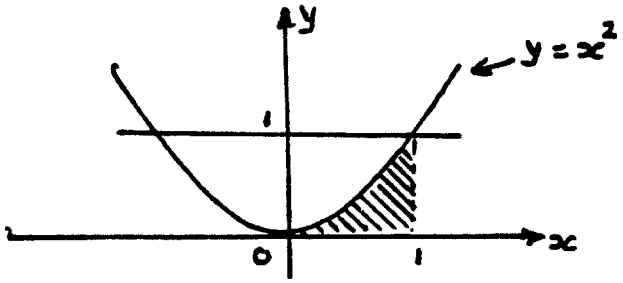
■ **71** $\displaystyle\int_0^1 \mathrm{d}y \int_{\sqrt{y}}^1 \frac{\mathrm{d}x}{\sqrt{y(1+x^2)}}$

$$= \int_0^1 \mathrm{d}x \int_0^{x^2} \frac{1}{\sqrt{1+x^2}} \frac{1}{\sqrt{y}}\, \mathrm{d}y$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} [2\sqrt{y}\,]_0^{x^2}\, \mathrm{d}x$$

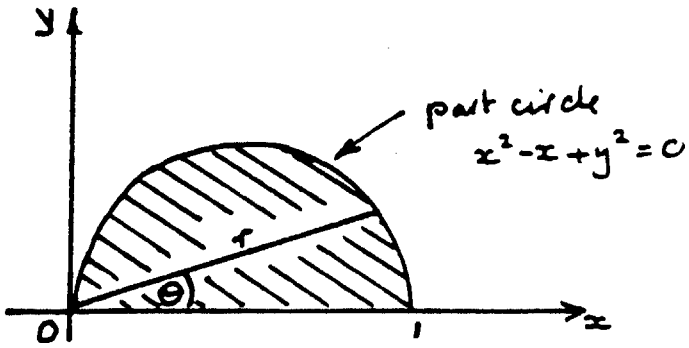
$$= \int_0^1 \frac{2x}{\sqrt{1+x^2}}\, \mathrm{d}x$$

$$= [2\sqrt{1+x^2}]_0^1 = 2(\sqrt{2}-1)$$



■ **72**

$$\int_0^1 \mathrm{d}x \int_0^{\sqrt{x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, \mathrm{d}y$$



Equation of circle in polar coordinates is $r = \cos\theta$

$$\int_0^1 \mathrm{d}x \int_0^{\sqrt{x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, \mathrm{d}y = \int_0^{\pi/2} \int_0^{\cos\theta} \frac{r\cos\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} r\, \mathrm{d}r\, \mathrm{d}\theta$$

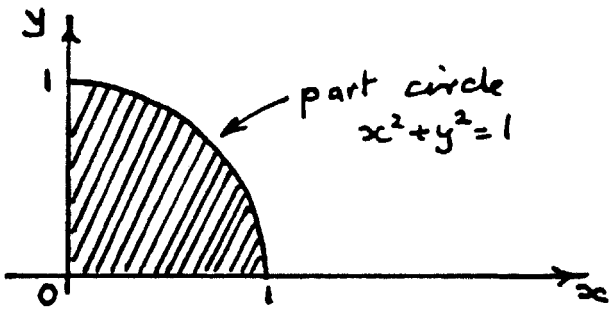
$$= \int_0^{\pi/2} \int_0^{\cos\theta} (\cos\theta) r\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[\frac{1}{2}r^2\right]_0^{\cos\theta} \mathrm{d}\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\cos^3\theta\, \mathrm{d}\theta = \frac{1}{2}.\frac{2}{3}\int_0^{\pi/2} \cos\theta\, \mathrm{d}\theta$$

$$= \frac{1}{3}$$

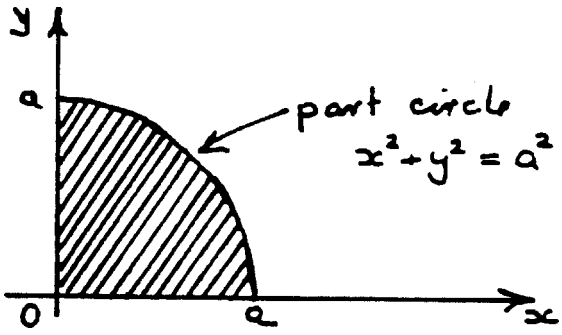
■ **73** $\displaystyle\int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \frac{x+y}{\sqrt{x^2+y^2}}\,\mathrm{d}y$



Change to polar coordinates

$$\begin{aligned}\int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \frac{x+y}{\sqrt{x^2+y^2}}\,\mathrm{d}y &= \int_0^{\pi/2}\int_0^1 \frac{r\cos\theta + r\sin\theta}{r} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^{\pi/2} (\cos\theta+\sin\theta)\,\mathrm{d}\theta \int_0^1 r\,\mathrm{d}r \\ &= [\sin\theta - \cos\theta]_0^{\pi/2}\left[\frac{1}{2}r^2\right]_0^1 \\ &= 1\end{aligned}$$

■ **74** $\displaystyle\int\int \frac{x+y}{x^2+y^2+a^2}\,\mathrm{d}x\,\mathrm{d}y$ over first quadrant of circle



using polar coordinates

$$\begin{aligned}\int\int \frac{x+y}{x^2+y^2+a^2}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\pi/2}\int_0^a \frac{r\cos\theta + r\sin\theta}{r^2+a^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^{\pi/2} (\cos\theta+\sin\theta)\,\mathrm{d}\theta \int_0^a \frac{r^2}{r^2+a^2}\,\mathrm{d}r \\ &= 2\int_0^a \left(1 - \frac{a^2}{r^2+a^2}\right)\mathrm{d}r \\ &= 2\left[r - a\tan^{-1}\frac{r}{a}\right]_0^a = 2a\left(1-\frac{\pi}{4}\right)\end{aligned}$$

■ **75** Using polar coordinates, parabola becomes

$$\begin{aligned} r^2\sin^2\theta &= 4 - 4r\cos\theta \\ r^2 &= 4 - 4r\cos\theta + r^2\cos^2\theta \\ &= (2 - r\cos\theta)^2 \\ r &= (2 - r\cos\theta), \qquad \text{positive root because } r = 2 \text{ at } \theta = \tfrac{\pi}{2} \\ &= \frac{2}{1+\cos\theta} \end{aligned}$$

$$\begin{aligned} \int\int \frac{x^2-y^2}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\pi/2}\int_0^{\frac{2}{1+\cos\theta}} (\cos^2\theta - \sin^2\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^{\pi/2} 2(\cos^2\theta - \sin^2\theta)\frac{1}{(1+\cos\theta)^2}\,\mathrm{d}\theta \\ &= \int_0^{\pi/2} \frac{2(2\cos^2\theta - 1)}{(1+\cos\theta)^2}\,\mathrm{d}\theta \\ &= (6\pi - 20)/3 = -0.3835 \end{aligned}$$

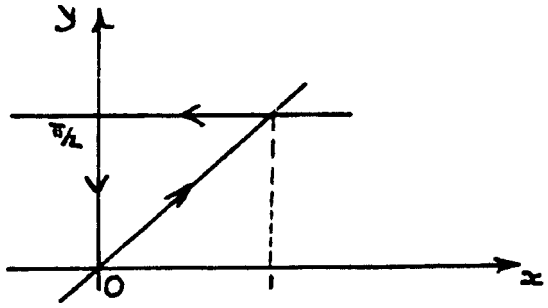
(use the substitution $t = \tan\frac{1}{2}\theta$).

■ **76** Circles are $r = a\cos\theta$, $r = b\sin\theta$ and intersect at $\theta = \tan^{-1}\frac{a}{b}$.

$$\begin{aligned} \int\int \frac{(x^2+y^2)^2}{(xy)^2}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\tan^{-1}\frac{a}{b}}\int_0^{b\sin\theta} \frac{1}{\sin^2\theta\cos^2\theta} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &\quad + \int_{\tan^{-1}\frac{a}{b}}^{\pi/2}\int_0^{a\cos\theta} \frac{1}{\sin^2\theta\cos^2\theta} r\,\mathrm{d}r\,\mathrm{d}\theta \end{aligned}$$

$$\begin{aligned} &= \int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{2}\frac{b^2\sin^2\theta}{\sin^2\theta\cos^2\theta}\,\mathrm{d}\theta + \int_{\tan^{-1}\frac{a}{b}}^{\pi/2} \frac{1}{2}\frac{a^2\cos^2\theta}{\sin^2\theta\cos^2\theta}\,\mathrm{d}\theta \\ &= \int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{2}b^2\sec^2\theta\,\mathrm{d}\theta + \int_{\tan^{-1}\frac{a}{b}}^{\pi/2} \frac{1}{2}a^2\mathrm{cosec}^2\theta\,\mathrm{d}\theta \\ &= \left[\frac{1}{2}b^2\tan\theta\right]_0^{\tan^{-1}\frac{a}{b}} + \left[-\frac{1}{2}a^2\cot\theta\right]_{\tan^{-1}\frac{a}{b}}^{\pi/2} \\ &= \frac{1}{2}ab + \frac{1}{2}ab = ab \end{aligned}$$

Exercises 3.4.6



■ **77** $\displaystyle\oint_C [\sin y\,\mathrm{d}x + (x - \cos y)\,\mathrm{d}y]$

$$= \int_0^1 \left[\sin\frac{1}{2}\pi x + (x - \cos\frac{1}{2}\pi x)\frac{1}{2}\pi\right]\,\mathrm{d}x + \int_1^0 \left[\sin\frac{1}{2}\pi\right]\,\mathrm{d}x + \int_{\pi/2}^0 [-\cos y]\,\mathrm{d}y$$

$$= \left[-\frac{2}{\pi}\cos\frac{1}{2}\pi x - \frac{1}{4}\pi x^2 - \sin\frac{1}{2}\pi x\right]_0^1 + \left[x\sin\frac{1}{2}\pi\right]_1^0 + [-\sin y]_{\pi/2}^0$$

$$= -1 + \frac{2}{\pi} + \frac{1}{4}\pi - 1 + 1$$

$$= -1 + \frac{2}{\pi} + \frac{\pi}{4}$$

$$\oint_C [\sin y\,\mathrm{d}x + (x - \cos y)\,\mathrm{d}y] = \iint_A [1 - \cos y]\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \mathrm{d}x \int_{\frac{1}{2}\pi x}^{\pi/2} (1 - \cos y)\,\mathrm{d}y$$

$$= \int_0^1 \left(\frac{\pi}{2} - \frac{\pi}{2}x - 1 + \sin\frac{1}{2}\pi x\right)\,\mathrm{d}x$$

$$= \frac{\pi}{2} - \frac{\pi}{4} - 1 + \frac{2}{\pi} = \frac{\pi}{4} - 1 + \frac{2}{\pi}$$

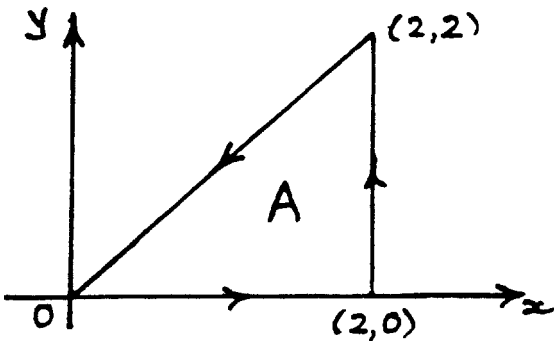
■ **78** $\displaystyle\oint [(x^2 y - y)\,\mathrm{d}x + (x + y^2)\,\mathrm{d}y]$

$$= \iint_A (1 - x^2 + 1)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^2 \mathrm{d}x \int_0^x (2 - x^2)\,\mathrm{d}y$$

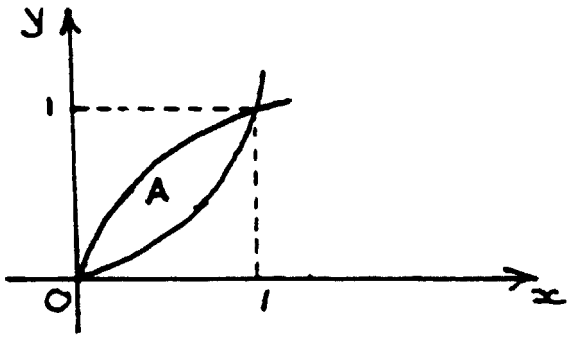
$$= \int_0^2 (2x - x^3)\,\mathrm{d}x = \left[x^2 - \frac{1}{4}x^4\right]_0^2$$

$$= 4 - 4 = 0$$



■ **79** $\displaystyle\oint_C (xy\,\mathrm{d}x + x\,\mathrm{d}y)$

$$= \int_0^1 (x^3 + 2x^2)\,\mathrm{d}x + \int_1^0 \left(x^{3/2} + \frac{1}{2}x^{1/2}\right)\mathrm{d}x = \frac{1}{4} + \frac{2}{3} - \left[\frac{2}{5} + \frac{1}{3}\right] = \frac{11}{60}$$



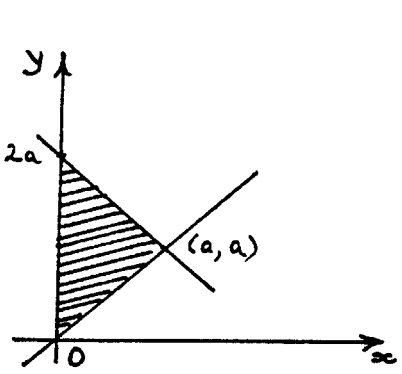
$$\oint_C (xy\,\mathrm{d}x + x\,\mathrm{d}y) = \iint_A (1-x)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^1 \mathrm{d}x \int_{x^2}^{\sqrt{x}} (1-x)\,\mathrm{d}y$$
$$= \int_0^1 (1-x)(\sqrt{x} - x^2)\,\mathrm{d}x$$
$$= \int_0^1 \sqrt{x} - x^{3/2} - x^2 + x^3\,\mathrm{d}x$$
$$= \frac{2}{3} - \frac{2}{5} - \frac{1}{3} + \frac{1}{4} = \frac{11}{60}$$

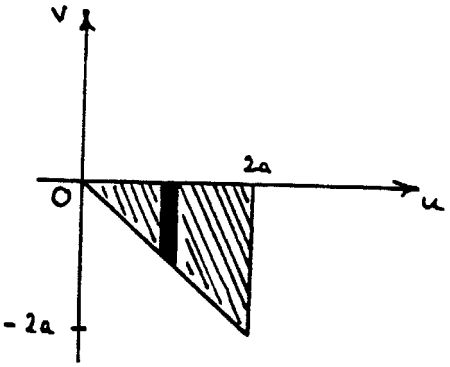
■ **80**

$$\oint_c \left[(\mathrm{e}^x - 3y^2)\,\mathrm{d}x + (\mathrm{e}^y + 4x^2)\,\mathrm{d}y\right] = \iint_A (8x + 6y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^{2\pi} \mathrm{d}\theta \int_0^2 (8r\cos\theta + 6r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \int_0^{2\pi} (8\cos\theta + 6\sin\theta)\,\mathrm{d}\theta \int_0^2 r\,\mathrm{d}r$$
$$= 0(2) = 0$$

■ **81** $\displaystyle\int_0^a \mathrm{d}x \int_x^{2a-x} \frac{y-x}{4a^2 + (y+x)^2}\,\mathrm{d}y = I$

$$\left.\begin{aligned} u &= x + y \\ v &= x - y \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} x &= (u+v)/2 \\ y &= (u-v)/2 \end{aligned} \quad \text{and} \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$





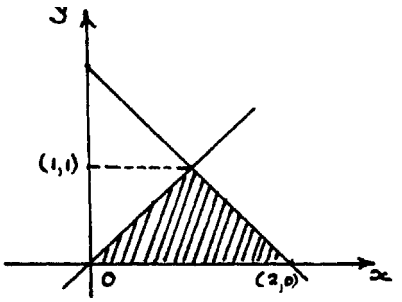
$$\begin{aligned} y = x \quad &\Rightarrow \quad u - v = u + v \quad \Rightarrow \quad v = 0 \\ y = 2a - x \quad &\Rightarrow \quad u - v = 4a - u - v \quad \Rightarrow \quad u = 2a \\ x = 0 \quad &\Rightarrow \quad u = -v \end{aligned}$$

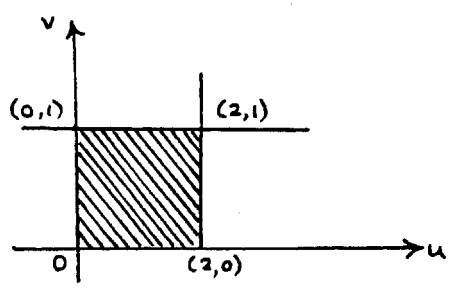
$$(0,0) \;\Rightarrow\; (0,0), \quad (a,a) \;\Rightarrow\; (2a,0), \quad (0,2a) \;\Rightarrow\; (2a,-2a)$$

$$\begin{aligned} I &= \int_0^{2a} \mathrm{d}u \int_{-u}^{0} \frac{-v}{4a^2+u^2} \left| -\frac{1}{2} \right| \mathrm{d}v = \frac{1}{4} \int_0^{2a} \frac{u^2}{4a^2+u^2}\, \mathrm{d}u \\ &= \frac{1}{4} \int_0^{2a} 1 - \frac{4a^2}{4a^2+u^2}\, \mathrm{d}u \\ &= \frac{1}{4} \left[u - 2a \tan^{-1} \frac{u}{2a} \right]_0^{2a} = \frac{a}{2} \left(1 - \frac{\pi}{4} \right) \end{aligned}$$

■ **82** $\displaystyle\int_0^1 \mathrm{d}y \int_y^{2-y} \frac{x+y}{x^2} \mathrm{e}^{x+y}\, \mathrm{d}x$

$$\left.\begin{matrix} u = x + y \\ v = y/x \end{matrix}\right\} \quad \Rightarrow \quad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -\frac{y}{x^2} \\ 1 & \frac{1}{x} \end{vmatrix} = \frac{1}{x^2}(x+y)$$





$$\begin{aligned} y = x \quad &\Rightarrow \quad v = 1 \\ y = 0 \quad &\Rightarrow \quad v = 0 \\ y = 2 - x \quad &\Rightarrow \quad u = 2 \end{aligned}$$

$$\int_0^1 \mathrm{d}y \int_y^{2-y} \frac{x+y}{x^2} \mathrm{e}^{x+y}\, \mathrm{d}x = \int_0^2 \mathrm{d}u \int_0^1 \mathrm{e}^u\, \mathrm{d}v$$
$$= \mathrm{e}^2 - 1$$

Exercises 3.4.8

■ **83** Surface area $= \displaystyle\iint_A \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \mathrm{d}x\, \mathrm{d}y$

where A is the domain $x^2 + y^2 \le 2,\ z = 0.$

$$z = 2 - x^2 - y^2 \ \Rightarrow\ \frac{\partial z}{\partial x} = -2x,\ \frac{\partial z}{\partial y} = -2y$$

Surface area $= \displaystyle\iint_A \sqrt{1 + 4x^2 + 4y^2}\, \mathrm{d}x\, \mathrm{d}y$

Set $x = r\cos\theta,\ y = r\sin\theta$, then

$$\begin{aligned} \text{Surface area} &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{2}} \sqrt{1 + 4r^2}\ r\, \mathrm{d}r \\ &= 2\pi \left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_0^{\sqrt{2}} \\ &= 2\pi \left[\frac{1}{12}(27 - 1)\right] \\ &= 52\pi/12 = 13\pi/3 \end{aligned}$$

■ **84(a)** Direction cosines of normal to S are $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$, so that

$$\iint_S (x^2 + y^2)\, \mathrm{d}S = \iint_A (x^2 + y^2) \frac{\mathrm{d}x\, \mathrm{d}y}{2/3}$$

where A is the area between $y = 6-2x$ and $y = 2-2x$ lying between $y = 0$ and $y = 3$.

$$\text{Thus} \iint_S (x^2+y^2)\,\mathrm{d}S = \int_0^3 \mathrm{d}y \int_{\frac{2-y}{2}}^{\frac{6-y}{2}} \frac{3}{2}(x^2+y^2)\,\mathrm{d}x$$

$$= \int_0^3 \frac{3}{2}\left[\frac{x^3}{3} + y^2 x\right]_{\frac{2-y}{2}}^{\frac{6-y}{2}} \mathrm{d}y$$

$$= \int_0^3 \left(13 - 6y + \frac{15}{4}y^2\right)\mathrm{d}y = \frac{183}{4}$$

84(b) $$\iint_S z\,\mathrm{d}S = \int_0^1 \mathrm{d}x \int_{-\sqrt{x-x^2}}^{+\sqrt{x-x^2}} z\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}y$$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z} \quad \text{and} \quad x^2+y^2+z^2 = 1$$

$$\iint_S z\,\mathrm{d}S = \int_0^1 \mathrm{d}x \int_{-\sqrt{x-x^2}}^{+\sqrt{x-x^2}} 1\,\mathrm{d}y$$

$$= 2\int_0^1 \sqrt{x-x^2}\,\mathrm{d}x$$

$$= \frac{\pi}{4}$$

(Use the substitution $x = \frac{1}{2} + \frac{1}{2}\sin t$. Alternatively, recognize area of circle of radius $\frac{1}{2}$.)

■ **85(a)**

$$\mathbf{v} = (xy, -x^2, x+z)$$

$$\mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}S = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)\,\mathrm{d}S$$

$$\Rightarrow \quad \iint_S \mathbf{v}\cdot\,\mathrm{d}\mathbf{S} = \iint_S \left(\frac{2}{3}xy - \frac{2}{3}x^2 + \frac{1}{3}(x+z)\right)\,\mathrm{d}S$$

$$= \int_0^3 \mathrm{d}x \int_0^{3-x} [xy - 2x^2 + (x + 6 - 2x - 2y)]\,\mathrm{d}y$$

$$= \int_0^3 \left[\frac{1}{2}xy^2 - 2x^2y - xy + 6y - y^2\right]_0^{3-x}\,\mathrm{d}x$$

$$= \int_0^3 \left\{\frac{x}{2}(3-x)^2 - 2x^2(3-x) - x(3-x) + 6(3-x)\right\}\,\mathrm{d}x$$

$$= 27/4$$

85(b) Use cylindrical polar coordinates, then $\mathrm{d}\mathbf{S} = (\mathbf{i}\cos\phi + \mathbf{j}\sin\phi)\,\mathrm{d}\phi\,\mathrm{d}z$ on cylinder and $\mathbf{v} = (3y, 2x^2, z^3) = (3\sin\phi, 2\cos^2\phi, z^3)$

$$\iint_S \mathbf{v}\cdot\,\mathrm{d}\mathbf{S}$$

$$= \int_0^{2\pi} \mathrm{d}\phi \int_0^1 (3\sin\phi\cos\phi + 2\cos^2\phi\sin\phi)\,\mathrm{d}z$$

$$= \left[\frac{3}{2}\sin^2\phi - \frac{2}{3}\cos^3\phi\right]_0^{2\pi} = 0$$

■ **86** $\displaystyle\iint_S z^2\,\mathrm{d}S = \iint_A z^2/\sqrt{(1-x^2-y^2)}\,\mathrm{d}x\,\mathrm{d}y$

where $x^2+y^2+z^2 = 1$ and A is the interior of the circle $x^2+y^2 = 1, \quad z = 0$

$$\iint_S z^2\,\mathrm{d}S = \iint_A \sqrt{1-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \sqrt{1-r^2}\,r\,\mathrm{d}r$$

$$= 2\pi\left[-\frac{1}{3}(1-r^2)^{3/2}\right]_0^1 = \frac{2\pi}{3}$$

■ **87(a)** $\displaystyle\iint_S 1 \, \mathrm{d}S = \iint_A 1\sqrt{1+4x^2+4y^2} \, \mathrm{d}x \, \mathrm{d}y$

where A is the interior of the circle $x^2 + y^2 = 2$

$$\iint_S \mathrm{d}S = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{2}} \sqrt{1+4r^2} \; r \, \mathrm{d}r$$

$$= 2\pi \left[\frac{1}{12}(1+4r^2)^{3/2} \right]_0^{\sqrt{2}} = \frac{2\pi}{12}(27-1)$$

$$= 13\pi/3 \qquad \text{Surface Area}$$

87(b)

$$\iint_S (x^2+y^2) \, \mathrm{d}S = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{2}} r^2\sqrt{1+4r^2} \; r \, \mathrm{d}r$$

$$= \frac{2\pi}{16} \left[\frac{1}{5}(1+4r^2)^{5/2} - \frac{1}{3}(1+4r)^{3/2} \right]_0^{\sqrt{2}}$$

$$= \frac{\pi}{8} \left[\frac{1}{5}(243-1) - \frac{1}{3}(27-1) \right] = 149\pi/30$$

2nd moment of surface area about z-axis.

87(c)

$$\iint_S z \, \mathrm{d}S = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{2}} (2-r^2)\sqrt{1+4r^2} \; r \, \mathrm{d}r$$

$$= 2\int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{2}} \sqrt{1+4r^2} \; r \, \mathrm{d}r - \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{2}} r^2\sqrt{1+4r^2} \, \mathrm{d}r$$

$$= \frac{26\pi}{3} - \frac{149\pi}{30} = \frac{111\pi}{30} = \frac{37\pi}{10}$$

■ **88** Direction cosines of normal to S are $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ so that

$$\iint_S \mathrm{d}S = \iint_A \frac{\mathrm{d}x \, \mathrm{d}y}{2/3}$$

where A is the interior of $x^2 + y^2 = 64$ lying in the first quadrant.

$$\iint_S \mathrm{d}S = \int_0^{\pi/2} \mathrm{d}\theta \int_0^8 \frac{3}{2} r \, \mathrm{d}r = \frac{\pi}{2} \times \frac{3}{2} \left[\frac{1}{2} r^2\right]_0^8 = 24\pi$$

■ **89** $\displaystyle\iint_S \mathrm{d}S = \iint_A \sqrt{1 + \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2} \, \mathrm{d}x \, \mathrm{d}y$

where A is the annulus between $x^2 + y^2 = 4$ and $x^2 + y^2 = 12$

$$\begin{aligned}\iint_S \mathrm{d}S &= \int_0^{2\pi} \mathrm{d}\theta \int_2^{\sqrt{12}} \sqrt{1 + \frac{1}{4} r^2} \; r \, \mathrm{d}r \\ &= 2\pi \left[\frac{4}{3}\left(1 + \frac{1}{4} r^2\right)^{3/2}\right]_2^{\sqrt{12}} = \frac{8\pi}{3}[4^{3/2} - 2^{3/2}] \\ &= \frac{16\pi}{3}(4 - 2^{1/2})\end{aligned}$$

■ **90** Using cylindrical polar coordinates, $\mathrm{d}\mathbf{S} = (4\mathbf{i}\cos\phi + 4\mathbf{j}\sin\phi) \, \mathrm{d}\phi \, \mathrm{d}z$

$$\text{and} \qquad \mathbf{V} = z\mathbf{i} + 2\cos\phi\mathbf{j} - 12\sin^2\phi z\mathbf{k}$$

$$\text{Thus} \qquad \mathbf{V} \cdot \mathrm{d}\mathbf{S} = (4z\cos\phi + 8\sin\phi\cos\phi) \, \mathrm{d}\phi \, \mathrm{d}z$$

$$\begin{aligned}\text{and} \qquad \iint_S \mathbf{V} \cdot \mathrm{d}\mathbf{S} &= \int_0^{\pi/2} \mathrm{d}\phi \int_0^5 (4z\cos\phi + 8\sin\phi\cos\phi) \, \mathrm{d}z \\ &= \int_0^{\pi/2} (50\cos\phi + 40\sin\phi\cos\phi) \, \mathrm{d}\phi \\ &= [50\sin\phi + 40\sin^2\phi]_0^{\pi/2} = 90\end{aligned}$$

■ **91** $\mathbf{F} = y\mathbf{i} + (x - 2xz)\mathbf{j} - xy\mathbf{k}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x - 2xz & -xy \end{vmatrix} = (x, y, -2z)$$

On sphere, $x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$

and $\mathrm{d}\mathbf{S} = a^2(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi$

$$\begin{aligned}\text{curl } \mathbf{F}\cdot\ \mathrm{d}\mathbf{S} &= a^3(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi - 2\cos^2\theta)\sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi \\ &= a^3(\sin^2\theta - 2\cos^2\theta)\sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi \\ \iint_S \text{curl } \mathbf{F}\cdot\ \mathrm{d}\mathbf{S} &= \int_0^{2\pi}\ \mathrm{d}\phi\ \int_0^{\pi} a^3[\sin^3\theta - 2\cos^2\theta\sin\theta]\ \mathrm{d}\theta \\ &= \int_0^{2\pi} a^3\ \mathrm{d}\phi \int_0^{\pi}\left[\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta - 2\cos^2\theta\sin\theta\right]\ \mathrm{d}\theta \\ &= \left[a^3\phi\right]_0^{2\pi}\left[-\frac{3}{4}\cos\theta + \frac{1}{12}\cos 3\theta + \frac{2}{3}\cos^3\theta\right]_0^{\pi} = 0\end{aligned}$$

Exercises 3.4.10

- **92(a)**

$$\begin{aligned}\int_0^1\ \mathrm{d}x\int_0^2\ \mathrm{d}y\int_1^3 x^2yz\ \mathrm{d}z &= \int_0^1 x^2\ \mathrm{d}x\int_0^2 y\ \mathrm{d}y\int_1^3 z\ \mathrm{d}z \\ &= \frac{1}{3}.\frac{1}{2}(4)\frac{1}{2}(3^2-1) = \frac{8}{3}\end{aligned}$$

92(b)

$$\begin{aligned}\int_0^2\int_1^3\int_2^4 xyz^2\ \mathrm{d}z\ \mathrm{d}y\ \mathrm{d}x &= \int_0^2\ \mathrm{d}x\int_1^3\ \mathrm{d}y\int_2^4 xyz^2\ \mathrm{d}z \\ &= \int_0^2 x\ \mathrm{d}x\int_1^3 y\ \mathrm{d}y\int_2^4 z^2\ \mathrm{d}z \\ &= \frac{1}{2}(4)\frac{1}{2}(3^2-1)\frac{1}{3}(4^3-2^3) \\ &= \frac{448}{3}\end{aligned}$$

- **93**

$$\begin{aligned}\int_{-1}^1\ \mathrm{d}z\int_0^z\ \mathrm{d}x\int_{x-z}^{x+z}(x+y+z)\ \mathrm{d}y &= \int_{-1}^1\ \mathrm{d}z\int_0^z\{2z(x+z)+2xz\}\ \mathrm{d}x \\ &= \int_{-1}^1 3z^3\ \mathrm{d}z = 0\end{aligned}$$

■ **94**

$$\begin{aligned}\iiint_V \sin(x+y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^{\pi}\,\mathrm{d}x\int_0^{\pi-x}\,\mathrm{d}y\int_0^{\pi-x-y}\sin(x+y+z)\,\mathrm{d}z\\ &= \int_0^{\pi}\,\mathrm{d}x\int_0^{\pi-x}[-\cos(x+y+z)]_0^{\pi-x-y}\,\mathrm{d}y\\ &= \int_0^{\pi}\,\mathrm{d}x\int_0^{\pi-x}[1+\cos(x+y)]\,\mathrm{d}y\\ &= \int_0^{\pi}[y+\sin(x+y)]_0^{\pi-x}\,\mathrm{d}x\\ &= \int_0^{\pi}(\pi-x+\sin\pi-\sin x)\,\mathrm{d}x\\ &= \left[\pi x-\frac{x^2}{2}+\cos x\right]_0^{\pi} = \frac{1}{2}\pi^2-2\end{aligned}$$

■ **95**

$$\begin{aligned}\iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^{1}\,\mathrm{d}x\int_0^{1-x}\,\mathrm{d}y\int_0^{1-x-y}xyz\,\mathrm{d}z\\ &= \int_0^{1}x\,\mathrm{d}x\int_0^{1-x}\frac{1}{2}y(1-x-y)^2\,\mathrm{d}y\\ &= \int_0^{1}\frac{1}{2}x\left[\frac{1}{2}(1-x)^2y^2-\frac{2}{3}(1-x)y^3+\frac{1}{4}y^4\right]_0^{1-x}\,\mathrm{d}x\\ &= \int_0^{1}\frac{1}{24}x(1-x)^4\,\mathrm{d}x\\ &= \frac{1}{24}\int_0^{1}(x-4x^2+6x^3-4x^4+x^5)\,\mathrm{d}x\\ &= \frac{1}{720}\end{aligned}$$

■ **96**

$$\begin{aligned}
V &= \iiint_V \mathrm{d}V = \iiint_V \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \\
&= \int_0^1 \mathrm{d}x \int_{x^2}^{\sqrt{x}} \mathrm{d}y \int_0^{2-x-y} \mathrm{d}z \\
&= \int_0^1 \mathrm{d}x \int_{x^2}^{\sqrt{x}} (2-x-y)\ \mathrm{d}y \\
&= \int_0^1 \left\{ 2\sqrt{x} - x^{3/2} - \frac{1}{2}x - 2x^2 + x^3 + \frac{1}{2}x^4 \right\} \mathrm{d}x \\
&= \left[\frac{4}{3} - \frac{2}{5} - \frac{1}{4} - \frac{2}{3} + \frac{1}{4} + \frac{1}{10} \right] = \frac{11}{30}
\end{aligned}$$

■ **97**

$$\begin{aligned}
\iiint_V (x^2+y^2+z^2)x\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z &= \int_0^{\pi/2} \mathrm{d}\phi \int_0^{\pi/2} \mathrm{d}\theta \int_0^1 r^2 r \sin\theta \cos\phi \sin\theta r^2\ \mathrm{d}r \\
&= \int_0^{\pi/2} \cos\phi\ \mathrm{d}\phi \int_0^{\pi/2} \sin^2\theta\ \mathrm{d}\theta \int_0^1 r^5\ \mathrm{d}r \\
&= 1 \times \frac{1}{2} \times \frac{\pi}{2} \times \frac{1}{6} = \frac{\pi}{24}
\end{aligned}$$

■ **98** $\displaystyle\iiint_V x^2y^2z^2(x+y+z)\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$

$$\begin{aligned}
&= \iiint_V x^3y^2z^2\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z + \iiint_V x^2y^2z^3\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z + \iiint_V x^2y^3z^2\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \\
&= \int_0^1 x^3\ \mathrm{d}x \int_0^{1-x} y^2\ \mathrm{d}y \int_0^{1-x-y} z^2\ \mathrm{d}z + \int_0^1 y^3\ \mathrm{d}y \int_0^{1-y} z^2\ \mathrm{d}z \int_0^{1-z-y} x^2\ \mathrm{d}x \\
&\qquad + \int_0^1 z^3\ \mathrm{d}z \int_0^{1-z} x^2\ \mathrm{d}x \int_0^{1-z-x} y^2\ \mathrm{d}y \\
&= 3\int_0^1 x^3\ \mathrm{d}x \int_0^{1-x} y^2\ \mathrm{d}y \int_0^{1-x-y} z^2\ \mathrm{d}z \ \text{ from symmetry}
\end{aligned}$$

$$
\begin{aligned}
&= 3\int_0^1 x^3\,\mathrm{d}x \int_0^{1-x} \frac{1}{3}y^2(1-x-y)^3\,\mathrm{d}y \\
&= \int_0^1 x^3\,\mathrm{d}x \int_0^{1-x} \{y^2[(1-x)^3 - 3(1-x)^2y + 3(1-x)y^2 - y^3]\}\,\mathrm{d}y \\
&= \int_0^1 x^3 \left[\frac{1}{3}(1-x)^6 - \frac{3}{4}(1-x)^6 + \frac{3}{5}(1-x)^6 - \frac{1}{6}(1-x)^6\right]\,\mathrm{d}x \\
&= \frac{1}{60}\int_0^1 x^3(1-x)^6\,\mathrm{d}x \\
&= \frac{1}{60}\int_0^1 x^6(1-x)^3\,\mathrm{d}x \\
&= \frac{1}{60}\int_0^1 x^6(1-3x+3x^2-x^3)\,\mathrm{d}x \\
&= \frac{1}{60}\left(\frac{1}{7} - \frac{3}{8} + \frac{3}{9} - \frac{1}{10}\right) \\
&= \frac{1}{50400}
\end{aligned}
$$

■ **99**

$$
\left.\begin{aligned} u &= x+y+z \\ uv &= y+z \\ uvw &= z \end{aligned}\right\} \qquad \begin{aligned} x &= u - uv \\ y &= uv - uvw \\ z &= uvw \end{aligned}
$$

$$
\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & v-vw & vw \\ -u & u-uw & uw \\ 0 & -uv & uv \end{vmatrix} = \begin{vmatrix} 1-v & v & vw \\ -u & u & uw \\ 0 & 0 & uv \end{vmatrix}
$$

$$
= \begin{vmatrix} 1 & v & vw \\ 0 & u & uw \\ 0 & 0 & uv \end{vmatrix} = u^2v
$$

$$
I = \iiint_V \exp(-(x+y+z)^3)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
$$

$$
= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{1-x-y} \exp(-(x+y+z)^3)\,\mathrm{d}z
$$

$$
x+y+z = 1 \quad \Rightarrow \quad u = 1
$$

$$
x = 0 \quad \Rightarrow \quad \left.\begin{aligned} u &= y+z \\ uv &= y+z \end{aligned}\right\} \quad \Rightarrow \quad v = 1
$$

$$y=0 \quad \Rightarrow \quad \left.\begin{array}{l} u=x+z \\ uv=z \\ uvw=z \end{array}\right\} \quad \Rightarrow \quad w=1$$

$$z=0 \quad \Rightarrow \quad \left.\begin{array}{l} u=x+y \\ uv=y \\ uvw=0 \end{array}\right\} \quad \Rightarrow \quad w=0$$

$$\begin{aligned} I &= \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v \int_0^1 \mathrm{e}^{-u^3} u^2 v \, \mathrm{d}v \\ &= \int_0^1 u^2 \mathrm{e}^{-u^3} \, \mathrm{d}u \int_0^1 v \, \mathrm{d}v \int_0^1 \mathrm{d}w \\ &= \left[-\frac{1}{3}\mathrm{e}^{-u^3}\right]_0^1 \left[\frac{1}{2}v^2\right]_0^1 [w]_0^1 \\ &= \frac{1}{6}[1-\mathrm{e}^{-1}] \end{aligned}$$

■ **100**

$$\begin{aligned} \iiint_V yz \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z &= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{2-x-y} yz \, \mathrm{d}z \\ &= \int_0^1 \mathrm{d}x \int_0^{1-x} \frac{1}{2}y(2-x-y)^2 \, \mathrm{d}y \\ &= \int_0^1 \mathrm{d}x \int_0^{1-x} \frac{1}{2}[y(2-x)^2 - 2y^2(2-x) + y^3] \, \mathrm{d}y \\ &= \int_0^1 \frac{1}{2}\left[\frac{1}{2}(1-x)^2(2-x)^2 - \frac{2}{3}(1-x)^3(2-x) + \frac{1}{4}(1-x)^4\right]\mathrm{d}x \\ &= \int_0^1 \frac{1}{2}\left[\frac{1}{2}t^2(1+t)^2 - \frac{2}{3}t^3(1+t) + \frac{1}{4}t^4\right] \, \mathrm{d}t \\ &= \int_0^1 \frac{1}{2}\left[\frac{1}{2}t^2 + t^3 + \frac{1}{2}t^4 - \frac{2}{3}t^3 - \frac{2}{3}t^4 + \frac{1}{4}t^4\right] \, \mathrm{d}t \\ &= \frac{1}{2}\left[\frac{1}{6} + \frac{1}{12} + \frac{1}{60}\right] = \frac{2}{15} \end{aligned}$$

$$\begin{aligned}
\text{Volume of prism} &= \iiint_V \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \\
&= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{2-x-y} \mathrm{d}z \\
&= \int_0^1 \mathrm{d}x \int_0^{1-x} (2-x-y)\ \mathrm{d}y \\
&= \int_0^1 \left[(2-x)(1-x) - \frac{1}{2}(1-x)^2\right]\ \mathrm{d}x \\
&= \int_0^1 \left(2 - 3x + x^2 - \frac{1}{2} + x - \frac{1}{2}x^2\right)\ \mathrm{d}x \\
&= \frac{3}{2} - 1 + \frac{1}{6} = \frac{2}{3}
\end{aligned}$$

Let the coordinates of the centroid be $(\overline{x},\ \overline{y},\ \overline{z})$, the taking moments about appropriate axes, we have

$$\frac{2}{3}\overline{x} = \iiint_V x\ dx\ dy\ dz, \frac{2}{3}\overline{y} = \iiint_V y\ dx\ dy\ dz,$$
$$\frac{2}{3}\overline{z} = \iiint_V z\ dx\ dy\ dz$$

From symmetry $\overline{y} = \overline{x}$.

$$\begin{aligned}
\overline{x} &= \frac{3}{2}\int_0^1 dx \int_0^{1-x} dy \int_0^{2-x-y} x\ dz \\
&= \frac{3}{2}\int_0^1 dx \int_0^{1-x} x(2-x-y)dy \\
&= \frac{3}{2}\int_0^1 \frac{3}{2}x - 2x^2 + \frac{1}{2}x^3 dx \\
&= \frac{3}{2}\left[\frac{3}{4} - \frac{2}{3} + \frac{1}{8}\right] = \frac{5}{16}
\end{aligned}$$

$$\begin{aligned}
\overline{z} &= \frac{3}{2}\int_0^1 dx \int_0^{1-x} dy \int_0^{2-x-y} z\, dz \\
&= \frac{3}{2}\int_0^1 dx \int_0^{1-x} \frac{1}{2}(2-x-y)^2 dy \\
&= \frac{3}{2}\int_0^1 \left[\frac{-1}{6}(2-x-y)^3\right]_{y=0}^{y=1-x} dx \\
&= \frac{3}{2}\int_0^1 -\frac{1}{6}\left[1-(2-x)^3\right] dx \\
&= \frac{3}{2}\left[-\frac{1}{6}x - \frac{1}{24}(2-x)^4\right]_0^1 \\
&= \frac{3}{2}\times\frac{11}{24} = \frac{11}{16}
\end{aligned}$$

■ **101**

$$\begin{aligned}
\iiint_V z\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z &= \int_0^{2\pi} \mathrm{d}\phi \int_{\pi/4}^{\pi/2} \mathrm{d}\theta \int_0^1 r^3 \cos\theta \sin\theta\ \mathrm{d}r \\
&= \int_0^{2\pi} \mathrm{d}\phi \int_{\pi/4}^{\pi/2} \frac{1}{2}\sin 2\theta \int_0^1 r^3\ \mathrm{d}r \\
&= [2\pi]\left[-\frac{1}{4}\cos 2\theta\right]_{\pi/4}^{\pi/2}\left[\frac{1}{4}\right] = \frac{\pi}{8}
\end{aligned}$$

■ **102**

$$\begin{aligned}
\iiint_V x\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z &= \int_0^{\pi/2} \mathrm{d}\phi \int_0^{\pi/2} \mathrm{d}\theta \int_0^a r\sin^2\theta . r\cos\phi\ \mathrm{d}r \\
&= \int_0^{\pi/2} \cos\phi\ \mathrm{d}\phi \int_0^{\pi/2} \sin^2\theta\ \mathrm{d}\theta \int_0^a r^3\ \mathrm{d}r \\
&= [1]\left[\frac{\pi}{4}\right]\left[\frac{1}{4}a^4\right] = \pi a^4/16
\end{aligned}$$

Exercises 3.4.13

■ **103** $\displaystyle\iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{S}$ $\qquad\qquad$ $\mathbf{F}=(4xz,-y^2,yz)$

S has six faces and the integral can be evaluated as the sum of six integrals.

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} =& \int_{\text{on } x=0}\int \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} + \int_{\text{on } x=1}\int \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} + \int_{\text{on } y=0}\int \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} \\
&+ \int_{\text{on } y=1}\int \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} + \int_{\text{on } z=0}\int \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} + \int_{\text{on } z=1}\int \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} \\
=& \int_0^1\int_0^1 (0,-y^2,yz)\cdot(-\mathbf{i}\,\mathrm{d}y\,\mathrm{d}z) \\
&+ \int_0^1\int_0^1 (4z,-y^2,yz)\cdot(\mathbf{i}\,\mathrm{d}y\,\mathrm{d}z) \\
&+ \int_0^1\int_0^1 (4xz,0,0)\cdot(-\mathbf{j}\,\mathrm{d}x\,\mathrm{d}z) \\
&+ \int_0^1\int_0^1 (4xz,-1,yz)\cdot(\mathbf{j}\,\mathrm{d}x\,\mathrm{d}z) \\
&+ \int_0^1\int_0^1 (0,-y^2,0)\cdot(-\mathbf{k}\,\mathrm{d}x\,\mathrm{d}y) \\
&+ \int_0^1\int_0^1 (4x,-y^2,y)\cdot(\mathbf{k}\,\mathrm{d}x\,\mathrm{d}y) \\
=& 0+\int_0^1\int_0^1 4z\,\mathrm{d}y\,\mathrm{d}z+0+\int_0^1\int_0^1(-1)\,\mathrm{d}x\,\mathrm{d}z+0+\int_0^1\int_0^1 y\,\mathrm{d}x\,\mathrm{d}y \\
=& 2-1+\frac{1}{2}=\frac{3}{2}
\end{aligned}$$

■ **104**

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} &= \iiint_V \operatorname{div}\mathbf{F}\,\mathrm{d}V \\
&= \iiint_V (z+z+2z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi/2}\mathrm{d}\theta\int_0^2 4r\cos\theta.r^2\sin\theta\,\mathrm{d}r \\
&= [2\pi]\,[-\cos 2\theta]_0^{\pi/2}\left[\frac{1}{4}r^4\right]_0^2 = 16\pi
\end{aligned}$$

■ **105** On $z = 0$, $\mathrm{d}\mathbf{S} = -\mathbf{k}\,\mathrm{d}x\,\mathrm{d}y$, $\mathbf{F} = (4x, -2y^2, 0)$, $\mathbf{F}\cdot\,\mathrm{d}\mathbf{S} = 0$

On $z = 3$, $\mathrm{d}\mathbf{S} = \mathbf{k}\,\mathrm{d}x\,\mathrm{d}y$, $\mathbf{F} = (4x, -2y^2, 9)$, $\mathbf{F}\cdot\,\mathrm{d}\mathbf{S} = 9\,\mathrm{d}x\,\mathrm{d}y$

On $x^2 + y^2 = 4$, $\mathrm{d}\mathbf{S} = (\mathbf{i}\cos\phi + \mathbf{j}\sin\phi)2\,\mathrm{d}\phi\,\mathrm{d}z$, $\mathbf{F} = (8\cos\phi, -8\sin\phi, z^2)$

and $\mathbf{F}\cdot\,\mathrm{d}\mathbf{S} = 16(\cos^2\phi - \sin^3\phi)\,\mathrm{d}\phi\,\mathrm{d}z$

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} &= \int\int_{(z=3)} 9\,\mathrm{d}x\,\mathrm{d}y + \int_0^{2\pi}\mathrm{d}\phi\int_0^3 16(\cos^2\phi - \sin^3\phi)\,\mathrm{d}z \\
&= 36\pi + 48\int_0^{2\pi}\cos^2\phi - \sin^3\phi\,\mathrm{d}\phi \\
&= 84\pi \\
\iiint_V \operatorname{div}\mathbf{F}\,\mathrm{d}V &= \iiint_V (4 - 4y + 2z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^{2\pi}\mathrm{d}\phi\int_0^2\mathrm{d}r\int_0^3 (4 - 4r\sin\phi + 2z)r\,\mathrm{d}z \\
&= \int_0^{2\pi}\mathrm{d}\phi\int_0^2 (21 - 12r\sin\phi)r\,\mathrm{d}r \\
&= \int_0^{2\pi}(42 - 32\sin\phi)\,\mathrm{d}\phi = 84\pi
\end{aligned}$$

■ **106** $\operatorname{div}(\mathbf{F}\times\operatorname{grad}\phi) = \operatorname{grad}\phi\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}(\operatorname{grad}\phi)$ and $\operatorname{curl}(\operatorname{grad}\phi) \equiv 0$ for all ϕ.

Hence $\iiint_V \operatorname{grad}\phi\cdot\operatorname{curl}\mathbf{F}\,\mathrm{d}V = \iint_S (\mathbf{F}\times\operatorname{grad}\phi)\cdot\,\mathrm{d}\mathbf{S}$

■ **107**

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} = &\int\int_{\text{on } x=0}\mathbf{F}\cdot\,\mathrm{d}\mathbf{S} + \int\int_{\text{on } y=0}\mathbf{F}\cdot\,\mathrm{d}\mathbf{S} + \int\int_{\text{on } z=0}\mathbf{F}\cdot\,\mathrm{d}\mathbf{S} + \int\int_{\text{on } z=1}\mathbf{F}\cdot\,\mathrm{d}\mathbf{S} \\
&+ \int\int_{\text{on } x^2+y^2=4}\mathbf{F}\cdot\,\mathrm{d}\mathbf{S}
\end{aligned}$$

On $x = 0$, $\mathbf{F} = (y^2, 0, 0)$, $\mathrm{d}\mathbf{S} = -\mathbf{i}\,\mathrm{d}y\,\mathrm{d}z$, $\int\int_{\text{on } x=0}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} = \int_0^2\,\mathrm{d}y\int_0^1 -y^2\,\mathrm{d}z = -\frac{8}{3}$

On $y = 0$, $\mathbf{F} = (0, 0, 0)$, so contribution is zero

On $z = 0$, $\mathbf{F} = (xy + y^2, x^2y, 0)$, $\mathrm{d}\mathbf{S} = -\mathbf{k}\,\mathrm{d}x\,\mathrm{d}y$, so contribution is zero

On $z = 1$, $\mathbf{F} = (xy + y^2, x^2y, 0)$, $\mathrm{d}\mathbf{S} = \mathbf{k}\,\mathrm{d}x\,\mathrm{d}y$, so contribution is zero

On $x^2+y^2=4$, $\mathbf{F}=(4(\sin\phi\cos\phi+\sin^2\phi), 8\cos^2\phi\sin\phi, 0)$, $\mathrm{d}\mathbf{S}=2(\cos\phi\mathbf{i}+\sin\phi\mathbf{j})\,\mathrm{d}\phi\,\mathrm{d}z$

$$\begin{aligned}
\int_{\text{on } x^2+y^2=4}\int \mathbf{F}\cdot\,\mathrm{d}\mathbf{S} &= \int_0^{\pi/2}\mathrm{d}\phi\int_0^1 8\sin\phi\cos^2\phi+8\sin^2\phi\cos\phi+16(\cos^2\phi\sin^2\phi)\,\mathrm{d}z\\
&= \left[-\frac{8}{3}\cos^3\phi+\frac{8}{3}\sin^3\phi+2\phi-\frac{1}{2}\sin 4\phi\right]_0^{\pi/2}\\
&= \frac{16}{3}+\pi\\
\iiint_V \operatorname{div}\mathbf{F}\,\mathrm{d}V &= \iiint_V (y+x^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&= \int_0^{\pi/2}\mathrm{d}\phi\int_0^2\mathrm{d}r\int_0^1 (r\sin\phi+r^2\cos^2\phi)r\,\mathrm{d}z\\
&= \int_0^{\pi/2}\mathrm{d}\phi\int_0^2 (r^2\sin\phi+r^3\cos^2\phi)\,\mathrm{d}r\\
&= \int_0^{\pi/2}\left(\frac{8}{3}\sin\phi+4\cos^2\phi\right)\,\mathrm{d}\phi\\
&= \frac{8}{3}+\pi
\end{aligned}$$

Hence, result

■ **108**

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 36xz+6y\cos x & 3+6\sin x+z\sin y & 18x^2-\cos y\end{vmatrix}\\
&= \mathbf{i}(\sin y-\sin y)+\mathbf{j}(36x-36x)+\mathbf{k}(6\cos x-6\cos x)\\
&= \mathbf{0}
\end{aligned}$$

Hence, there is a function $\phi(x,y,z)$, such that $\mathbf{F}=\operatorname{grad}\phi$

■ **109**

$$\oint_C \mathbf{A}\cdot\ \mathrm{d}\mathbf{r} = \iint_S \text{curl }\mathbf{A}\cdot\ \mathrm{d}\mathbf{S}$$

$$\text{curl }\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\mathbf{k}$$

$$\oint_C \mathbf{A}\cdot\ \mathrm{d}\mathbf{r} = \iint_S 2\mathbf{k}\cdot\ \mathrm{d}\mathbf{S}$$

Let S be the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \ z=0$
Then, $\mathrm{d}\mathbf{S} = \mathbf{k}\cdot\ \mathrm{d}x\ \mathrm{d}y$, and

$$\oint \mathbf{A}\cdot\ \mathrm{d}\mathbf{r} = 2\iint_S\ \mathrm{d}x\ \mathrm{d}y = 2\pi ab$$

■ **110**

$$\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \mathbf{i}(-2yz+2yz)+\mathbf{j}(0)+\mathbf{k}(1)$$
$$= \mathbf{k}$$

$$\iint_S \text{curl }\mathbf{F}\cdot\ \mathrm{d}\mathbf{S} = 16\int_0^{2\pi}\ \mathrm{d}\phi\int_0^{\pi/2}\mathbf{k}\cdot(\sin\theta\cos\phi\mathbf{i}+\sin\theta\sin\phi\mathbf{j}+\cos\theta\mathbf{k})\sin\theta\ \mathrm{d}\theta$$
$$= 16\int_0^{2\pi}\ \mathrm{d}\phi\int_0^{\pi/2}\sin\theta\cos\theta\ \mathrm{d}\theta = 16[2\pi]\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$
$$= 16\pi$$

$$\oint_C \mathbf{F}\cdot\ \mathrm{d}\mathbf{r}$$

On circle $x^2+y^2=16,\ \ z=0,\ \ x=4\cos\phi,\ \ y=4\sin\phi,\ \ r=4(\cos\phi,\sin\phi,0)$

$$\mathbf{F} = (8\cos\phi-4\sin\phi,0,0),\qquad \mathrm{d}\mathbf{r} = 4(-\sin\phi,\cos\phi,0)\ \mathrm{d}\phi$$

$$\oint_C \mathbf{F}\cdot\ \mathrm{d}\mathbf{r} = \int_0^{2\pi}(-32\cos\phi\sin\phi+16\sin^2\phi+0)\ \mathrm{d}\phi$$
$$= 16\pi$$

■ **111**

$$\text{curl } (\mathbf{a}f(\mathbf{r})) = -\mathbf{a} \times \text{grad } f(\mathbf{r})$$

$$\iint_S -(\mathbf{a} \times \text{grad } f(\mathbf{r})) \cdot \mathbf{n} \, \mathrm{d}S = \int_C \mathbf{a}f(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

$$\Rightarrow \quad \iint_S -(\text{grad } f(\mathbf{r}) \times \mathbf{n}) \cdot \mathbf{a} \, \mathrm{d}S = \int_C \mathbf{a}f(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

$$\Rightarrow \quad \mathbf{a} \cdot \iint_S (\mathbf{n} \times \text{grad } f(\mathbf{r})) \, \mathrm{d}S = \mathbf{a} \cdot \int_C f(\mathbf{r}) \, \mathrm{d}\mathbf{r}$$

$$\Rightarrow \quad \iint_S \mathbf{n} \times \text{grad } f(\mathbf{r}) \, \mathrm{d}S = \int_C f(\mathbf{r}) \, \mathrm{d}\mathbf{r}$$

$$f(\mathbf{r}) = 3xy^2 \quad \Rightarrow \quad \text{grad } f(\mathbf{r}) = (3y^2, 6xy, 0)$$

$$\mathbf{n} \times \text{grad } f(\mathbf{r}) = \mathbf{k} \times \text{grad } f(\mathbf{r}) = (-6xy, 3y^2, 0)$$

$$\int_0^1 \mathrm{d}x \int_0^2 (-6xy, 3y^2, 0) \, \mathrm{d}y = \int_0^1 (-12x, 8, 0) \, \mathrm{d}x$$
$$= (-6, 8, 0)$$

$$\int_C f(\mathbf{r}) \, \mathrm{d}\mathbf{r} = \int_0^1 0.\mathbf{i} \, \mathrm{d}x + \int_0^2 3y^2\mathbf{j} \, \mathrm{d}y + \int_1^0 12x\mathbf{i} \, \mathrm{d}x + \int_2^0 0.\mathbf{j} \, \mathrm{d}y$$
$$= (-6, 8, 0)$$

■ **112** $\displaystyle\iint_S \text{curl } \mathbf{F} \cdot \, \mathrm{d}S = \oint_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{r}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y + z & x - z & y - x \end{vmatrix} = (2, 2, -1)$$

$$\iint_S \text{curl } \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = \iint_S (2, 2, -1) \cdot (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta$$
$$= \int_0^{\pi/2} \mathrm{d}\phi \int_0^{\pi/2} (2\sin\theta\cos\phi + 2\sin\theta\sin\phi - \cos\theta) \sin\theta \, \mathrm{d}\theta$$
$$= \int_0^{\pi/2} \left(\pi\sin\phi - \frac{1}{2}\right) \mathrm{d}\phi = \frac{3\pi}{4}$$

Here, C has three portions:
On $z = 0$, $\mathbf{r} = (\cos\phi, \sin\phi, 0) \quad \Rightarrow \quad d\mathbf{r} = (-\sin\phi, \cos\phi, 0)\ d\phi$
and $\mathbf{F} = (2\sin\phi, \cos\phi, \sin\phi - \cos\phi)$

$$\int \mathbf{F} \cdot\ d\mathbf{r} = \int_0^{\pi/2} (-2\sin^2\phi + \cos^2\phi)\ d\phi = -\frac{\pi}{4}$$

On $y = 0$, $\mathbf{r} = (\sin\theta, 0, \cos\theta) \quad \Rightarrow \quad d\mathbf{r} = (\cos\theta, 0, -\sin\theta)\ d\theta$
and $\mathbf{F} = (\cos\theta, \sin\theta - \cos\theta, -\sin\theta)$

$$\int \mathbf{F} \cdot\ d\mathbf{r} = \int_0^{\pi/2} (\cos^2\theta + \sin^2\theta)\ d\theta = \frac{\pi}{2}$$

On $x = 0$, $\mathbf{r} = (0, \sin\theta, \cos\theta) \quad \Rightarrow \quad d\mathbf{r} = (0, \cos\theta, -\sin\theta)\ d\theta$
and $\mathbf{F} = (2\sin\theta + \cos\theta, -\cos\theta, \sin\theta)$

$$\int \mathbf{F} \cdot\ d\mathbf{r} = \int_{\pi/2}^{0} (-\cos^2\theta - \sin^2\theta)\ d\theta = \frac{\pi}{2}$$

Review Exercises 3.7

■ **1**

$$\frac{\partial u}{\partial x} = nx^{n-1} f(t) + x^n f'(t)\left(\frac{-y}{x^2}\right)$$
$$\frac{\partial u}{\partial y} = x^n f'(t)\left(\frac{1}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f(t) = nu \qquad (1)$$

Differentiate (1) w.r.t. x

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x} \qquad (2)$$

Differentiate (1) w.r.t. y

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} \qquad (3)$$

$x \times (2) + y \times (3) \quad \Rightarrow$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow \qquad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$u(x,y) = x^4 + y^4 + 16x^2y^2$$

$$x\frac{\partial u}{\partial x} = x(4x^3 + 32xy^2)$$

$$y\frac{\partial u}{\partial y} = y(4y^3 + 32x^2y^2)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4(x^4 + 16x^2y^2 + y^4)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = x^2(12x^2 + 32y^2) + 2xy(64xy) + y^2(12y^2 + 32x^2)$$
$$= 12(x^4 + y^4 + 16x^2y^2)$$

■ **2**

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}, \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2} + 2\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = a\frac{\partial^2 f}{\partial u^2} + b\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}b + \frac{\partial^2 f}{\partial v \partial u}a$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}a + \frac{\partial f}{\partial v}b, \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2}a^2 + 2\frac{\partial^2 f}{\partial u \partial v}ab + \frac{\partial^2 f}{\partial v^2}b^2$$

$$9\frac{\partial^2 f}{\partial x^2} - 9\frac{\partial^2 f}{\partial x \partial y} + 2\frac{\partial^2 f}{\partial y^2} = (9 - 9a + 2a^2)\frac{\partial^2 f}{\partial u^2} + (9 - 9b + 2b^2)\frac{\partial^2 f}{\partial v^2}$$
$$+ 2\left(9 - \frac{9}{2}(a+b) + 2ab\right)\frac{\partial^2 f}{\partial u \partial v}$$

$$\Rightarrow \left.\begin{aligned} 9 - 9a + 2a^2 &= 0 \\ 9 - 9b + 2b^2 &= 0 \\ 9 - \tfrac{9}{2}(a+b) + 2ab &\neq 0 \end{aligned}\right\} \quad \Rightarrow \quad a \neq b \quad \Rightarrow \quad a = 3, \quad b = \frac{3}{2}$$

$$\frac{\partial^2 f}{\partial u \partial v} = 0 \quad \Rightarrow \quad f = F(u) + G(v)$$

$$\text{i.e.} \quad f(x,y) = F(x+3y) + G(x+3y/2)$$

$$f(x,0) = F(x) + G(x) = \sin x$$

$$\frac{\partial f}{\partial y}(x,0) = 3F'(x) + \frac{3}{2}G'(x) = 3\cos x$$

$$\Rightarrow \qquad F(x) + \frac{1}{2}G(x) = \sin x + k$$

$$\Rightarrow \qquad \frac{1}{2}G(x) = -k \quad \text{and} \quad F(x) = \sin x + 2k$$

$$\Rightarrow \qquad f(x,y) = \sin(x+3y)$$

■ **3**

$$\nabla \times (P,Q,R) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) + \mathbf{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \mathbf{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

$$= \mathbf{0}$$

$$\Rightarrow \qquad \nabla \times (\nabla f) \equiv 0$$

■ **4(a)** $xy = c$, hyperbolas

$$\text{grad } f = (y,x) = \left(\frac{c}{x}, x\right) \text{ on hyperbola}$$

$$y = \frac{c}{x} \qquad \Rightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c}{x^2}$$

That is, tangent in direction of $\left(1, -\frac{c}{x^2}\right) = \mathbf{t}$

$$\mathbf{t} \cdot \text{grad } f = \frac{c}{x} - \frac{c}{x} = 0 \qquad \text{that is, orthogonal}$$

4(b) $\dfrac{x}{x^2+y^2} = c$ circles, centres on x-axis, through (0,0)

$$\text{grad } f = \left(\frac{y^2 - x^2}{(x^2+y^2)^2}, \frac{-2y}{(x^2+y^2)^2}\right)$$

■ **5(a)**

$$\begin{aligned}\text{grad}(\boldsymbol{\omega}\cdot\mathbf{r}) &= \boldsymbol{\omega}\times(\nabla\times\mathbf{r}) + \mathbf{r}\times(\nabla\times\boldsymbol{\omega}) + (\boldsymbol{\omega}\cdot\nabla)\mathbf{r} + (\mathbf{r}\cdot\nabla)\boldsymbol{\omega}\\ &= \boldsymbol{\omega}\times\mathbf{0} + \mathbf{0} + \boldsymbol{\omega} + \mathbf{0}\\ &= \boldsymbol{\omega}\end{aligned}$$

■ **5(b)**

$$\begin{aligned}\text{curl }(\boldsymbol{\omega}\times\mathbf{r}) &= -(\boldsymbol{\omega}\cdot\nabla)\mathbf{r} + \boldsymbol{\omega}(\nabla\cdot\mathbf{r})(+\mathbf{r}\cdot\nabla)\boldsymbol{\omega} - \mathbf{r}(\nabla\cdot\boldsymbol{\omega})\\ &= -\boldsymbol{\omega} + 3\boldsymbol{\omega} + \mathbf{0} + \mathbf{0}\\ &= 2\boldsymbol{\omega}\end{aligned}$$

■ **6(a)** See problem **3** above.

6(b)

$$\begin{aligned}\text{div }\mathbf{v} &= \text{div }\{\text{grad }[zf(\mathbf{r})] + \alpha f(\mathbf{r})\mathbf{k}\}\\ &= \text{div }\{\mathbf{k}f(\mathbf{r}) + z\text{ grad }f(\mathbf{r})\} + \alpha\mathbf{k}\cdot\nabla f(\mathbf{r})\\ &= \mathbf{k}\cdot\nabla f(\mathbf{r}) + \mathbf{k}\cdot\text{grad }f(\mathbf{r}) + z\nabla^2 f(\mathbf{r}) + \alpha\mathbf{k}\cdot\nabla f(\mathbf{r})\\ &= (2+\alpha)\frac{\partial f}{\partial z}\\ \nabla^2\mathbf{v} &= \nabla(\nabla\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v})\\ &= (2+\alpha)\nabla\left(\frac{\partial f}{\partial z}\right) - \nabla\times(\nabla\times(\nabla(zf) + \alpha f\mathbf{k}))\end{aligned}$$

$$\nabla\times\nabla(zf) \equiv \mathbf{0}$$

$$\nabla\times(\alpha f\mathbf{k}) = \alpha\nabla f\times\mathbf{k}$$

$$\begin{aligned}\nabla\times(\nabla\times\alpha f\mathbf{k}) &= \alpha(\mathbf{k}\cdot\nabla)\nabla f - \alpha\mathbf{k}(\nabla^2 f)\\ &= \alpha\frac{\partial}{\partial z}(\nabla f) = \alpha\nabla\left(\frac{\partial f}{\partial z}\right)\end{aligned}$$

$$\Rightarrow \qquad \nabla^2\mathbf{v} = 2\nabla\left(\frac{\partial f}{\partial z}\right)$$

■ **7** $\mathbf{F} = (x^2 - y^2 + x)\mathbf{i} - (2xy + y)\mathbf{j}$

$$\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 - y^2 + x & -2xy - y & 0\end{vmatrix} = (0, 0, -2y + 2y) = \mathbf{0}$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (x^2 - y^2 + x, -2xy - y, 0)$$

$$\Rightarrow \qquad f(x, y, z) = \frac{x^3}{3} - y^2 x + \frac{x^2 - y^2}{2} + c$$

$$\int_{(1,2)}^{(2,1)} \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = \int_{(1,2)}^{(2,1)} \text{grad } f \cdot \, \mathrm{d}\mathbf{r} = [f]_{(1,2)}^{(2,1)} = \frac{22}{3}$$

$$\mathrm{d}\mathbf{r} = \mathbf{i} \, \mathrm{d}x + \mathbf{j} \, \mathrm{d}y = (\mathbf{i} - \mathbf{j}) \, \mathrm{d}x$$

as on $y = 3 - x, \quad \mathrm{d}y = -\, \mathrm{d}x$

$$\int_1^2 (x^2 - y^2 + x + 2xy + y) \, \mathrm{d}x = \int_1^2 (x^2 - (3 - x)^2 + x + 2(3 - x) + 3 - x) \, \mathrm{d}x$$
$$= \frac{22}{3}$$

■ **8**

$$W = \int_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{r}$$
$$\mathbf{r} = (1 - \cos\theta)\mathbf{i} + \sin\theta\mathbf{j}$$
$$\mathrm{d}\mathbf{r} = (\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) \, \mathrm{d}\theta$$

8(a) $\mathbf{F} = 2\sin\frac{1}{2}\theta\mathbf{i}$

$$\int_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = \int_0^\pi 4\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} \, \mathrm{d}\theta$$
$$= \left[\frac{8}{3}\sin^3\frac{\theta}{2}\right]_0^\pi = \frac{8}{3}$$

8(b) $\mathbf{F} = 2\sin\frac{\theta}{2}\hat{\mathbf{n}} = 2\sin\frac{\theta}{2}(\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$

$$\int_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = \int_0^\pi 2\sin\frac{\theta}{2} \, \mathrm{d}\theta = 4\left[-\cos\frac{\theta}{2}\right]_0^\pi = 4$$

■ **9**

$$\mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k})t \qquad 0 \le t \le 1$$
$$\mathrm{d}\mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\ \mathrm{d}t$$
$$W = \int_0^1 \mathbf{F}\cdot\ \mathrm{d}\mathbf{r}$$
$$\mathbf{F} = (xy, -y, 1) = (t^2, -t, 1)$$
$$\Rightarrow \qquad W = \int_0^1 (t^2 - t + 1)\ \mathrm{d}t = \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}$$

■ **10**

$$\mathbf{F} = I\oint_C \mathrm{d}\mathbf{r}\times\mathbf{B}$$
$$\mathbf{r} = \sin\theta\mathbf{i} + \cos\theta\mathbf{j} + \sin\frac{\theta}{2}\mathbf{k}$$
$$\mathrm{d}\mathbf{r} = \left(\cos\theta\mathbf{i} - \sin\theta\mathbf{j} + \frac{1}{2}\cos\frac{\theta}{2}\mathbf{k}\right)\ \mathrm{d}\theta$$
$$\mathbf{B} = \sin\theta\mathbf{i} - \cos\theta\mathbf{j} + \mathbf{k}$$
$$\mathbf{F} = I\oint_0^{2\pi}\left\{\mathbf{i}\left(\frac{1}{2}\cos\frac{\theta}{2}\cos\theta - \sin\theta\right) + \mathbf{j}\left(\frac{1}{2}\cos\frac{\theta}{2}\sin\theta - \cos\theta\right)\right.$$
$$\left. + \mathbf{k}(\sin^2\theta - \cos^2\theta)\right\}\ \mathrm{d}\theta = \frac{4}{3}I\mathbf{j}$$

■ **11**

$$\text{Circulation} = \oint_C \mathbf{v}\cdot\ \mathrm{d}\mathbf{r}$$
$$= \int_1^{-1}\frac{-y\ \mathrm{d}x}{x^2+y^2} + \int_1^{-1}\frac{x\ \mathrm{d}y}{x^2+y^2} + \int_{-1}^{1}\frac{y\ \mathrm{d}x}{x^2+y^2} + \int_{-1}^{1}\frac{x\ \mathrm{d}y}{x^2+y^2}$$
$$\text{on } y=1 \qquad \text{on } x=-1 \qquad \text{on } y=-1 \qquad \text{on } x=1$$
$$= \int_{-1}^{1}\frac{1}{1+x^2}\ \mathrm{d}x + \int_1^{-1}\frac{\mathrm{d}y}{1+y^2} - \int_{-1}^{1}\frac{\mathrm{d}x}{1+x^2} + \int_{-1}^{1}\frac{\mathrm{d}y}{1+y^2}$$
$$= 0$$

■ **12**

$$I_z = \iint_A \rho(x^2+y^2)\ \mathrm{d}A, \qquad \text{where density } \rho = kxy$$
$$= \int_0^c \mathrm{d}x\int_{x^2/c}^{c}(x^2+y^2)kxy\ \mathrm{d}y$$

$$
\begin{aligned}
&= \int_0^c \left[\frac{1}{4}kx(x^2+y^2)^2\right]_{x^2/c}^{c} \,\mathrm{d}x \\
&= \int_0^c \left[-\frac{1}{4}kx\left(x^2+\frac{x^4}{c^2}\right)^2 + \frac{1}{4}kx(x^2+c^2)^2\right] \,\mathrm{d}x \\
&= \frac{1}{4}k\int_0^c -\left(\frac{x^9}{c^4}+\frac{2x^7}{c^2}-2x^3c^2-xc^4\right) \,\mathrm{d}x \\
&= \frac{13}{80}kc^6
\end{aligned}
$$

■ **13** Equation of cone is $x^2+y^2=\frac{a^2}{h^2}(z-h)^2$

$$
\begin{aligned}
V &= 2\int_c^a \,\mathrm{d}x \int_0^{\sqrt{a^2-x^2}} z \,\mathrm{d}y \\
&= 2\int_c^a \,\mathrm{d}x \int_0^{\sqrt{a^2-x^2}} \left\{h+\frac{h}{a}\sqrt{x^2+y^2}\right\} \,\mathrm{d}y \\
&= 2\int_c^a \left[hy+\frac{hx^2}{2a}\sinh^{-1}\frac{y}{x}-\frac{hy}{2a}\sqrt{x^2+y^2}\right]_0^{\sqrt{a^2-x^2}} \,\mathrm{d}x \\
&= 2\int_c^a \left[\frac{h}{2}\sqrt{a^2-x^2}+\frac{hx^2}{2a}\sinh^{-1}\frac{\sqrt{a^2-x^2}}{x}\right] \,\mathrm{d}x \\
&= \frac{2ha^2}{3}\left(\frac{\pi}{2}-\sin^{-1}\left(\frac{c}{a}\right)\right)-\frac{hc}{3}\sqrt{a^2-c^2}-\frac{hc^3}{3a}\tanh^{-1}\left[\frac{\sqrt{a^2-c^2}}{a}\right]
\end{aligned}
$$

■ **14** Volume is $8\int\limits_x\int\limits_y\int\limits_{z\geq 0} \mathrm{d}V$

$x^2+y^2=a^2$ is a cylinder with z-axis as axis of symmetry and radius a.
$z^2+y^2=a^2$ is a cylinder with x-axis as axis of symmetry and radius a.

$$
\begin{aligned}
\Rightarrow \qquad V &= 8\int_0^a \,\mathrm{d}y \int_0^{\sqrt{a^2-y^2}} \,\mathrm{d}x \int_0^{\sqrt{a^2-y^2}} \,\mathrm{d}z \\
&= 8\int_0^a [x]_0^{\sqrt{a^2-y^2}}[z]_0^{\sqrt{a^2-y^2}} \,\mathrm{d}y \\
&= 8\int_0^a (a^2-y^2) \,\mathrm{d}y = 8\left[a^2y-\frac{1}{3}y^3\right]_0^a = \frac{16a^3}{3}
\end{aligned}
$$

■ **15** Elastic energy of ΔV is $q^2\Delta V/(2EI)$ where $q = q_0\rho/r$ and ρ is the distance from the centre and r is radius of cylinder.

$$\begin{aligned}\text{Total energy} &= \int_0^{2\pi} \mathrm{d}\phi \int_0^r \mathrm{d}\rho \int_0^l \frac{q_0^2\rho^3}{2EIr^2}\ \mathrm{d}z \\ &= 2\pi l \int_0^r \frac{q_0^2\rho^3\ \mathrm{d}\rho}{2EIr^2} \\ &= \frac{\pi q_0^2 r^2 l}{4EI}\end{aligned}$$

■ **16** On $x = 0$, $\quad \mathrm{d}\mathbf{S} = -\mathbf{i}\ \mathrm{d}y\ \mathrm{d}z \quad$ and $\quad \mathbf{v}\cdot\ \mathrm{d}\mathbf{S} = -3x^2y\ \mathrm{d}y\ \mathrm{d}z \equiv 0$
On $y = 0$, $\quad \mathbf{v}\cdot\ \mathrm{d}\mathbf{S} = 0 \quad$ On $\quad z = 0, \mathbf{v}\cdot\ \mathrm{d}\mathbf{S} = 0$
On $z = 1$, $\quad \mathbf{v}\cdot\ \mathrm{d}\mathbf{S} = 0 \quad$ On $\quad x + y = 1,\ \mathrm{d}\mathbf{S} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})\ \mathrm{d}S$

$$\begin{aligned}\Rightarrow \qquad \iint \mathbf{v}\cdot\ \mathrm{d}\mathbf{S} &= \int_0^1 \mathrm{d}x \int_0^1 \left\{\frac{3}{\sqrt{2}}x^2(1-x) + \frac{1}{\sqrt{2}}x(1-x)^2\right\}\sqrt{2}\ \mathrm{d}z \\ &= \int_0^1 (2x^2 + x)(1-x)\ \mathrm{d}x \\ &= \frac{1}{3}\end{aligned}$$

■ **17** $\displaystyle\iint_S \mathbf{v}\cdot\ \mathrm{d}\mathbf{S}$

On S, $\quad \mathrm{d}\mathbf{S} = (\mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta)a^2\sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi$
and $\quad \mathbf{v} = \mathbf{i}2a\sin^2\theta\cos\phi\sin\phi - \mathbf{j}a^2\sin^2\theta\sin^2\phi + \mathbf{k}(a\sin\theta\cos\phi + a\sin\theta\sin\phi)$

$$\begin{aligned}\iint_S \mathbf{v}\cdot\ \mathrm{d}\mathbf{S} = \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} &\{2a^3\sin^4\theta\cos^2\phi\sin\phi - a^4\sin^4\theta\sin^3\phi \\ &+ a^3\sin^2\theta\cos\theta\cos\phi + a^3\cos\theta\sin^2\theta\sin\phi\}\ \mathrm{d}\phi \\ = 0\end{aligned}$$

■ **18** $\displaystyle\oint_C \mathbf{F}\cdot\ \mathrm{d}\mathbf{r}$

C is the circle $x^2 + y^2 = 16,\ z = 0$, so that, on the circle

$$\mathbf{F} = \left(x^2 + y - 4,\ 3xy,\ 0\right), \quad \mathbf{r} = 4\left(\cos\theta, \sin\theta, 0\right)$$

and $\mathrm{d}\mathbf{r} = (-4\sin\theta, 4\cos\theta, 0)\ \mathrm{d}\theta$

$$\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{\theta=0}^{2\pi} \left[-16\left(4\cos^2\theta + \sin\theta - 1\right)\sin\theta + 192\cos\theta\sin^2\theta\right]\ \mathrm{d}\theta$$
$$= \int_0^{2\pi} -16\sin^2\theta\ \mathrm{d}\theta \qquad \text{(from symmetries)}$$
$$= -16\pi$$

$$\text{curl}\ \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix} = (0,\ -2z,\ 3y-1)$$

On the hemisphere

$$\mathbf{r} = 4\,(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$
$$d\mathbf{S} = 16\,(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\sin\theta\ \ \mathrm{d}\phi\ \ \mathrm{d}\theta$$

$$\int\int_S \text{curl}\ \ \mathbf{F}\ .\ \ \mathrm{d}\mathbf{S} =$$

$$\int_0^{\pi/2} \mathrm{d}\theta \int_0^{2\pi} 16(-8\sin^2\theta\cos\theta\sin\phi + 12\sin^2\theta\cos\theta\cos\phi - \cos\theta\sin\theta)\mathrm{d}\phi$$

$$= \int_0^{\pi/2} -16\cos\theta\sin\theta\,[2\pi]\ \ \mathrm{d}\theta = -16\pi$$

■ **19**

$$\int\int_S \mathbf{a}\cdot\mathrm{d}\mathbf{S} = \int\int\int_V \text{div}\ \mathbf{a}\ \ \mathrm{dV} - \int\int_{S_1} \mathbf{a}\cdot\mathrm{d}\mathbf{S}$$

where V is the hemisphere $x^2+y^2+z^2 = a^2$ (different a from the vector $\mathbf{a}$), S_1 is the circle $x^2+y^2 = a^2,\ z = 0$. div $\mathbf{a} = 0$ and $\mathrm{d}\mathbf{S} = -\mathbf{k}\ \ \mathrm{d}x\ \ \mathrm{d}y$ on S_1

$$\int\int_{S_1} \mathbf{a}\cdot\mathrm{d}\mathbf{S} = \int\int_S (x\mathbf{i}+y\mathbf{j})\ \cdot\ (-\mathbf{k}\ \ \mathrm{d}x\ \mathrm{d}y) = 0$$

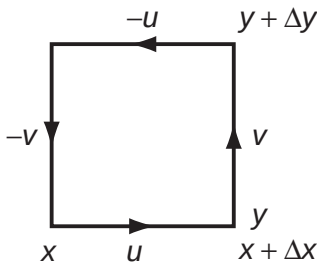
Hence

$$\int\int_S \mathbf{a}\cdot\mathrm{d}\mathbf{S} = 0$$

■ **20**

$$\begin{aligned}\int\int\int_V xyz \ \mathrm{d}V &= \int_0^1 \int_0^{1-x} \int_0^{2-x} xyz \ \mathrm{d}z \ \mathrm{d}y \ \mathrm{d}x \\ &= \int_0^1 \int_0^{1-x} \frac{1}{2}xy\,(2-x)^2 \ \mathrm{d}y \ \mathrm{d}x \\ &= \int_0^1 \frac{1}{4}x\,(1-x)^2\,(2-x)^2 \ \mathrm{d}x \\ &= \int_0^1 \frac{1}{4}\left(x^5 - 6x^4 + 13x^3 - 12x^2 + 4x\right) \ \mathrm{d}x \\ &= \frac{1}{24} - \frac{3}{10} + \frac{13}{16} - 1 + \frac{1}{2} = \frac{13}{240}\end{aligned}$$

■ **21**



Net circulation (anti clockwise) is

$$-u(x,y)\Delta x - v(x+\Delta x, y)\Delta y + u(x, y+\Delta y)\Delta x + v(x,y)\Delta y$$

If net circulation is zero then, dividing by $\Delta x \Delta y$,

$$\frac{u(x,y+\Delta y) - u(x,y)}{\Delta y} - \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} = 0$$

$\Delta x, \Delta y \to 0$ gives

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0.$$

Since $u = -\frac{\partial \psi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$ we obtain

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{Laplace equation.}$$

4

Functions of a Complex Variable

Exercises 4.2.2

- **1(a)** If $| z-2+j | = | z-j+3 |$, so that

$$| x+jy-2+j | = | x+jy-j+3 |$$

or

$$(x-2)^2+(y+1)^2 = (x+3)^2+(y-1)^2$$
$$x^2-4x+4+y^2+2y+1 = x^2+6x+9+y^2-2y+1$$

Then cancelling the squared terms and tidying up,

$$y = \frac{5}{2}x + \frac{5}{4}$$

1(b) $z+z^*+4j(z-z^*)=6$
Using, $z+z^*=2x,\ z-z^*=2jy$ gives

$$2x+4j2jy = 6$$
$$y = \frac{1}{4}x - \frac{3}{4}$$

- **2** The straight lines are

$$| z-1-j | = | z-3+j |$$
$$| z-1+j | = | z-3-j |$$

which, in Cartesian form, are

$$(x-1)^2+(y-1)^2 = (x-3)^2+(y+1)^2$$
$$\text{that is,}\quad x^2-2x+1+y^2-2y+1 = x^2-6x+9+y^2+2y+1$$
$$y = x-2$$

and $(x-1)^2+(y+1)^2=(x-3)^2+(y-1)^2$

$$\text{i.e.}\quad x^2-2x+1+y^2+2y+1=x^2-6x+9+y^2-2y+1$$
$$y=-x+2$$

These two lines intersect at $\pi/2$ (the products of their gradients is -1) and $y=0,\ x=2$ at their intersection, that is, $z=2+j0$.

■ **3** $w=jz+4-3j$ can be written as

$$w=e^{j\pi/2}z+4-3j\ (\text{since } j=\cos\frac{\pi}{2}+j\sin\frac{\pi}{2}=e^{j\pi/2})$$

which is broken down as follows:

z	$\longrightarrow$	$e^{j\pi/2}z$	$\longrightarrow$	$e^{j\pi/2}z+4-3j=w$
	rotate anticlockwise by $\frac{1}{2}\pi$		translation $(0,0)\rightarrow(4,-3)$	

Let $w=u+jv$ and $z=x+jy$

$$\text{so that,}\quad u+jv=j(x+jy)+4-3j$$
$$=jx-y+4-3j$$
$$\text{that is,}\quad u=-y+4 \qquad (1)$$
$$v=x-3 \qquad (2)$$

Taking 6 times equation (2) minus equation (1) gives,

$$6v-u=6x+y-22$$

so that, if $6x+y=22$, we must have $6v-u=0$ so that, $u=6v$ is the image of the line

$$6x+y=22$$

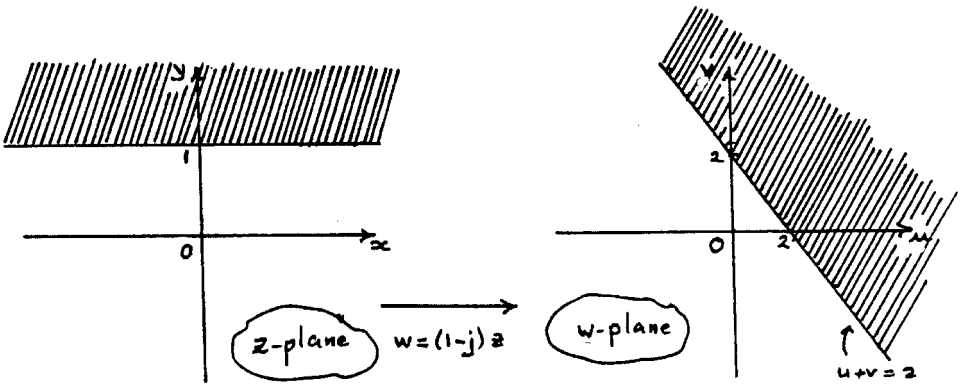
■ **4** Splitting the mapping $w = (1-j)z$ into real and imaginary parts gives

$$\begin{aligned} u + jv &= (1-j)(x+jy) \\ &= x + y + j(y-x) \end{aligned}$$

$$\text{that is,} \quad \begin{aligned} u &= x + y \\ v &= y - x \end{aligned}$$

$$\text{so that,} \quad u + v = 2y$$

Therefore $y > 1$ corresponds to $u + v > 2$.



■ **5** Since $w = jz + j$

$$x = v - 1, \; y = -u$$

so that $x > 0$ corresponds to $v > 1$.

■ **6** Since $w = jz + 1$

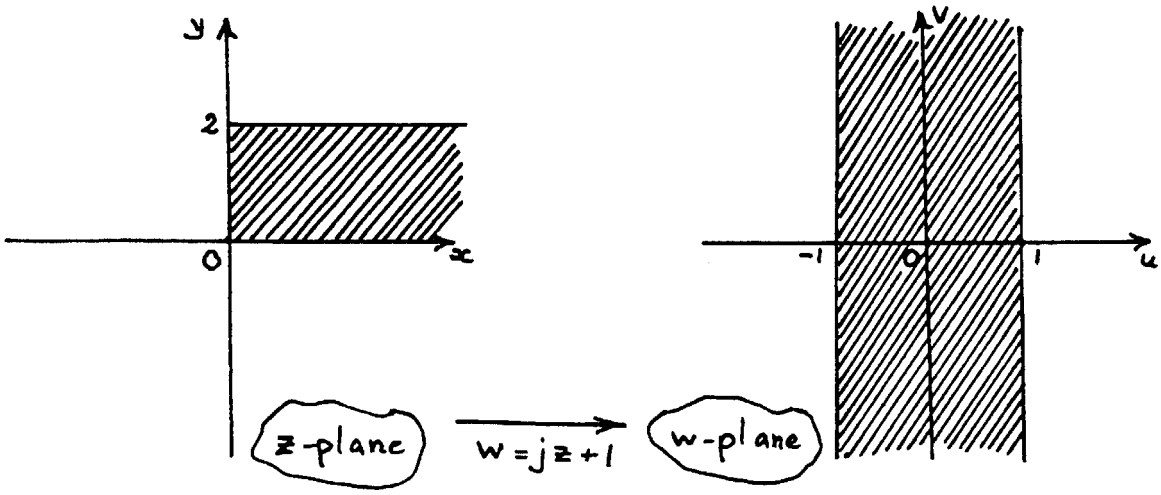
$$v = x$$

$$u = -y + 1$$

so that $x > 0 \Rightarrow v > 0$

and $0 < y < 2 \Rightarrow -1 < u < 1$ or $|\, u \,| < 1$.

This is illustrated below



■ **7** Given $w = (j + \sqrt{3})z + j\sqrt{3} - 1$, we obtain, on equating real and imaginary parts,

$$u = x\sqrt{3} - y - 1, \quad v = x + y\sqrt{3} + \sqrt{3}$$

$$\text{or} \quad v\sqrt{3} - u = 4y + 4, \quad \text{and} \quad v + u\sqrt{3} = 4x$$

on rearranging.

Thus **7(a)** $y = 0$ corresponds to $v\sqrt{3} - u = 4$ or $u = v\sqrt{3} - 4$

7(b) $x = 0$ corresponds to $v + u\sqrt{3} = 0$ or $v = -u\sqrt{3}$

7(c) Since $u + 1 = x\sqrt{3} - y$ and $v - \sqrt{3} = x + y\sqrt{3}$ squaring and adding gives

$$(u+1)^2 + (v-\sqrt{3})^2 = (x\sqrt{3} - y)^2 + (x + y\sqrt{3})^2$$
$$= 4x^2 + 4y^2$$

Thus, $x^2 + y^2 = 1 \Rightarrow (u+1)^2 + (v-\sqrt{3})^2 = 4$

7(d) Since $v\sqrt{3} - u = 4y + 4$ and $v + u\sqrt{3} = 4x$, squaring and adding gives

$$4v^2 + 4u^2 = 16(y+1)^2 + 16x^2$$
$$\text{or} \quad u^2 + v^2 = 4(x^2 + y^2 + 2y + 1)$$

Thus, $x^2 + y^2 + 2y = 1$ corresponds to $u^2 + v^2 = 8$

■ **8(a)** $w = \alpha z + \beta$

Inserting $z = 1+j,\ w = j$ and $z = -1,\ w = 1+j$ gives the following two equations for α and β

$$j = \alpha(1+j) + \beta \quad \text{or} \quad 1 + j = -\alpha + \beta$$

from which, by subtraction,

$$-1 = (2+j)\alpha \quad \text{or} \quad \alpha = \frac{1}{5}(-2+j)$$

so that, $\beta = 1 + j + \alpha = \dfrac{1}{5}(3+6j)$ gives $5w = (-2+j)z + 3 + 6j$

8(b) Writing $w = u + jv,\ z = x + jy$ and equating real and imaginary parts gives

$$5u = -2x - y + 3$$
$$5v = x - 2y + 6$$

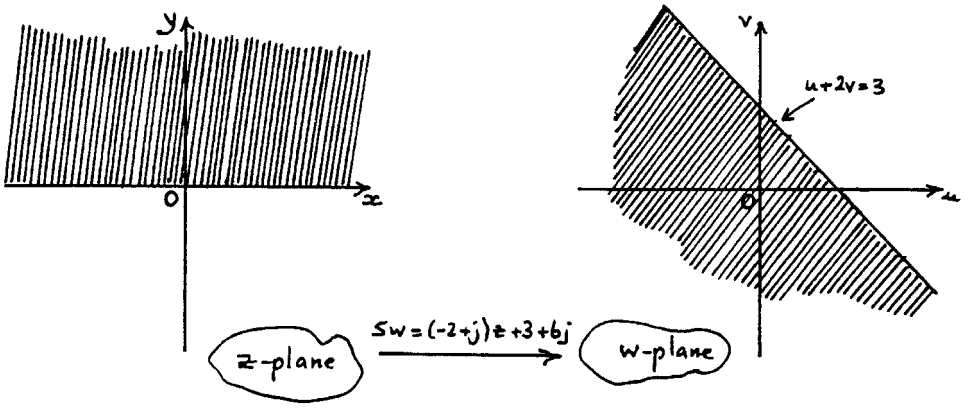
Eliminating y yields

$$5v - 10u = 5x \quad \text{or} \quad v - 2u = x$$

Eliminating x yields

$$5u + 10v = -5y + 15 \quad \text{or} \quad u + 2v = -y + 3$$

so that $y > 0$ corresponds to $u + 2v < 3$



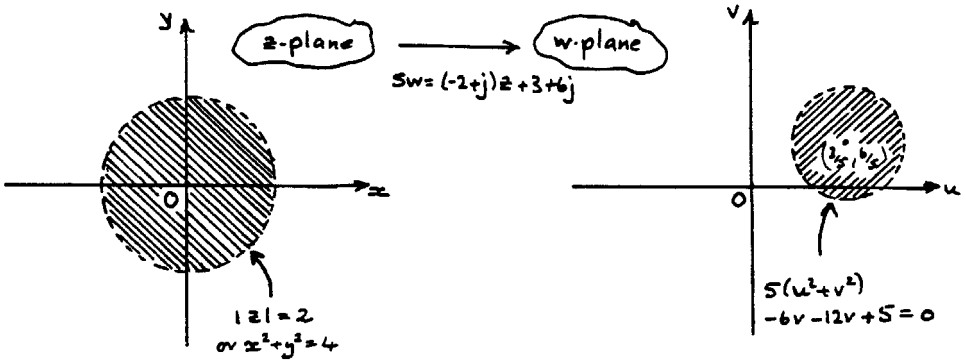
8(c) From part (b),

$$x = v - 2u$$
$$y = 3 - u - 2v$$

Squaring and adding gives

$$x^2 + y^2 = (v - 2u)^2 + (3 - u - 2v)^2$$
$$= 5(u^2 + v^2) - 6u - 12v + 9$$
$$|\, z \,| < 2 \Rightarrow x^2 + y^2 < 4$$

so that, $5(u^2 + v^2) - 6u - 12v + 5 < 0$
or $(5u - 3)^2 + (5v - 6)^2 < 20$
that is, $\left(v - \frac{3}{5}\right)^2 + \left(v - \frac{6}{5}\right)^2 < \frac{20}{25} = \frac{4}{5} = \left(\frac{2}{5}\sqrt{5}\right)^2$



8(d) The fixed point(s) are given by

$$5z = (-2 + j)z + 3 + 6j$$
$$\text{so that,} \quad z = \frac{3 + 6j}{7 - j} = \frac{(3 + 6j)(7 + j)}{50}$$
$$= \frac{3}{10}(1 + 3j)$$

Exercises 4.2.5

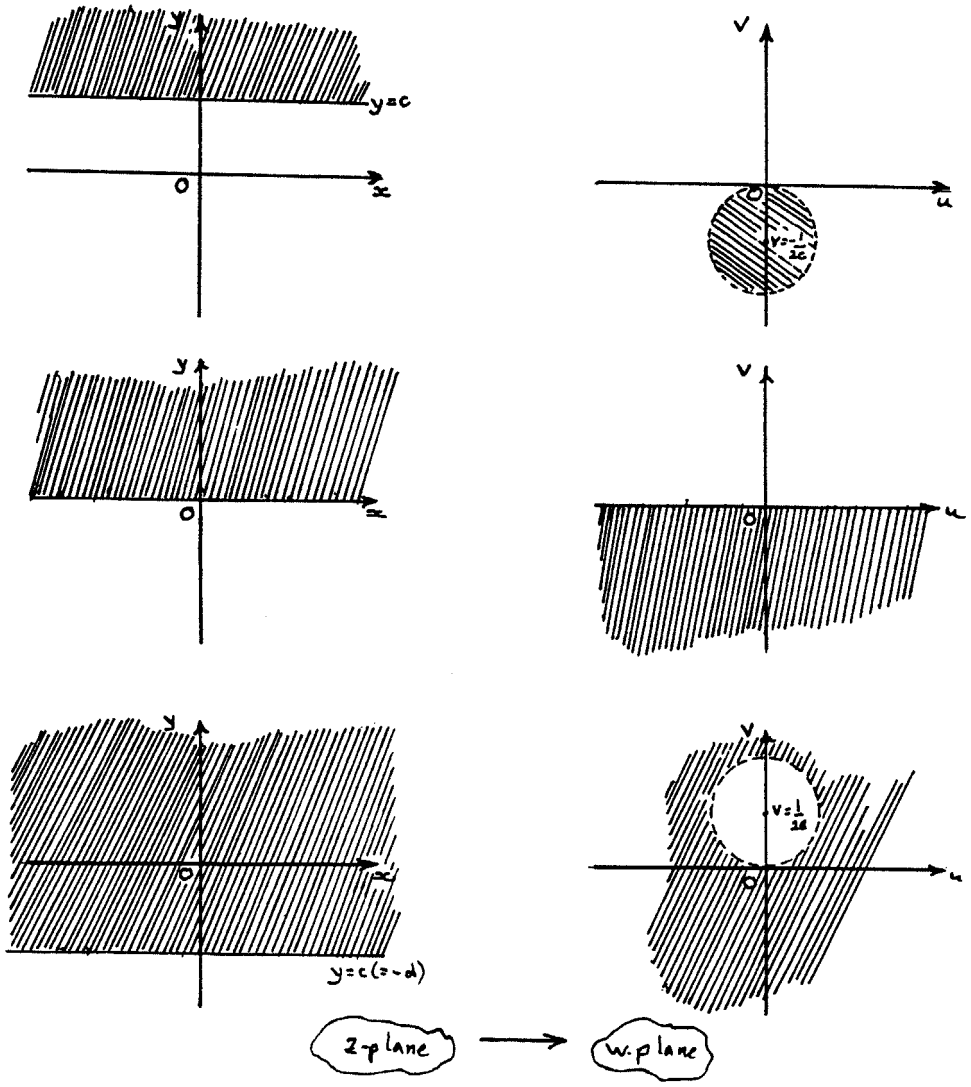
■ **9** Writing $w = \frac{1}{z}$, $z = \frac{1}{w}$

$$\Rightarrow x + jy = \frac{1}{u + jv} = \frac{u - jv}{u^2 + v^2}$$
$$\text{so that} \quad y = -\frac{v}{u^2 + v^2}$$

If $y > c$ then, $-\dfrac{v}{u^2+v^2} > c$
or rearranging $u^2 + v^2 + \dfrac{v}{c} < 0$

$$u^2 + \left(v + \frac{1}{2c}\right)^2 < \left(\frac{1}{2c}\right)^2$$

If $c = 0, -\dfrac{v}{u^2+v^2} > c \Rightarrow v < 0$
If $c < 0$, put $c = -d$ and $-\dfrac{v}{u^2+v^2} > -d$
or, on rearranging, $u^2 + \left(v - \dfrac{1}{2d}\right)^2 > \left(\dfrac{1}{2d}\right)^2$



■ **10** Putting $z = \dfrac{1}{w}$ in $\left|z + \dfrac{3}{4} + j\right| = \dfrac{7}{4}$ gives $\left|\dfrac{1}{w} + \dfrac{3}{4} + j\right| = \dfrac{7}{4}$ or $\left|1 + (\dfrac{3}{4} + j)w\right| = \dfrac{7}{4} \mid w \mid$ which, writing $w = u + jv$ and expanding, gives

$$\left(1 + \frac{3}{4}u - v\right)^2 + \left(\frac{3}{4}v + u\right)^2 = \frac{49}{16}(u^2 + v^2)$$

or, on rearranging gives

$$u^2 + v^2 - u + \frac{4}{3}v - \frac{2}{3} = 0$$

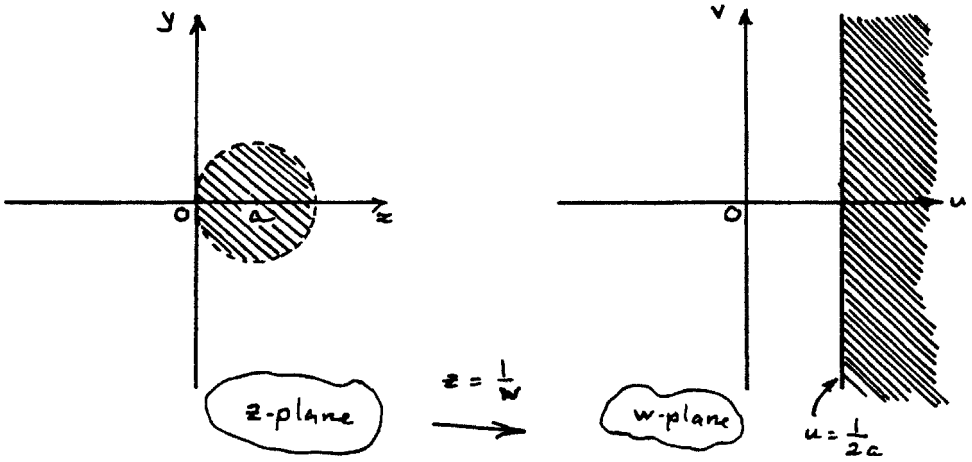
$$\left(u - \frac{1}{2}\right)^2 + \left(v + \frac{2}{3}\right)^2 = \left(\frac{7}{6}\right)^2$$

a circle centre $(\dfrac{1}{2}, -\dfrac{2}{3})$ and radius $\dfrac{7}{6}$.

■ **11** Putting $z = \dfrac{1}{w}$ in $\mid z - a \mid = a$ gives $\mid 1 - aw \mid = a \mid w \mid$ from which $u = \dfrac{1}{2a}$ can be obtained (on writing $w = u^2 + v^2$).

Hence $\mid z - a \mid = a$ maps to $Re\{w\} = \dfrac{1}{2a}$ under $w = \dfrac{1}{z} \cdots = \dfrac{1}{2a}$; that is, the half plane $\mathrm{Re}\{w\} > \dfrac{1}{2a}$.

Moreover, the interior of $\mid z - a \mid = a$ maps to right of the line $Re\{w\} = \dfrac{1}{2a}$. The point $z = \dfrac{1}{2}a$ mapping to $w = \dfrac{2}{a}$ confirms this.



■ **12** The general bilinear mapping is

$$w = \frac{az + b}{cz + d}$$

with $z = 0, w = j \Rightarrow b = jd$,

$z = -j, w = 1 \Rightarrow d - jc = b - ja$
and $z = -1,\ w = 0 \Rightarrow a = b$
Hence $b = a,\ d = -ja$ and $c = ja$
and the mapping is

$$w = \frac{z+1}{j(z-1)}$$

Making z the subject of this formula, we obtain

$$z = \frac{jw+1}{jw-1}$$

Writing $z = x + jy,\ w = u + jv$ and equating real and imaginary parts

$$x = \frac{u^2+v^2-1}{u^2+(v+1)^2},\ y = \frac{-2u}{u^2+(v+1)^2}$$

Lines $x =$ constant $= k$, say, transform to

$$k[u^2+(v+1)^2] = u^2+v^2-1$$
$$\text{or} \quad u^2+v^2+\frac{2k}{k-1}v = -\frac{1}{k-1}$$

This can be rewritten as

$$u^2+\left(v+\frac{k}{k-1}\right)^2 = \frac{k}{(k-1)^2} - \frac{1}{k-1} = \frac{1}{(k-1)^2}$$

which are circles (except $k = 1$ which is $v = -1$).
Lines $y =$ constant $= l$, say, transform to

$$u^2+(v+1)^2+\frac{2u}{l} = 0$$
$$\text{or} \quad \left(u+\frac{1}{l}\right)^2+(v+1)^2 = \frac{1}{l^2}$$

which are circles (except $l = 0$ which is $u = 0$).

The fixed points are given by

$$z = \frac{z+1}{jz-j}$$

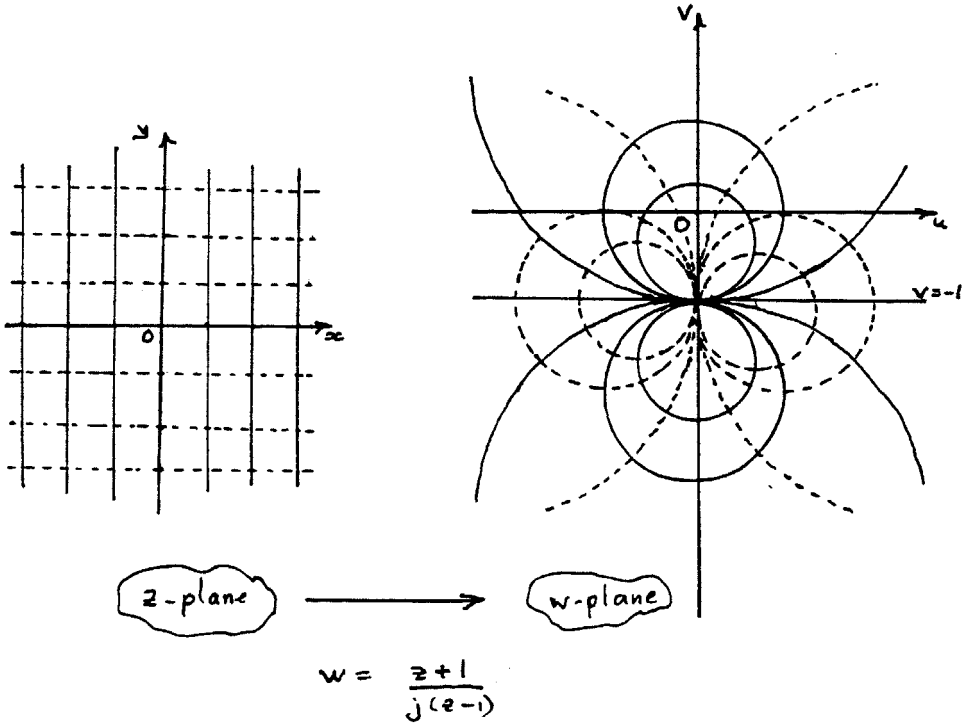
$$\text{or} \quad jz^2 - (j+1)z - 1 = 0$$

$$z = \frac{(j+1) \pm \sqrt{(j+1)^2 + 4j}}{2j}$$

$$= \frac{(j+1) \pm \sqrt{6j}}{2j}$$

$$= \frac{1}{2}(j-1)(-1 \pm \sqrt{3})$$

(since $\sqrt{6j} = \pm(1+j)\sqrt{3}$).



■ **13** $w = \dfrac{1+j}{z}$

13(a)

$$z = 1 \Rightarrow w = 1 + j$$

$$z = 1 - j \Rightarrow w = \frac{1+j}{1-j} = j$$

$$z = 0 \Rightarrow w = \infty$$

13(b) $\quad | w | = \dfrac{| 1 + j |}{| z |} = \dfrac{\sqrt{2}}{| z |}$

so that $| z | = \dfrac{\sqrt{2}}{| w |} < 1 \Rightarrow | w | > \sqrt{2}$

that is, interior of the unit circle maps to the exterior of the circle, centre as the origin and radius $\sqrt{2}$.

13(c) $\quad z = \dfrac{1+j}{w}$

$$\Rightarrow x + jy = \frac{(1+j)}{u^2+v^2}(u - jv) \text{ so that}$$
$$x = \frac{u+v}{u^2+v^2} \ , \ y = \frac{u-v}{u^2+v^2}$$

Therefore $x = y$ corresponds to $v = 0$ (the real axis) and $x + y = 1$ corresponds to $\dfrac{2u}{u^2+v^2} = 1$ that is $(u-1)^2 + v^2 = 1$ a circle, centre (1.0) and radius 1.

13(d) The fixed point of the mapping is given by $z^2 = 1 + j$. Using the polar form $1 + j = \sqrt{2}e^{\pi j/4}$, so $z = \pm 2^{1/4} e^{\pi j/8}$

■ **14** The bilinear transformation

$$w = \frac{z+1}{z-1}$$

Writing $z = x + jy$, $w = u + jv$ and equating real and imaginary parts gives

$$u = \frac{x^2+y^2-1}{(1+x)^2+y^2} \ , \ v = \frac{2y}{(1+x)^2+y^2}$$

Hence, all points on the circle $x^2 + y^2 = 1$ correspond to $u = 0$.
From the point $(0, -1)$ to the point $(0, 1)$ on the circle $x^2 + y^2 = 1$ we use the Parameterization $x = \cos\theta, y = \sin\theta$, $\pi/2 \leq \theta \leq 3\pi/2$. Using $v = \dfrac{2y}{(1+x)^2+y^2} = \dfrac{2y}{1+x^2+y^2+2x}$ we note that $v = \dfrac{y}{1+x}$ on $x^2 + y^2 = 1$, so that $v = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = \tan\dfrac{1}{2}\theta$ and between $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}, \tan\dfrac{1}{2}\theta$ ranges from 1 to ∞ and from $-\infty$ to -1 hence $| v | \geq 1$.

■ **15(a)** With $w = u + jv$ and $z = x + jy$

The transformation $w = \dfrac{z+j}{z-3}$ implies $z = \dfrac{3w+j}{w-1}$ from which we deduce that

$$x = \frac{3(u^2+v^2) - 3u + v}{(u-1)^2 + v^2}\ ,\ y = \frac{u - 3v - 1}{(u-1)^2+v^2}$$

$$\text{and} \quad u = \frac{x^2+y^2-3x+y}{(x-3)^2+y^2}\ ,\ v = \frac{x-3y-3}{(x-3)^2+y^2}$$

The line $y = 0$ corresponds to the line $u - 3v - 1 = 0$ in the w plane. The line $x = y$ corresponds to the curve

$$3(u^2+v^2) - 3u + v = u - 3v - 1$$

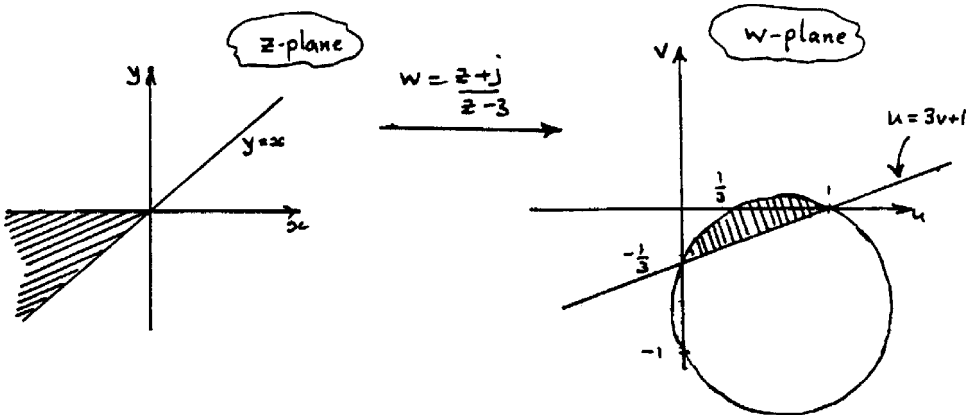
$$\text{that is,} \quad \left(u - \frac{2}{3}\right)^2 + \left(v + \frac{2}{3}\right)^2 = \frac{5}{9} \tag{1}$$

a circle centre $(\frac{2}{3}, -\frac{2}{3})$ and radius $\frac{1}{3}\sqrt{5}$ in the w plane.

The origin in the z plane (the intersection of the line $y = 0$ and $x = y$) corresponds to the point $w = -j\frac{1}{3}$ in the w plane. The point at infinity in the z plane (the other 'intersection') corresponds to the point $w = 1$ in the w plane.

The origin (in the w plane) lies *outside* the circle (1), and is also outside the wedge shaped region in the z plane ($z = -j3$ is its image).

So, the following figure can be drawn:



The point $w = \dfrac{2}{3}$ lies inside the shaded region in the w plane, and corresponds to the point $z = \dfrac{3\cdot\frac{2}{3} + j}{\frac{2}{3} - 1} = -3(2+j) = -6 - 3j$ inside the shaded region of the z plane. (This is a useful check.)

15(b) The fact that $w = 1$ does not correspond to any finite value of z has already been established.
Consider the equation $w = \dfrac{z+j}{z-3}$.
Taking the modulus of both sides gives

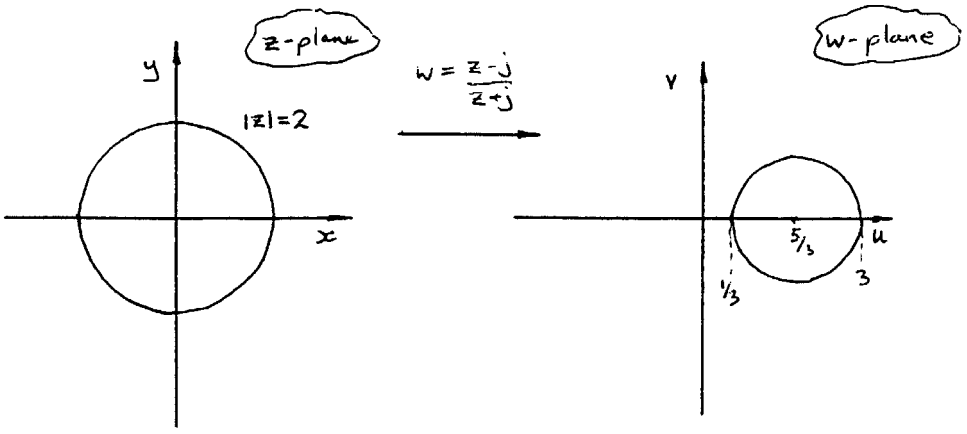
$$| w | = \left| \frac{z+j}{z-3} \right|$$

$$\begin{aligned} \text{If} \quad | w | = 1 \Rightarrow & | z+j | = | z-3 | \\ \text{or} \quad & x^2 + (y+1)^2 = (x-3)^2 + y^2 \\ & x^2 + y^2 + 2y + 1 = x^2 - 6x + 9 + y^2 \\ \text{so that,} \quad & 2y = -6x + 8 \\ \text{or} \quad & y + 3x = 4 \end{aligned}$$

Hence the unit circle in the w plane, $| w | = 1$, corresponds to the line $y + 3x = 4$.

■ **16** If $w = \dfrac{z-j}{z+j}$
then $z = \dfrac{-wj - j}{w-1}$
so that $| z | = \left| \dfrac{w+1}{w-1} \right|$.
So if $| z | = 2$, $| w+1 | = 2 | w-1 |$
or $(u+1)^2 + v^2 = 4(u-1)^2 + 4v^2$
which simplifies to

$$\left(u - \frac{5}{3}\right)^2 + v^2 = \frac{16}{9}, \quad \text{a circle centre } \left(\frac{5}{3}, 0\right) \text{ and radius } \frac{4}{3}$$



■ **17** If $w = e^{j\theta_0}\dfrac{z - z_0}{z - z_0^*}$ then,

taking modulus

$$| w |= \Big|\frac{z - z_0}{z - z_0^*}\Big| \text{ since } | e^{j\theta_0} |= 1$$

If z is real (i.e. z is on the real axis) then

$$| z - z_0 |=| z - z_0^* |= \left[(x - x_0)^2 + y_0^2\right]^{1/2} \text{ and } z_0 = x_0 + jy_0$$

Hence $| w |= 1$. Thus the real axis in the z plane corresponds to the unit circle $| w |= 1$ in the w plane. Making z the subject of the transformation gives

$$z = \frac{wz_0^* - e^{j\theta_0}z_0}{w - e^{j\theta_0}}$$

Hence the origin in the w plane maps to $z = z_0$.

Thus the inside of the unit circle in the w plane corresponds to the upper half of the z plane provided

$$Im\{z_0\} > 0$$

Since $w = 0$ maps to $z = z_0$ and $\underline{z_0 = j}$ and $z = \infty$ maps to $w = e^{j\theta_0} = -1$ it gives $\underline{\theta_0 = \pi}$.

■ **18** For the transformation

$$w = \frac{2jz}{z + j}$$

the fixed points are given by

$$z = \frac{2jz}{z + j}$$

$$z^2 + jz = 2jz$$

$$\text{or} \quad z(z - j) = 0, \; z = 0 \text{ or } j$$

Hence circular arcs or straight lines through $z = 0, j$ are transformed to circular arcs or straight lines through $w = 0, j$ by the properties of bilinear transformation (section 4.2.4).

The inverse transformation is

$$z = \frac{jw}{2j - w}$$

$| z - \frac{1}{2} | < \frac{1}{2}$ becomes $\left|\frac{jw}{2j-w} - \frac{1}{2}\right| < \frac{1}{2}$ which simplifies to $| w - 1 | < 1$ (use $w = u + jv$ and split into real and imaginary parts).
Similarly, $| z - \frac{1}{2} | < \frac{1}{2}$ becomes $| w - \frac{4}{3} | > \frac{2}{3}$

■ **19** The general bilinear mapping is

$$w = \frac{az + b}{cz + d}$$

if $w = 0$ corresponds to $z = z_0$ then

$$w = \frac{(z - z_0)e^{j\theta_0}}{cz + d}$$

If, additionally, $| w | = 1$ is mapped to $| z | = 1$ then $\left|\frac{z - z_0}{cz + d}\right| = 1$ and the inverse of z_0 is also mapped to the inverse of $w = 0$ that is, $w = \infty$.
Hence $cz + d$ can be replaced by $z_0^* z - 1$ giving the mapping as

$$w = e^{j\theta_0}\left(\frac{z - z_0}{z_0^* z - 1}\right)$$

where θ_0 is any real number.

Exercises 4.2.7

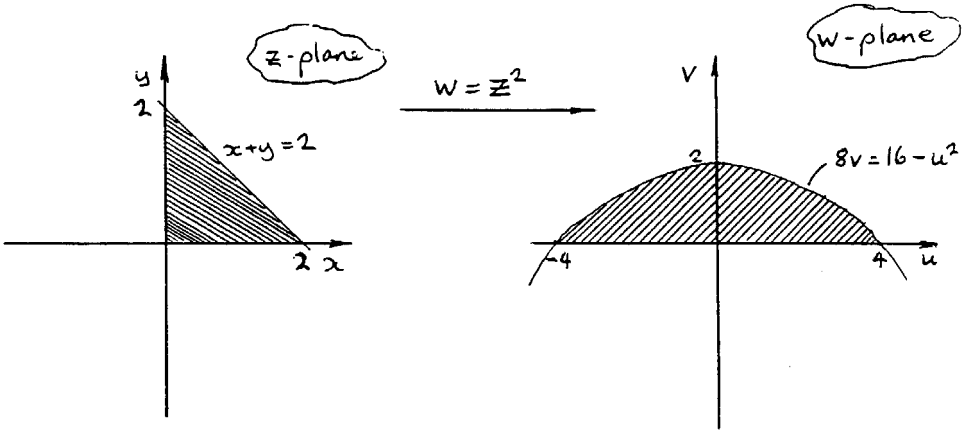
■ **20** Under the mapping $w = z^2,\ u = x^2 - y^2, v = 2xy$
It is not possible to achieve formulae of the type $x = \phi(u, v),\ y = \psi(u, v)$, however we can use $u = x^2 - y^2,\ v = 2xy$ to determine images. Points $(0 + j0), (2 + j0)$ and $(0 + j2)$ transform to $(0 + j0), (4 + j0)$ and $(-4 + j0)$ respectively.
The positive real axis $y = 0, x \geq 0$ transforms to the (positive) real axis $v = 0, u = x^2$.
The positive imaginary axis $x = 0, y \geq 0$ transforms to the (negative) real axis $v = 0, u = -y^2$.
The line joining the point $2 + j^0$ to the point $0 + j^2$ has equation $x + y = 2$.
By using the equations $u = x^2 - y^2$ and $v = 2xy$ we obtain

$$u = 4(1 - y),\ v = 2y(2 - y)$$

from which, eliminating y we get

$$8v = 16 - u^2$$

Hence we deduce the following picture:



■ **21** Under the transformation $w = z^2$, $u = x^2 - y^2$ and $v = 2xy$.
Hence the line $y = x$ transforms to $u = 0, v > 0$
and the line $y = -x$ transforms to $u = 0, v < 0$.
The line $y = mx$ transforms to $v = \dfrac{2m}{1-m^2}u$.
Putting $m = \tan\theta_0$, $\dfrac{2m}{1-m^2} = \tan 2\theta_0$.
Hence $y = x\tan\theta_0$ transforms to $v = u\tan 2\theta$.
Thus lines through the origin of slope θ_0 in the z plane transform to lines through the origin of slope $2\theta_0$ in the w plane. Hence the angle between the lines through the origin in the z plane is doubled by the transformation $w = z^2$.

■ **22** $w = z^n$
Writing $z = re^{j\theta}$, $w = r^n e^{nj\theta}$

22(a) Circles $\mid z \mid = r$ are transformed to circles $\mid w \mid = r^n$

22(b) Straight lines passing through the origin intersecting with angle θ_0 are $\theta = k$ and $\theta = k + \theta_0$. These are transformed to $w = r^n e^{njk}$ and $w = r^n e^{nj(k+\theta_0)}$ that is, lines $\phi = nk$ and $\phi = nk + n\theta_0$ as required.

■ **23** If $w = \dfrac{1+z^2}{z} = z + \dfrac{1}{z} = z + \dfrac{z^*}{\mid z \mid^2}$
then $u = x + \dfrac{x}{x^2+y^2}$ and $v = y - \dfrac{y}{x^2+y^2}$

If $|\,z\,| = r$, then $u = x\big(1 + \frac{1}{r^2}\big)$ and $v = y\big(1 - \frac{1}{r^2}\big)$

$$x = \frac{r^2 u}{r^2 + 1}\ ,\ y = \frac{r^2 v}{r^2 - 1}$$

Squaring and adding gives $\big(\frac{r^2 u}{r^2+1}\big)^2 + \big(\frac{r^2 v}{r^2-1}\big)^2 = r^2\ (r \neq 1)$ (I).
If $r = 1$ and $v = 0, |\,x\,| \leq 1$ (because $x^2 = 1 - y^2$) and $u = 2x$. Hence the image of the unit circle $|\,z\,| = 1$, that is, $-2 \leq u \leq 2$, $v = 0$ and the portion of the real axis in the w plane is between -2 and $+2$.
The curves (I) are ellipses, major axis $\frac{1+r^2}{r^2}$ and minor axis $\frac{|\,r^2 - 1\,|}{r^2}$ if r is very large, and both of these quantities tend to 1. Hence the image curve I tends to a circle $u^2 + v^2 = r^2$.

Exercises 4.3.3

- **24(a)**

$$\begin{aligned} ze^z &= (x + jy)e^{x+jy} \\ &= e^x(x + jy)(\cos y + j \sin y) \\ &= e^x(x\cos y - y\sin y) + je^x(y\cos y + x\sin y) \\ \text{so}\quad u &= (x\cos y - y\sin y)e^x\ ,\ v = (y\cos y + x\sin y)e^x \end{aligned}$$

We need to check the Cauchy–Riemann equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= (x\cos y - y\sin y + \cos y)e^x \\ \frac{\partial u}{\partial y} &= (-x\sin y - y\cos y - \sin y)e^x \\ \frac{\partial v}{\partial x} &= (y\cos y + x\sin y + \sin y)e^x \\ \frac{\partial v}{\partial y} &= (-y\sin y + \cos y + x\cos y)e^x \end{aligned}$$

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
Thus the Cauchy–Riemann equations are valid and

$$\frac{d}{dz}(ze^z) = (z + 1)e^z$$

24(b) Following the same procedure as in (a), we deduce that $\sin 4z$ is analytic with derivative $4\cos 4z$.

24(c) This time, $zz^* = x^2 + y^2$ which is real.
Obviously, therefore, $\dfrac{\partial u}{\partial x}(= 2x) \neq \dfrac{\partial v}{\partial y}(= 0)$.
Thus zz^* is not analytic.

24(d) Similarly to part (a), $\cos 2z$ is analytic with derivative $-2\sin 2z$.

■ **25** $w = x^2 + ay^2 - 2xy + j(bx^2 - y^2 + 2xy) = u + jy$

$$\frac{\partial u}{\partial x} = 2x - 2y\ ,\ \frac{\partial u}{\partial y} = 2ay - 2x$$
$$\frac{\partial v}{\partial x} = 2bx + 2y\ ,\ \frac{\partial v}{\partial y} = -2y + 2x$$

The Cauchy–Riemann equations are $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$.
The second is satisfied and the first only holds if $a = -1, b = 1$. Since $w(z) = w(x+jy)$ we simply put $y = 0$ which gives $w(x) = x^2 + jx^2$ and hence $w(z) = z^2 + jz^2$ and $\dfrac{dw}{dz} = 2(1 + j)z$

■ **26** With $u = 2x(1 - y) = 2x - 2xy$

$$\frac{\partial u}{\partial x} = 2 - 2y\ ,\ \frac{\partial u}{\partial y} = -2x$$

The Cauchy–Riemann equations demand

$$\frac{\partial v}{\partial y} = 2 - 2y\ ,\ \frac{\partial v}{\partial x} = 2x$$

Integrating and comparing, these give

$$v = x^2 - y^2 + 2y + C \text{ (take } C = 0)$$

Form $u + jv = 2x - 2xy + j(x^2 - y^2 + 2y) = w(z)$.

Since $z = x + jy$, if we put $y = 0$, we can find $w(x)$ which will give the functional form of w. Thus

$$w(x) = 2x + jx^2$$

Hence $w(z) = 2z + jz^2$

■ **27**

$$\begin{aligned}
\phi(x, y) &= e^x(x\cos y - y\sin y) \\
\frac{\partial \phi}{\partial x} &= e^x(x\cos y - y\sin y + \cos y) \\
\frac{\partial \phi}{\partial y} &= e^x(-x\sin y - y\cos y - \sin y) \\
\frac{\partial^2 \phi}{\partial x^2} &= e^x(x\cos y - y\sin y + 2\cos y) \\
\frac{\partial^2 \phi}{\partial y^2} &= e^x(-x\cos y + y\sin y - 2\cos y)
\end{aligned}$$

hence $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ and ϕ is harmonic.
Writing $z = \phi(x, y) + j\psi(x, y)$, the Cauchy–Riemann equations demand

$$\begin{aligned}
\frac{\partial \psi}{\partial x} &= -\frac{\partial \phi}{\partial y} = e^x(x\sin y + y\cos y + \sin y) \\
\frac{\partial \psi}{\partial y} &= \frac{\partial \phi}{\partial x} = e^x(x\cos y - y\sin y + \cos y)
\end{aligned}$$

Integrating $\frac{\partial \psi}{\partial x}$ with respect to x (using integration by parts for the first term) gives $\psi = e^x(x\sin y + y\cos y) + f(y)$. Examining $\phi(x, y)$ demands that $f(y) = 0$ because all terms will be multiplied by e^x.
Hence $w(z) = \phi(x, y) + j\psi(x, y) = e^x(x\cos y - y\sin y) + je^x(x\sin y + y\cos y)$ and $w(x + j0) = w(x) = xe^x$. Hence $w(z) = ze^z$.

■ **28** Here we have $u(x, y) = \sin x \cosh y$

$$\begin{aligned}
\text{so that} \quad & \frac{\partial u}{\partial x} = \cos x \cosh y \quad \text{and} \quad \frac{\partial u}{\partial y} = \sin x \sinh y \\
\text{hence} \quad & \frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \sin x \cosh y
\end{aligned}$$

so that $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and u is harmonic.

Using the Cauchy–Riemann equations gives $v = \cos x \sinh y$ so that $u + jv = w(z) = \sin x \cosh y + j \cos x \sinh y$. Putting $y = 0$ gives $w(x + j0) = \sin x$ so that $w(z) = \sin z$.

■ **29** The orthogonal trajectories of a family of curves $\phi(x, y) = \alpha$ are $\psi(x, y) = \beta$ where ϕ and ψ are conjugate functions: that is, $\phi(x, y) + j\psi(x, y) = w(z)$ which is an analytic function.
Proceeding as in the previous examples.

29(a) If $\phi(x, y) = x^3y - xy^3$ then $\psi(x, y) = \dfrac{1}{4}(x^4 + y^4) - \dfrac{3}{2}x^2y^2$

29(b) If $\phi(x, y) = e^{-x} \cos y + xy$ then $\psi(x, y) = e^{-x} \sin y + \dfrac{1}{2}(x^2 - y^2)$.

Hence the orthogonal trajectories are,
29(a) $x^4 - 6x^2y^2 + y^4 = \beta$, a constant
29(b) $2e^{-x} \sin y + x^2 - y^2 = \beta$, a constant.

■ **30(a)** z^2e^{2z}

$= (x + jy)^2 e^{2(x+jy)}$
$= (x^2 - y^2 + 2jxy)(e^{2x}(\cos 2y + j \sin 2y))$
$= e^{2x}((x^2 - y^2) \cos 2y - 2xy \sin 2y) + je^{2x}((x^2 - y^2) \sin 2y + 2xy \cos 2y)$

30(b) $\sin 2z$
$= \sin(2x + j2y)$
$= \sin 2x \cosh 2y + j \cos 2x \sinh 2y$

Straightforward calculus reveals that both functions obey the Cauchy–Riemann equations and are thus analytic. Their derivatives are **(a)** $(2z^2 + 2z)e^{2z}$ and **(b)** $2 \cos 2z$ respectively.

■ **31** Writing $w = \sin^{-1} z$ we can say that

$$z = \sin w = \sin(u + jv) = \sin u \cosh v + j \cos u \sinh v$$

so that, equating real and imaginary parts,

$$x = \sin u \cosh v$$
$$\text{and} \quad y = \cos u \sinh v$$

Squaring and adding gives

$$\begin{aligned} x^2 + y^2 &= \sin^2 u \cosh^2 v + \cos^2 u \sinh^2 v \\ &= \sin^2 u \cosh^2 v + (1 - \sin^2 u)(\cosh^2 v - 1) \\ &= \sin^2 u + \cosh^2 v - 1 \end{aligned}$$

from which

$$x^2 + y^2 + 1 = \sin^2 u + \frac{x^2}{\sin^2 u} \tag{I}$$

Solving for $\sin^2 u$ gives

$$\sin^2 u = \frac{1}{2}(1 + x^2 + y^2) - \frac{1}{2}\sqrt{(1 + x^2 + y^2)^2 - 4x^2}$$

where the minus sign is taken, since with $u = \pi/2$ (i.e. $x = \cosh v, y = 0$) inconsistencies result otherwise. From $\cosh^2 v = \frac{x^2}{\sin^2 u}$ we obtain

$$\cosh^2 v = \frac{1}{2}(1 + x^2 + y^2) + \frac{1}{2}\sqrt{(1 + x^2 + y^2)^2 - 4x^2}$$

(This is most easily found by solving equation (I) for $\frac{1}{\sin^2 u}$ then using $\cosh^2 v = \frac{x^2}{\sin^2 u}$.)

Square rooting and inverting give u and v in terms of x and y. It can be shown that the expression under the square root sign is positive, for $1 + x^2 + y^2 - 2x = (x-1)^2 + y^2 \geq 0$ for all real x and y thus $(1 + x^2 + y^2)^2 \geq 4x^2$. Hence $w = \sin^{-1} z$ is an analytic function with derivative $\frac{1}{\sqrt{1-z^2}}$.

■ **32**

$$\begin{aligned} |\sin z|^2 &= |\sin x \cosh y + j \cos x \sinh y|^2 \\ &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \cosh^2 y - \cos^2 x = \sinh^2 y + \sin^2 x \end{aligned}$$

The result follows immediately from the last two expressions.

Exercises 4.3.5

- **33** Mappings are not conformal at the points where $\dfrac{dw}{dz} = 0$

- **33(a)** $\dfrac{dw}{dz} = 2z = 0$ when $z = 0$. $z = 0$ is the only point where the mapping fails to be conformal.

33(b) $\dfrac{dw}{dz} = 6z^2 - 42z + 72 = 0$ when $z^2 = 7z + 12 = 0$ that is, non-conformal points are $z = 4, z = 3$ (both real).

33(c) $\dfrac{dw}{dz} = 8 - \dfrac{1}{z^3} = 0$ when $z^3 = \dfrac{1}{8}$ giving $\dfrac{1}{2}, \dfrac{1 \pm j\sqrt{3}}{4}$ as non-conformal points.

- **34** Proceeding as in Example 4.13, the mapping

$$w = z - \frac{1}{z}$$

has a fixed point at $z = \infty$, is analytic everywhere except at $z = 0$ and conformal except where $\dfrac{dw}{dz} = 0$

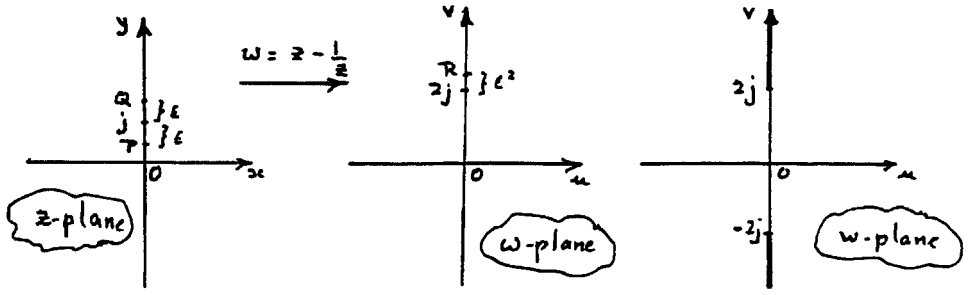
$$\text{that is,} \quad 1 + \frac{1}{z^2} = 0, \; z = \pm j$$

Since both of these occur on the imaginary axis, consideration of this axis is adequate to completely analyse this mapping.

The image of $z = j$ is $w = 2j$, and the image of $z = -j$ is $w = -2j$. Writing $z = j + j\varepsilon$, ε real, we find that

$$\begin{aligned} w &= j\varepsilon - \frac{1}{j + j\varepsilon} \\ &= j[1 + \varepsilon - (1 + \varepsilon)^{-1}] \\ &= j[1 + \varepsilon + 1 - \varepsilon + \varepsilon^2 + \ldots] \\ &\simeq j[2 + \varepsilon^2] \end{aligned}$$

So, no matter whether $\varepsilon > 0$ or $\varepsilon < 0$, the image point of $z = j + j\varepsilon$ is <u>above</u> $w = j2$ on the imaginary axis.



that is, points Q and P in the z plane both map to R in the w plane in a manner to Example 4.13, the non-conformality of $z = \pm j$ is confirmed and as the imaginary axis (in the z plane) is traversed from $-jz$ to 0, the imaginary axis (in similar the w plane) is traversed from $-jz$ to $-j2$ and back to $-j\infty$ (when $z = -j, w$ reaches $-j2$). Similarly, as the imaginary axis (in the z plane) is traversed from $+j\infty$ to 0, the imaginary axis (in the w plane) is traversed from $+j\infty$ to $+j2$ and back to $+j\infty$ again.

Finally, points on the imaginary axis in the w plane such that $w = aj, -2 < a < 2$, do not arise from any points on the imaginary axis in the z plane. This point is obvious once one solves

$$aj = z - \frac{1}{z}$$

to obtain

$$z = \frac{1}{2}aj \pm \frac{1}{2}\sqrt{4 - a^2}$$

■ **35** If $w = e^z$

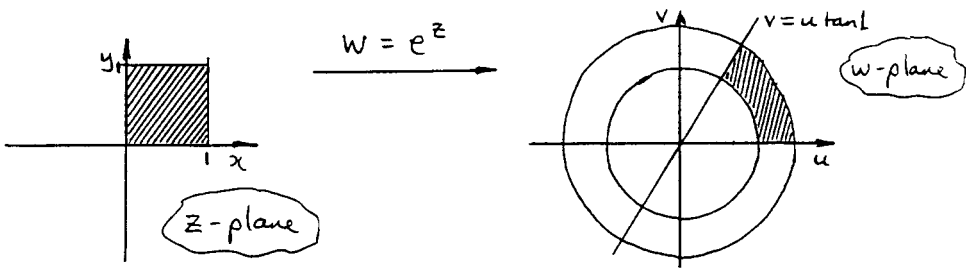
then $u = e^x \cos y$ and $v = e^x \sin y$

Hence the expressions $u^2 + v^2 = e^{2x}$ and $v = u \tan y$ can be derived.

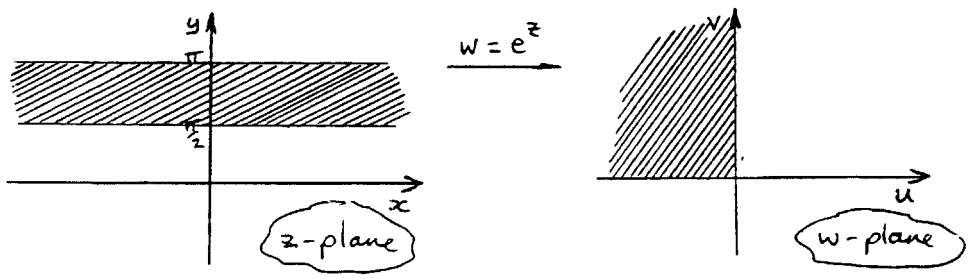
35(a) $0 \leq x < \infty$ is mapped to the exterior of the unit circle $u^2 + v^2 = 1$

35(b) $0 \leq x \leq 1$ is mapped to the annulus $1 \leq u^2 + v^2 \leq e^2$

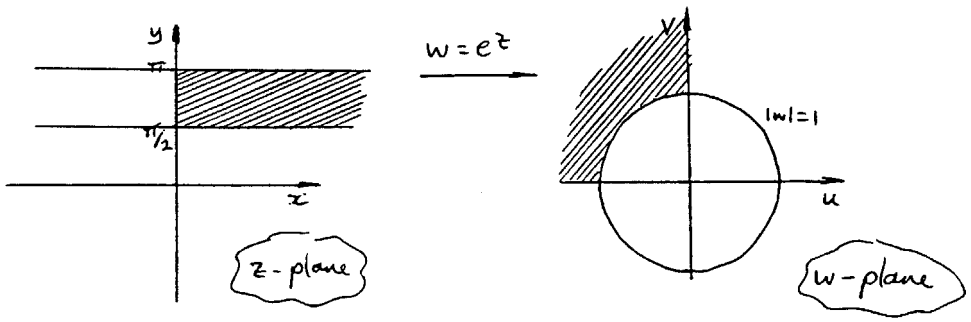
$0 \leq y \leq 1$ is mapped to the region between the radiating lines $v = 0$ and $v = u \tan 1$.



35(c) $\frac{1}{2}\pi \leq y \leq \pi$ is mapped to the region between $u = 0(v > 0)$ and $v = 0(u < 0)$
that is,



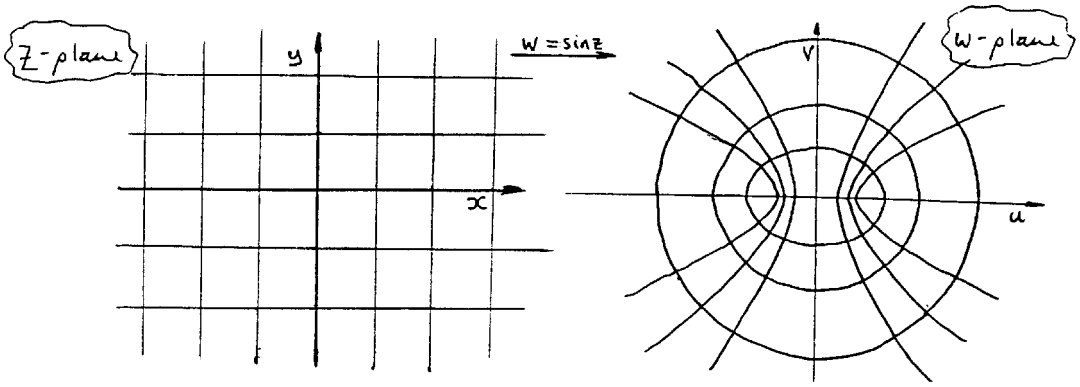
Thus if $0 \leq x < \infty$ then the image region in the w plane is in the shaded quadrant, but outside the unit circle.



■ **36** If $w = \sin z$ then $\frac{dw}{dz} = \cos z$.
Since $\cos z = 0$ when $z = (2n+1)\pi/2$ these are the points where the mapping is not conformal

$$w = \sin z \Rightarrow u + jv = \sin x \cosh y + j \cos x \sinh y$$

Hence $v = \sin x \cosh y,\ v = \cos x \sinh y$
thus lines $x = k$ transform to $\left(\frac{u}{\sin k}\right)^2 - \left(\frac{v}{\cos k}\right)^2 = 1$ (hyperbolae)
and lines $y = k$ transform to $\left(\frac{u}{\cosh l}\right)^2 + \left(\frac{v}{\sinh l}\right)^2 = 1$ (ellipses)



■ **37** If $z = \zeta + \frac{a^2}{\zeta}$ and $\zeta = Re^{j\theta}$
then $z = Re^{j\theta} + \frac{a^2}{R}e^{-j\theta}$
so that $x = \left(R + \frac{a^2}{R}\right)\cos\theta$ and $y = \left(R - \frac{a^2}{R}\right)\sin\theta$.
If $R = a, x = 2a\cos\theta$ and $y = 0$ then the real line between $\pm 2a$ is traversed.
Length of line segment $= 4a$.
For a circle of radius b,

$$x = \left(b + \frac{a^2}{b}\right)\cos\theta,\ y = \left(b - \frac{a^2}{b}\right)\sin\theta$$

Hence the image in the z plane is an ellipse of the form

$$\frac{b^2x^2}{(a^2+b^2)^2} + \frac{b^2y^2}{(b^2-a^2)^2} = 1$$

Exercises 4.4.2

■ **38(a)**

$$\frac{1}{z-j} = (z-j)^{-1} = j\left(1 - \frac{z}{j}\right)^{-1} = j\left(1 + \frac{z}{j} + \left(\frac{z}{j}\right)^2 + \ldots\right)$$
$$= j + z - jz^2 - z^3 + jz^4 \ldots$$

38(b)

$$\frac{1}{z-j} = \frac{1}{z}\left(1 - \frac{j}{z}\right)^{-1} = \frac{1}{z}\left(1 + \frac{j}{z} + \left(\frac{j}{z}\right)^2 + \ldots\right)$$
$$= \frac{1}{z} + \frac{j}{z^2} - \frac{1}{z^3} - \frac{j}{z^4} + \ldots$$

38(c) In order that $| z-1-j | < \sqrt{2}$ we write

$$\frac{1}{z-j} = \frac{1}{z-1-j+1} = (1+z-1+j)^{-1}$$
$$= 1-(z-1-j)+(z-1-j)^2-(z-1-j)^3+\ldots$$

Which is valid inside $| z-1-j | < | 1-j | = \sqrt{2}$.

■ **39**

$$\frac{1}{z^2+1} = (z^2+1)^{-1} = 1-z^2+z^4-z^6+\ldots$$
$$\text{where } | z | < 1.$$

Using the fact that we can differentiate power series term-by-term and the radius of convergence remains unaltered

$$-\frac{2z}{(z^2+1)^2} = -2z+4z^3-6z^5+\ldots$$

so

39(a)

$$\frac{1}{(z^2+1)^2} = 1-2z^2+3z^4-4z^6+5z^8+\ldots$$
$$| z | < 1$$

Operating on $\dfrac{1}{(z^2+1)^2}$ in a similar fashion gives

$$-\frac{4z}{(z^2+1)^3} = -4z+12z^3-24z^5+40z^7\ldots$$

so

39(b)

$$\frac{1}{(z^2+1)^3} = 1-3z^2+6z^4-10z^6+15z^8+\ldots$$
$$| z | < 1$$

Exercises 4.4.4

■ **40** Taylor's theorem is

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \dots$$

We thus compute $f(z)$ and its first few derivatives then evaluate them at $z = a$.

40(a)

$$f(z) = \frac{1}{1+z},\ f'(z) = -\frac{1}{(1+z)^2},\ f''(z) = \frac{2}{(1+z)^3} \text{ and } f'''(z) = -\frac{6}{(1+z)^4}$$

Hence

$$f(1) = \frac{1}{2}, f'(1) = -\frac{1}{4},\ f''(1) = \frac{2}{8} = \frac{1}{4} \text{ and } f'''(1) = -\frac{3}{8}$$

thus

$$\frac{1}{1+z} = \frac{1}{2} - \frac{1}{4}(z-1) + \frac{1}{8}(z-1)^2 - \frac{1}{16}(z-1)^3 + \dots$$

The radius of convergence is the distance between the nearest singularity of $f(z)$ to the point about which the expansion is made. The point $z = -1$ is the only singularity and the distance between this and $z = 1$ is 2 (along the real axis).

40(b)

$$f(z) = \frac{1}{z(z-4j)} = \frac{j}{4}\left(\frac{1}{z} - \frac{1}{z-4j}\right) \text{ using partial fractions}$$

$$f'(z) = \frac{j}{4}\left(-\frac{1}{z^2} + \frac{1}{(z-4j)^2}\right) \qquad f(1) = \frac{1}{4}; f'(1) = 0$$

$$f''(z) = \frac{j}{4}\left(-\frac{2}{z^3} + \frac{2}{(z-4j)^3}\right) \qquad f''(1) = -\frac{1}{8}; f'''(1) = 0$$

$$f'''(z) = \frac{j}{4}\left(-\frac{6}{z^4} + \frac{6}{(z-4j)^4}\right) \qquad f^{iv}(1) = +\frac{3}{8}; f^{v}(1) = 0$$

$$f^{iv}(z) = \frac{j}{4}\left(-\frac{24}{z^4} + \frac{24}{(z-4j)^5}\right) \qquad f^{vi}(1) = -\frac{45}{16}$$

$$f^{v}(z) = \frac{j}{4}\left(-\frac{120}{z^6} + \frac{120}{(z-4j)^6}\right) \text{ etc.}$$

Thus

$$\frac{1}{z(z-4j)} = \frac{1}{4} - \frac{1}{16}(z-2j)^2 + \frac{1}{64}(z-2j)^4 - \frac{1}{256}(z-2j)^6 + \ldots$$

The radius of convergence is 2 since $z = 2j$ is 2 from the singularities at $z = 0$ and $z = 4j$.

40(c) $f(z) = \frac{1}{z^2}$

gives $f'(z) = -\frac{2}{z^3}$, $f''(z) = \frac{6}{z^4}$ and $f'''(z) = -\frac{24}{z^5}$.
Putting $z = 1 + j$ gives

$$f(1+j) = -\frac{j}{2},\ f'(1+j) = -\frac{2}{(1+j)^3} = \frac{1+j}{2}$$
$$f''(1+j) = \frac{6}{(1+j)^4} = \frac{3}{2} \text{ and } f'''(1+j) = -\frac{24}{(1+j)^5} = -3(1-j)$$

Hence

$$\frac{1}{z^2} = -\frac{j}{2} + \frac{1}{2}(1+j)(z-1-j) + \frac{3}{4}(z-1-j)^2 - \frac{1}{2}(1-j)(z-1-j)^3 + \ldots$$

The radius of convergence is the distance between the origin (a double pole) and $1 + j$ that is, $\sqrt{2}$.

■ **41** With $f(z) = \dfrac{1}{1+z+z^2}$
we could use the binomial expansion

$$f(z) = (1+z+z^2)^{-1} \text{ gathering terms to } \mathrm{O}(z^3)$$

This is certainly more efficient than using the derivatives of $f(z)$. However, the best way is to use the fact that $(z^3 - 1) = (z-1)(z^2+z+1)$. That is

$$\frac{1}{1+z+z^2} = \frac{z-1}{z^3-1} = \frac{1-z}{1-z^3}$$
$$= (1-z)(1-z^3)^{-1}$$

$$= 1 - z + z^3 \ldots$$

to order z^3

valid in the region $|\, z \,| < 1$.

■ **42** If $f(z) = \frac{1}{z^4 - 1}$ the singularities are at the points where $z^4 = 1$ that is, $z = 1, -1, -j$ and j. The radii of convergence are the minimum distances of the points $z = 0,\ 1 + j$ and $2 + j2$ from these singularities.
$z = 0$ is equidistant (1) from each radius of convergence $= 1$.
$z = 1 + j$ is distance (1) from $z = 1$ and $z = j$ with radius of convergence $= 1$.
$z = z + j2$ is a distance $|\, 2 + j2 - 1 \,|$ from 1 and a distance $|\, 2 + j2 - j \,|$ from j.
Both of these distances $= [2^2 + (2-1)^2]^{1/2} = \sqrt{5}$.

■ **43** If $f(z) = \tan z$ then $f'(z) = \sec^2(z)$ and $f'' = 2\sec^2 z \tan z$ but subsequent derivatives get cumbersome to compute (except by using a Computer Algebra package). Since $\tan z = \dfrac{\sin z}{\cos z}$, we can use the series for $\sin z$ and $\cos z$ as follows:

$$\begin{aligned}
\tan z &= \frac{z - \frac{z^3}{6} + \frac{z^5}{120}}{1 - \frac{z^2}{2} + \frac{z^4}{24}} \\
&= z\left(1 - \frac{z^2}{6} + \frac{z^4}{120}\right)\left(1 - (\frac{z^2}{2} - \frac{z^4}{24})\right)^{-1} \\
&= z\left(1 - \frac{z^2}{6} + \frac{z^4}{120}\right)\left(1 + \frac{z^2}{2} - \frac{z^4}{24} + (\frac{z^2}{2} - \frac{z^4}{24})^2\right) + \ldots \\
&= z + \frac{1}{3}z^3 + \left(\frac{1}{120} - \frac{1}{12} - \frac{1}{24} + \frac{1}{4}\right)z^5 + \ldots \\
\tan z &= z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots
\end{aligned}$$

Since $z = \pi/2$ is the closest singularity, the radius of convergence is $\frac{\pi}{2}$.

Exercises 4.4.6

■ **44** The function $\dfrac{1}{z(z-1)^2}$ has a simple pole at $z = 0$ and a double pole at $z = 1$. In order to find the Laurent expansions, we simply find the following binomial expansions

$$\frac{1}{z}(1-z)^{-2} = \frac{1}{z}(1+2z+3z^2+4z^3+\ldots)$$
$$= \frac{1}{z}+2+3z+4z^2+\ldots \text{ about } z=0$$
$$\text{valid for } 0 <| z |< 1$$

and

$$\frac{1}{(z-1)^2}(1-(1-z))^{-1}$$
$$= \frac{1}{(z-1)^2}[1+(1-z)+(1-z)^2+(1-z)^3+\ldots]$$
$$= \frac{1}{(1-z)^2}+\frac{1}{1-z}+1+(1-z)+(1-z)^2+\ldots$$
$$\text{valid for } 0 <| 1-z |< 1$$

■ **45(a)** With $f(z) = z^2 \sin\left(\frac{1}{z}\right)$, there is a singularity at $z=0$ and another at $z=\infty$. Expanding $\sin\left(\frac{1}{z}\right)$ as a power series in $\frac{1}{z}$ we find

$$z^2 \sin\left(\frac{1}{z}\right) = z^2\left(\frac{1}{z}-\frac{1}{3!z^3}+\frac{1}{5!z^5}-\cdots\right)$$
$$= z-\frac{1}{3!z}+\frac{1}{5!z^3}-\cdots$$
$$= \cdots+\frac{1}{5!z^3}-\frac{1}{3!}\frac{1}{z}+z.$$

Since the principal part is infinite, there must be an essential singularity at $z=0$.

(b) Writing $z=\frac{1}{w}$ in order to investigate $z=\infty$ we obtain

$$z^2 \sin\left(\frac{1}{z}\right) = \frac{1}{w^2}\sin w = \frac{1}{w^2}\left(w-\frac{w^3}{3!}+\frac{w^5}{5!}-\cdots\right)$$
$$= \frac{1}{w}-\frac{w}{3!}+\frac{w^3}{5!}-\cdots$$
$$= z-\frac{1}{z3!}+\frac{1}{z^35!}-\cdots$$

which implies a simple pole at $z=\infty$. (The expansion is the same as that about $z=0$, but re-interpreted.)

(c) At any other point $z^2 \sin\frac{1}{z}$ is regular and has a Taylor series of the form $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$. specifically, about $z=a$, $z^2 \sin\left(\frac{1}{z}\right) = a^2 \sin\frac{1}{a} + a_1 z + a_2 z^2 + \ldots$ where $a_1 = f'(a)$, $a_2 = f''(a)$, etc.

■ **46** With $f(z) = \frac{z}{(z-1)(2-z)}$
there are simple poles at $z = 1$ and $z = 2$.

46(a) Inside the unit circle $|\, z \,| = 1$, therefore there is a Taylor series

$$\begin{aligned}
&z(1-z)^{-1}(2-z)^{-1} \\
&= \frac{z}{2}(1-z)^{-1}\big(1-\frac{z}{2}\big)^{-1} \\
&= \frac{z}{2}(1+z+z^2+z^3+\ldots)\left(1+\frac{z}{2}+\big(\frac{z}{2}\big)^2+\big(\frac{z}{2}\big)^3+\cdots\right) \\
&= \frac{z}{2}+\frac{3}{4}z^2+\frac{z}{2}\left(z^2+\frac{1}{2}z^2+\frac{1}{4}z^2+\ldots\right)+\frac{z}{2}\left(z^3+\frac{1}{2}z^3+\frac{1}{4}z^3+\frac{1}{8}z^3+\ldots\right) \\
&= \frac{1}{2}z+\frac{3}{4}z^2+\frac{7}{8}z^3+\frac{15}{16}z^4+\cdots \quad |\, z \,| < 1
\end{aligned}$$

46(b) In the annulus $1 < |\, z \,| < 2$ we rearrange $f(z)$ to obtain a Laurent series as follows

$$\begin{aligned}
\frac{z}{(z-1)(z-2)} &= \frac{2}{z-2}-\frac{1}{z-1} = -\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \\
&= -\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) \\
&= \ldots-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{z}-1-\frac{z}{2}-\frac{z^2}{4}-\cdots
\end{aligned}$$

46(c) For $|\, z \,| > 2$ we rearrange as follows

$$\begin{aligned}
\frac{z}{(z-1)(z-2)} &= \frac{2}{z-2}-\frac{1}{z-1} \\
&= \frac{2}{z}\big(1-\frac{2}{z}\big)^{-1}-\frac{1}{z}\big(1-\frac{1}{z}\big)^{-1} \\
&= \frac{2}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) \\
&= \frac{1}{z}+\frac{3}{z^2}+\frac{7}{z^3}+\frac{15}{z^4}+\cdots
\end{aligned}$$

46(d) For $|\, z-1 \,| > 1$ we write $\frac{1}{z-1} = w$ and find a Taylor's series in w.
If $w = \frac{1}{z-1}$ then $wz - w = 1$ or $z = \frac{1+w}{w}$

so that

$$\begin{aligned}\frac{z}{(z-1)(z-2)} &= w\left(\frac{1+w}{1-w}\right)\\ &= w(1+w)(1+w+w^2+w^3+\ldots)\\ &= w+2w^2+2w^3+\ldots\\ &= \frac{1}{z-1}+\frac{2}{(z-1)^2}+\frac{2}{(z-1)^3}+\cdots\end{aligned}$$

46(e) For $0 <| z-2 |< 1$ we write $w = z-2$ hence

$$\begin{aligned}\frac{z}{(z-1)(z-2)} &= \frac{w+2}{w(w+1)} = \big(1+\frac{2}{w}\big)(1+w)^{-1}\\ &= \big(1+\frac{2}{w}\big)(1-w+w^2-w^3+\ldots)\\ &= \frac{2}{w}-1+w-w^2+w^3-\ldots\\ &= \frac{2}{(z-2)}-1+(z-2)-(z-2)^2+(z-2)^3\ldots\end{aligned}$$

Exercises 4.5.2

■ **47** The point at infinity is ignored in this question. Most if not all can be found immediately by inspection.

47(a) $\dfrac{\cos z}{z^2}$: double pole at $z = 0$, zeros whenever $\cos z = 0$ that is, $z = \frac{1}{2}(2n+1)\pi, n =$ integer.

47(b) $\dfrac{1}{(z+j)^2(z-j)}$: has a double pole at $z = -j$, a simple pole at $z = j$ and no zeros in the finite z plane.

47(c) $\dfrac{z}{z^4-1}$: simple poles at $z^4 = 1$ that is, $z = 1, -1, j, -j$ and a zero at $z = 0$.

47(d) $\cosh z$: since $\coth z = \dfrac{\cosh z}{\sinh z}$ this has simple poles at those points where $z = jn\pi$ and zero at those points where $z = \frac{1}{2}j(2n+1)\pi, n =$ integer.

47(e) $\dfrac{\sin z}{z^2+\pi^2}$: simple poles at $z=\pm j\pi$ and zeros at $z=n\pi, n=$ integer.

47(f) $e^{z/(1-z)}$: this has an essential singularity at $z=1$ and no zeros.

47(g) $\dfrac{z-1}{z^2+1}$: this has simple poles at $z=\pm j$ and a zero at $z=1$.

47(h) $\dfrac{z+j}{(z+2)^3(z-3)}$: this has a triple pole at $z=-2$, a simple pole at $z=3$ and a zero at $z=-j$.

47(i) $\dfrac{1}{z^2(z^2-4z+5)}$: this has simple poles at $z^2-4z+5=0$, that is, $z=5,-1$ and a double pole at $z=0$.

■ **48(a)** $\dfrac{1-\cos z}{z^2}$. In order to investigate this, we expand $\cos z$. Only $z=0$ is a possible (finite) singularity

$$\frac{1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)}{z^2}=\frac{1}{2!}-\frac{z^2}{4!}+\cdots$$

The RHS is a power series, thus the singularity at $z=0$ is removable.

48(b) $\dfrac{e^{z^2}}{z^3}$. Using the power series for e^{z^2} gives the expansion

$$\begin{aligned}\frac{e^{z^2}}{z^3}&=\frac{1}{z^3}\left(1+z^2+\frac{z^4}{2!}+\cdots\right)\\&=\frac{1}{z^3}+\frac{1}{z}+\frac{z}{2!}+\cdots\end{aligned}$$

$z=0$ is thus a pole of order 3.

48(c) $\frac{1}{z}\cosh\left(\frac{1}{z}\right)$. Obviously the point $z=0$ is a problem.
$\frac{1}{z}\left(1+\frac{1}{z^2 2!}+\frac{1}{z^4 4!}+\cdots\right)$ is the Laurent series which indicates that $z=0$ is an essential singularity.

48(d) $\tan^{-1}(z^2+2z+2)$. For this problem the easiest way to proceed is to find the Maclaurin series from first principles. At $z=0, \tan^{-1}(z^2+2z+2) = \tan^{-1} 2$ which is finite. This means that $z=0$ is a regular point, hence it is not actually necessary to find the Laurent series (in this case Maclaurin series) for the function. In fact

$$\tan^{-1}(z^2+2z+2) = \tan^{-1} 2 + \frac{2}{5}z - \frac{6}{25}z^2 + \cdots$$

■ **49** If $f(z) = \frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomials, then the only singularities of $f(z)$ are the algebraic zeros of $q(z)$. These zeros are either distinct or multiple. The distinct zeros give rise to simple poles of $f(z)$ whereas the multiple zeros give rise to poles of higher order. $f(z)$ can only have these kinds of singularity, although it may have none if q divides p, so that $f(z)$ is polynomial. $f(z)$ therefore cannot have an essential singularity.

Exercises 4.5.4

■ **50(a)** $\dfrac{2z+1}{(z^2-z-2)} = \dfrac{2z+1}{(z-2)(z+1)}$ hence the singularities are simple poles at $z=2, z=-1$.

Using the formula residue $= \lim_{z\to z_0}[(z-z_0)f(z)]$ the residues are $\frac{5}{3}$ at $z=2$ and $\frac{1}{3}$ at $z=-1$.

50(b) $\dfrac{1}{z^2(1-z)}$ has a simple pole at $z=1$ and a double pole at $z=0$.

The residue at $z=1$ is $\lim_{z\to 1}\left(-\dfrac{1}{z^2}\right) = -1$

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2}(1+z+z^2+\ldots) = \frac{1}{z^2} + \frac{1}{z} + 1 + \cdots$$

Hence the residue at $z=0$ is 1.

50(c) $\dfrac{3z^2+2}{(z-1)(z^2+9)} = \dfrac{3z^2+2}{(z-1)(z+j3)(z-j3)}$

Hence there are simple poles at $z = 1, j3, -j3$

$$\begin{aligned}
\text{At } z=1,\ \text{residue } &= \frac{3+2}{(1+j3)(1-j3)} = \frac{5}{10} = \frac{1}{2} \\
\text{at } z=j3,\ \text{residue } &= \frac{-3\times 9+2}{(j3-1)j6} = \frac{-25}{6(-3-j)} = \frac{5}{12}(3-j) \\
\text{at } z=-j3,\ \text{residue } &= \frac{5}{12}(3+j) \text{ by symmetry.}
\end{aligned}$$

50(d) $\dfrac{z^3-z^2+z-1}{z^3+4z} = \dfrac{z^3-z^2+z-1}{z(z+j2)(z-j2)} = \dfrac{(z-1)(z^2+4)}{z(z+j2)(z-j2)}$

which has simple poles at $z = 0, j2, -j2$.

$$\begin{aligned}
\text{At } z=0,\ \text{residue } &= -\frac{1}{4} \\
\text{at } z=j2,\ \text{residue } &= \lim_{z\to j2}\frac{(z-1)(z^2+4)}{(z+j2)} = \frac{3}{8}(-1+2j) \\
\text{at } z=-j2,\ \text{residue } &= \frac{3}{8}(-1-2j) \text{ similarly.}
\end{aligned}$$

50(e) $\dfrac{z^6+4z^4+z^3+1}{(z-1)^5}$ has a pole of order 5 at $z = 1$.

The formula for calculating residues is convenient for this problem.

$$\begin{aligned}
\text{Residue } &= \frac{1}{4!}\lim_{z\to 1}\frac{d^4}{dz^4}(z^6+4z^4+z^3+1) \\
&= \frac{1}{24}(6.5.4.3+4.4.3.2) \\
&= \frac{456}{24} = 19
\end{aligned}$$

50(f) $\left(\dfrac{z+1}{z-1}\right)^2$ has a double pole at $z = 1$.

$$\text{Residue } = \lim_{z\to 1}\frac{d}{dz}(z+1)^2 = 4$$

50(g) $\dfrac{z+1}{(z-1)^2(z+3)}$ has a simple pole at $z=-3$ and a double pole at $z=1$.

$$\text{Residue at } z=-3 \text{ is } -\frac{1}{8}$$
$$\text{Residue at } z=1 \text{ is } \frac{d}{dz}\Big(\frac{z+1}{z+3}\Big)\Big|_{z=1} = \frac{1}{8}$$

50(h) $\dfrac{3+4z}{z^3+3z^2+2z} = \dfrac{3+4z}{z(z+1)(z+2)}$ has simple poles at $z=-2,-1$ and 0. Residues are, respectively, $-\dfrac{5}{2}, 1$ and $\dfrac{3}{2}$ following the same procedure as part (c).

■ **51(a)** The pole of $\dfrac{\cos z}{z}$ at $z=0$ is simple, thus the residue is $\cos(0)=1$.

51(b) The poles of $\dfrac{\sin z}{z^4+z^2+1}$ are all simple, and the residue at $z=e^{\pi j/3}$ is

$$\lim_{z\to e^{\pi j/3}}(z-e^{\pi j/3})\frac{\sin z}{z^4+z^2+1} = \sin e^{\pi j/3}\lim_{z\to e^{\pi j/3}}\left(\frac{z-e^{\pi j/3}}{z^4+z^2+1}\right)$$

Using L'Hôpital's rule, the limit is

$$\frac{1}{4e^{\pi j}+2e^{\pi j/3}}$$
$$= \frac{1}{-4+2\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)} = \frac{1}{-3+j\sqrt{3}} = \frac{1}{12}(-3-j\sqrt{3})$$

giving the residue $-\frac{1}{12}(3+j\sqrt{3})\sin\left(\frac{1}{2}(1+j\sqrt{3})\right)$

51(c) The pole of $\dfrac{z^4-1}{z^4+1}$ at $z=e^{\pi j/4}$ is simple and we proceed as in the last part. The residue is

$$-2\lim_{z\to e^{\pi j/4}}\left(\frac{z-e^{\pi j/4}}{z^4+1}\right)$$
$$= -2\frac{1}{4e^{3\pi j/4}} = -\frac{1}{2}\frac{1}{-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}} = \frac{1}{2}\left(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right).$$

Hence the residue is $\frac{\sqrt{2}}{4}(1+j)$.

51(d) $\dfrac{z}{\sin z}$ has a simple pole at $z = \pi$

$$\text{Residue } = \pi \lim_{z\to\pi} \big(\frac{z-\pi}{\sin z}\big) = -\pi.$$

51(e) $\dfrac{1}{(z^2+1)^2}$ has a double pole at $z = j$

$$\begin{aligned}\text{Residue } &= \lim_{z\to j} \frac{d}{dz}\big\{(z-j)^2 \frac{1}{(z-j)^2(z+j)^2}\big\} \\ &= \lim_{z\to j}\big\{-\frac{2}{(z+j)^3}\big\} \\ &= -\frac{2}{8j^3} = -\frac{j}{4}.\end{aligned}$$

■ **52(a)** $\dfrac{\cos z}{z^3}$ has a triple pole at $z = 0$.

$$\frac{\cos z}{z^3} = \frac{1}{z^3} - \frac{1}{2z} + \frac{1}{24}z - \cdots \qquad \text{residue } = -\frac{1}{2}$$

52(b) $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ has a double pole at $z = -1$.

$$\begin{aligned}\text{Residue } &= \frac{d}{dz}\frac{(z+1)^2(z^2-2z)}{(z+1)^2(z^2+4)} \text{ at } z=-1 \\ &= \frac{(2z-2)(z^2+4)-2z(z^2-2z)}{(z^2+4)^2}\Big|_{z=-1} = -\frac{14}{25}\end{aligned}$$

52(c) The function $\dfrac{e^z}{\sin^2 z}$ has a double pole wherever

$$\sin z = 0 \text{ that is at } z = n\pi,\ n = \text{ an integer}$$

In order to find the residue, we need to compute

$$\lim_{z\to n\pi} \frac{d}{dz}\big[\frac{(z-n\pi)^2 e^z}{\sin^2 z}\big]$$

Now

$$\frac{d}{dz}\left[\frac{(z-n\pi)^2 e^z}{\sin^2 z}\right] = e^z \frac{d}{dz}\left[\frac{(z-n\pi)^2}{\sin^2 z}\right] + \frac{(z-n\pi)^2}{\sin^2 z} e^z \qquad \text{(I)}$$

and

$$\begin{aligned}\frac{d}{dz}\left(\frac{(z-n\pi)^2}{\sin^2 z}\right) &= 2\frac{z-n\pi}{\sin z}\cdot\frac{d}{dz}\left(\frac{z-n\pi}{\sin z}\right) \\ &= \frac{2(z-n\pi)}{\sin z}\cdot\frac{\sin z-(z-n\pi)\cos z}{\sin^2 z}\end{aligned}$$

As $z \to n\pi$, $\dfrac{z-n\pi}{\sin z} \to 1$ and

$$\frac{\sin z-(z-n\pi)\cos z}{\sin^2 z} \to \frac{\cos z-\cos z+(z-n\pi)\sin z}{2\sin z\cos z} \text{ (using L'Hôpital's rule)}$$
$$\to 0 \text{ as } z \to n\pi$$

Hence the RHS of equation (I) $\to e^{n\pi}$ as $z \to n\pi$.
Thus the residue is $e^{n\pi}$.

Exercises 4.6.3

■ **53** $\int_C (z^2+3z)dz$ with $z = x+jy$ and $dz = dx+jdy$

$$\begin{aligned}\text{hence } (z^2+3z)dz &= (x^2-y^2+j2xy+3x+j3y)(dx+jdy) \\ &= (x^2-y^2+3x)dx-(2xy+3y)dy \\ &\quad + j[(x^2-y^2+3x)dy+(2xy+3y)dx]\end{aligned}$$

53(a) The straight line joining $2+j0$ to $0+j2$ has equation $x+y=2$ in Cartesian coordinates. This has parametric equation $x=t$ and $y=2-t$ from which $dx=dt$ and $dy=-dt$, and using the above expression for $(z^2+3z)dz$

$$\begin{aligned}(z^2+3z)dz &= (t^2-(2-t)^2+3t)dt+(2t(2-t)+3(2-t))dt \\ &\quad + j[-(t^2-(2-t)^2+3t)dt+(2t(2-t)+3(2-t))dt]\end{aligned}$$

and the range of integration is from $t=2$ to $t=0$.

Hence

$$\begin{aligned}\int_C (z^2+3z)dz &= \int_2^0 (8t-2t^2+2)dt \\ &\quad + j\int_2^0 (-6t-2t^2+10)dt \\ &= \big[4t^2-\frac{2}{3}t^3+2t\big]_2^0 + j\big[-3t^2-\frac{2}{3}t^3+10t\big]_2^0\end{aligned}$$

$$\text{so } \int_C (z^2+3z)dz = -\frac{44}{3} - j\frac{8}{3}$$

53(b) On the straight line from $2+j0$ to $2+j2, x=2$ and y goes from 0 to 2, so that $dx=0$.

Therefore

$$\begin{aligned}\int_{C_1} (z^2+3z)dz &= \int_0^2 -(4t+3t)dt \\ &\quad + j\int_0^2 (4-t^2+6)dt \\ &= \big[-\frac{7}{2}t^2\big]_0^2 + j\big[10t-\frac{1}{3}t^3\big]_0^2 \\ &= -14 + j\frac{52}{3}\end{aligned}$$

On the straight line from $2+j2$ to $0+j2, y=2$ and x goes from 2 to 0, so that $dy=0$.

Therefore

$$\begin{aligned}\int_{C_2} (z^2+3z)dz &= \int_2^0 (t^2-4+3t)dt \\ &\quad + j\int_2^0 (4t+6)dt \\ &= \big[\frac{1}{3}t^3-4t+\frac{3}{2}t^2\big]_2^0 + j\big[2t^2+3t\big]_2^0 \\ &= -\frac{2}{3} - j14\end{aligned}$$

Thus $\int\limits_C (z^2+3z)dz = \int\limits_{C_1} (z^2+3z)dz + \int\limits_{C_2} (z^2+3z)dz = -\frac{44}{3} - j\frac{8}{3}$.

53(c) For this part, we use $z = 2e^{j\theta}$ on $\mid z \mid = 2$ and θ varies between 0 and $\pi/2$ on the quarter circle joining $2+j0$ to $0+j2$. Thus $(z^2+3z)dz = (4e^{2j\theta}+6e^{j\theta})2je^{j\theta}d\theta$ so that

$$\begin{aligned}\int_C (z^2+3z)dz &= \int_0^{\pi/2} \left[8je^{3j\theta} + 12je^{2j\theta}\right]d\theta \\ &= \left[\frac{8}{3}e^{3j\theta} + 6e^{2j\theta}\right]_0^{\pi/2} \\ &= \left[-\frac{8}{3} - 6\right] - \left[\frac{8}{3} + 6\right] \\ &= -\frac{8j}{3} - \frac{44}{3}\end{aligned}$$

Hence the integrals are all the same.

■ **54(a)** On $\mid z \mid = 1,\ z = e^{j\theta},\ 0 \le \theta \le 2\pi$

$$\begin{aligned}&\text{so that } \oint (5z^4 - z^3 + 2)dz \\ &= \int_0^{2\pi} (5e^{4j\theta} - e^{3j\theta} + 2)je^{j\theta}d\theta \\ &= \left[\frac{5}{5}e^{5j\theta} - \frac{1}{4}e^{4j\theta} + 2e^{j\theta}\right]_0^{2\pi} = 0\end{aligned}$$

hence $e^{2\pi j} = e^0 = 1$

54(b) Integrating around the square in the order $0 + j0,\ 1 + j0,\ 1 + j1$ and $0 + j1$ gives the answers $\frac{11}{4}, 3 + \frac{j}{4}, -\frac{11}{4}$ and $-3 - \frac{j}{4}$. Adding these together gives 0.

54(c) On the parabola $y = x^2,\ x = t$ and $y = t^2$ so that $z = t + jt^2$ and $dz = (1 + 2jt)dt$.
On the parabola $y^2 = x,\ x = t^2$ and $y = t$ so that $z = t^2 + jt$ and $dz = (2t + j)dt$.
The computation of $\oint_C (5z^4 - z^3 + 2)dz$ is extremely long winded but straightforward and gives the answer 0.

■ **55** In order to evaluate

$$\int_C \frac{dz}{(z - z_0)^n}$$

we surround the point $z = z_0$ with a circle of radius ε on which $z = z_0 + \varepsilon e^{j\theta}$, $0 \leq \theta < 2\pi$.
Using equation (4.45) the integral around C is the same as the integral around the circle on which $z = z_0 + \varepsilon e^{j\theta} d\theta$. Thus

$$\int_C \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{j\varepsilon e^{j\theta}}{\varepsilon^n e^{nj\theta}} d\theta$$

If $n \neq 1$, then the integral integrates to

$$\left[j\varepsilon^{(1-n)} \frac{e^{(1-n)j\theta}}{(1-n)j} \right]_0^{2\pi} = 0$$

as in Example 4.30.
If $n = 1$,

$$\int_C \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} j d\theta = 2\pi j$$

- **56(a)** If $z = 4$ is outside C, by Cauchy's theorem,

$$\int_C \frac{dz}{z-4} = 0$$

56(b) If $z = 4$ is inside C, by problem 55

$$\int_C \frac{dz}{z-4} = 2\pi j$$

- **57** In order to use Cauchy's integral theorem, we split into partial fractions

$$\frac{2z}{(2z-1)(z+2)} = \frac{2/5}{2z-1} + \frac{4/5}{z+2}$$

- **57(a)** If C is the circle $\mid z \mid = 1$

$$\begin{aligned} \int_C \frac{2z dz}{(2z-1)(z+2)} &= \frac{2}{5} \int_C \frac{dz}{2z-1} + \frac{4}{5} \int_C \frac{dz}{z+2} \\ &= \frac{2}{5} \cdot 2\pi j + \frac{4}{5} \cdot 0 \end{aligned}$$

since $z = \frac{1}{2}$ is inside $| z | = 1$ whereas $z = -2$ is outside. Hence $\int_C \dfrac{2zdz}{(2z-1)(z+2)}$ $= \dfrac{4}{5}\pi j$

57(b) If C is the circle $| z | = 3$, both singularities (poles) are inside C, and hence

$$\int_C \frac{2zdz}{(2z-1)(z+2)} = \frac{2}{5}2\pi j + \frac{4}{5}\cdot 2\pi j$$
$$= \frac{12}{5}\pi j$$

■ **58** This follows a pattern similar to Exercise 57.
Using partial fractions gives

$$\frac{5z}{(z+1)(z-2)(z+4j)} = \frac{\frac{5}{51}(-1-4j)}{z+1} + \frac{\frac{1}{3}(1-2j)}{z-2} + \frac{\frac{2}{17}(-2+9j)}{z+4j}$$

58(a) Only the first two poles $(z = -1,\ z = 2)$ are inside $| z | = 3$ hence

$$\int_C \frac{5zdz}{(z+1)(z-2)(z+4j)} = 2\pi j\left(\frac{5}{51}(-1-4j) + \frac{1}{3}(1-2j)\right)$$
$$= \frac{4\pi}{17}(9+2j)$$

58(b) All three poles are inside $| z | = 5$ hence

$$\int_C \frac{5zdz}{(z+1)(z-2)(z+4j)} = 2\pi j\left(\frac{5}{51}(-1-4j) + \frac{1}{3}(1-2j)\right.$$
$$\left. + \frac{2}{17}(-2+9j)\right)$$
$$= 0$$

■ **59** Equation (4.48) gives the general form of Cauchy's integral theorem

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi j}{n!}f^{(n)}(z_0)$$

where C is a contour enclosing the point $z = z_0$.

59(a)

$$\oint_C \frac{z^3+z}{(2z+1)^3}dz = \frac{1}{8}\oint_C \frac{z^3+z}{(z+\frac{1}{2})^3}dz$$
$$= \frac{1}{8}\frac{2\pi j}{2}\frac{d^2}{dz^2}\bigg|_{-\frac{1}{2}}(z^3+z)(z=-\tfrac{1}{2} \text{ is inside } \mid z \mid = 1)$$
$$= \frac{\pi}{8}j\big(6(-\frac{1}{2})\big) = -\frac{3\pi j}{8}$$

59(b) First of all we need to separate the integrand using partial fractions

$$\frac{4z}{(z-1)(z+2)^2} = \frac{\frac{4}{9}}{z-1} - \frac{\frac{4}{9}}{z+2} + \frac{\frac{8}{3}}{(z+2)^2}$$

Hence

$$\oint_C \frac{4zdz}{(z-1)(z+2)^2} = 2\pi j\frac{4}{9} - 2\pi j\frac{4}{9} + 0 = 0$$

using Cauchy's integral theorem (the derivative of $\frac{8}{3}$ is of course zero). All poles of the integrand are inside the circle $\mid z \mid = 3$.

Exercises 4.6.6

■ **60** $\dfrac{z}{z^2+1}$ has poles at $z = \pm j$

60(a) Since $\dfrac{z}{z^2+1}$ is regular inside $\mid z \mid = \dfrac{1}{2}$

$$\oint_C \frac{zdz}{z^2+1} = 0 \text{ if } C \text{ is the circle } \mid z \mid = \frac{1}{2}$$

60(b) The residues of $\dfrac{z}{z^2+1}$ at $z = \pm j$ (both inside $\mid z \mid = 2$) are

$$\lim_{z\to+j}\frac{(z-j)z}{(z+j)(z-j)} = \lim_{z\to+j}\frac{z}{z+j} = \frac{1}{2}$$

and

$$\lim_{z\to-j}\frac{(z+j)z}{(z+j)(z-j)} = \lim_{z\to-j}\frac{z}{z-j} = \frac{1}{2}$$

Hence, using the residue theorem

$$\oint_C \frac{zdz}{z^2+1} = 2\pi j\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi j$$

■ **61** The singularities of $\dfrac{z^2+3jz-2}{z^3+9z}$ are at $z^3+9z=0$, that is, $z=0, 3j, -3j$ (all simple poles). Only $z=0$ is inside $|\,z\,|=1$ but all three are inside $|\,z\,|=4$. Hence we shall find all the residues.

At $z=0$, residue is $\lim\limits_{z\to 0}\left(\dfrac{z^2+3jz-2}{z^2+9}\right) = -\dfrac{2}{9}$

At $z=3j$, the residue is

$$\lim_{z\to 3j}\frac{(z-3j)(z^2+3jz-2)}{z(z-3j)(z+3j)}$$
$$= \frac{(3j)^2+3j3j-2}{3j(3j+3j)} = \frac{-9-9-2}{-18} = \frac{10}{9}$$

At $z=-3j$, the residue is

$$\lim_{z\to -3j}\frac{(z+3j)(z^2+3jz-2)}{z(z-3j)(z+3j)}$$
$$= \frac{(-3j)^2+(3j)(-3j)-2}{(-3j)(-3j-3j)} = -\frac{-9+9-2}{-18} = \frac{1}{9}$$

61(a) For this part, since only the residue at $z=0$ is inside $C(|\,z\,|=1)$

$$\therefore \oint_C \frac{z^2+3jz-2}{z^3+9z}dz = 2\pi j\left(-\frac{2}{9}\right) = -\frac{4\pi j}{9}$$

61(b) For this part, all residues need to be taken into account since all the poles of $f(z)$ are inside $C(|\,z\,|=4)$

$$\therefore \oint_C \frac{z^2+3jz-2}{z^3+9z}dz = 2\pi j\left(-\frac{2}{9}+\frac{10}{9}+\frac{1}{9}\right) = 2\pi j$$

Note that in this case, all the zeros of the denominator were obviously poles. In general, we would need to check if they were not removable by factorizing the numerator.

■ **62** $f(z) = \dfrac{(z^2+2)(z^2+4)}{(z^2+1)(z^2+6)}$ has poles at $z = \pm j,\ z = \pm j\sqrt{6}$.

Residue at $z = j$ is

$$\lim_{z\to j} \frac{(z-j)(z^2+2)(z^2+4)}{(z-j)(z+j)(z^2+6)} = \frac{(-1+2)(-1+4)}{2j(-1+6)} = -\frac{3j}{10}$$

Residue at $z = -j$ is

$$\lim_{z\to -j} \frac{(z+j)(z^2+2)(z^2+4)}{(z+j)(z-j)(z^2+6)} = \frac{(-1+2)(-1+4)}{(-2j)(-1+6)} = \frac{3j}{10}$$

Residue at $z = j\sqrt{6}$ is

$$\lim_{z\to j\sqrt{6}} \frac{(z-j\sqrt{6})(z^2+2)(z^2+4)}{(z-j\sqrt{6})(z+j\sqrt{6})(z^2+1)} = \frac{(-6+2)(-6+4)}{2j\sqrt{6}(-6+1)} = \frac{8}{2j\sqrt{6}(-5)}$$
$$= \frac{2}{15}j\sqrt{6}$$

Residue at $z = -j\sqrt{6}$ is thus $= -\dfrac{2}{15}j\sqrt{6}$

62(a) The circle $\mid z \mid = 2$ contains the poles at $z = \pm j$ but not those at $z = \pm j\sqrt{6}$. The sum of the residues inside $C = -\dfrac{3j}{10} + \dfrac{3j}{10} = 0$.
Hence the integral $= 0$.

62(b) The circle $\mid z - j \mid = 1$ contains the residue only at $z = j$.
Hence

$$\oint_C \frac{(z^2+2)(z^2+4)}{(z^2+1)(z^2+6)}dz = 2\pi j\big(-\frac{3j}{10}\big) = \frac{3\pi}{5}$$

62(c) The circle $\mid z \mid = 4$ contains all the poles. Since the sum of the residues is zero, so is the integral.

■ **63** The function $\dfrac{1}{z^2(1+z^2)^2}$ has double poles at $z = 0$ and $z = \pm j$.
Residue at $z = 0$ is

$$\frac{d}{dz}\left[\frac{1}{(1+z^2)^2}\right]\Bigg|_{z=0} = -\frac{4z}{(1+z^2)^3}\Bigg|_{z=0} = 0$$

Residue at $z = j$ is

$$\frac{d}{dz}\left[\frac{1}{z^2(z+j)^2}\right]\Bigg|_{z=j} = -\frac{2(2z+j)}{(z^2+jz)^3} = -\frac{3}{4}j$$

Residue at $z = -j$ is $\dfrac{3}{4}j$

63(a) If C is the circle $|\, z \,| = \dfrac{1}{2}$, only the residue at $z = 0$ is in C. Thus

$$\oint_C \frac{dz}{z^2(1+z^2)^2} = 2\pi j(0) = 0$$

63(b) All the singularities are inside $|\, z \,| = 2$, but since they sum to 0,

$$\oint_C \frac{dz}{z^2(1+z^2)^2} = 0$$

■ **64(a)** The singularities of $\dfrac{3z^2+2}{(z-1)(z^2+4)}$ are at $z = 1,\ \pm 2j$.
They are all simple poles. Using the formula (4.37) the residues are :-
at $z = 1$: 1
at $z = 2j$: $\dfrac{1}{8}(2-j)$
at $z = -2j$: $\dfrac{1}{8}(2+j)$

(i) If C is $|\, z-2 \,| = 2$ only the residue at $z = 1$ is included, hence

$$\oint_C \frac{3z^2+2}{(z-1)(z^2+4)}dz = 2\pi j$$

(ii) If C is $|\, z \,| = 4$, all the residues are included, hence

$$\oint_C \frac{3z^2+2}{(z-1)(z^2+4)}dz = \frac{5}{2}\pi j$$

64(b) The singularities of $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ are at $z = -1, \pm 2j$. A double pole is at $z = -1$ and simple poles are at $z = \pm 2j$.

Residues are:-
at $z=-1$: $-\dfrac{14}{25}$
at $z=2j$: $-1+j$
at $z=-2j$: $-1-j$

(i) If C is $|\,z\,|=3$, all singularities are inside C.
Hence
$$\oint_C \frac{z^2-2z}{(z+1)^2(z+4)}dz = 2\pi j\left(-\frac{14}{25}-2\right) = -\frac{128}{25}\pi j$$

(ii) If C is $|\,z+j\,|=2$ then $z=-1$ and $z=-2j$ are inside C, but $z=2j$ is not.
Hence
$$\oint_C \frac{z^2-2z}{(z+1)^2(z^2+4)}dz = 2\pi j\left(-\frac{14}{25}-1-j\right) = \frac{2\pi}{25}(25-j39)$$

64(c) The function $\dfrac{1}{(z+1)^3(z-1)(z-2)}$

Simple poles at $z=1,2$, triple poles at $z=-1$.
Residues :-
$z=1$: $-\dfrac{1}{8}$
$z=2$: $\dfrac{1}{27}$
$z=-1$: $\dfrac{1}{27}-\dfrac{1}{8}\left(=-\dfrac{19}{216}\right)$

(i) The circle $|\,z\,|=\frac{1}{2}$ contains none of the singularities and therefore
$$\oint_C \frac{dz}{(z+1)^3(z-1)(z-2)} = 0$$

(ii) The circle $|\,z+1\,|=1$ contains the singularity $z=-1$ and therefore
$$\oint_C \frac{dz}{(z+1)^3(z-1)(z-2)} = 2\pi j\left(-\frac{19}{216}\right) = -\frac{19\pi j}{108}$$

(iii) The rectangle and vertices $\pm j, 3\pm j$, contains the singularities at $z=1, z=2$ and therefore
$$\oint_C \frac{dz}{(z+1)^3(z-1)(z-2)} = -\frac{19\pi j}{108}$$

64(d) The function $\dfrac{z-1}{(z^2-4)(z+1)^4}$ has a pole of order 4 at $z=-1$ and simple poles at $z=\pm 2$.

Residue at $z=2$ is $\dfrac{1}{324}$

Residue at $z=-2$ is $-\dfrac{3}{4}$

Residue at $z=-1$ is given by $\dfrac{1}{3!}\dfrac{d^3}{dz^3}\left(\dfrac{z-1}{z^2-4}\right)$

This residue is calculated by using partial fractions

$$\frac{z-1}{z^2-4}=\frac{\frac{1}{4}}{z-2}+\frac{\frac{3}{4}}{z+2}$$

whence

$$\begin{aligned}\frac{1}{3!}\frac{d^3}{dz^3}\left(\frac{z-1}{z^2-4}\right)&=\frac{1}{24}\frac{d^3}{dz^3}\left(\frac{1}{z-2}\right)+\frac{1}{8}\frac{d^3}{dz^3}\left(\frac{1}{z+2}\right)\\&=-\frac{1}{4(z-2)^4}-\frac{3}{4(z+2)^4}\end{aligned}$$

putting $z=-1$ gives the residue $-\dfrac{1}{4.3^4}-\dfrac{3}{4}=-\dfrac{61}{81}$

(i) The circle $|\,z\,|=\frac{1}{2}$ contains none of the singularities hence the integral

$$\oint_C \frac{(z-1)}{(z^2-4)(z+1)^4}dz=0$$

(ii) The circle $|\,z+\frac{3}{2}\,|=2$ contains the singularities at $z=-1$ and $z=-2$ but not that at $z=2$

Hence

$$\oint_C \frac{(z-1)}{(z^2-4)(z+1)^4}dz=2\pi j\left(-\frac{3}{4}-\frac{61}{81}\right)=-\frac{487\pi j}{162}$$

(iii) The triangle with vertices $-\dfrac{3}{2}+j,-\dfrac{3}{2}-j,3+j0$ contains all the singularities, hence

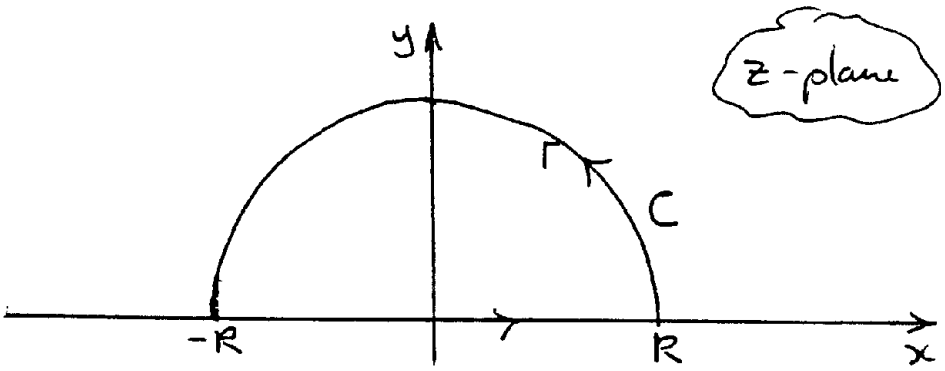
$$\begin{aligned}\oint_C \frac{z-1}{(z^2-4)(z+1)^4}dz&=2\pi j\left(\frac{1}{4.3^4}-\frac{3}{4}-\frac{1}{4.3^4}-\frac{3}{4}\right)\\&=-3\pi j\end{aligned}$$

■ **65(a)** $\int_{-\infty}^{\infty} \frac{dx}{x^2+x+1}$

Since the integrand satisfies the condition for Type 1 infinite real integrals, given in section 4.6.5, we consider

$$\oint_C \frac{dz}{z^2+z+1} \quad \text{where} \quad C \text{ is a semicircle with radius}$$

R and centre the origin in the upper half z-plane



By the residue theorem

$$\oint_C \frac{dz}{z^2+z+1} = 2\pi j \ \{\text{sum of residues of poles of integrand inside } C\}$$

$$z^2+z+1=0 \Rightarrow z = \frac{-1 \pm \sqrt{1-4}}{2} = -\tfrac{1}{2} \pm j\tfrac{1}{2}\sqrt{3}$$

Only one of these simple poles lies inside C (the one with positive imaginary part)

$$\text{Residue there } = \lim_{z \to z_0} (z-z_0)\frac{1}{z^2+z+1} \ , \ z_0 = -\tfrac{1}{2} + \tfrac{1}{2}j\sqrt{3}$$

That is, residue $= \lim_{z \to z_0} \left(\frac{1}{2z+1}\right)$ using L'Hôpital's rule (for simplicity) $= \frac{1}{j\sqrt{3}}$

Thus

$$\oint_C \frac{dz}{z^2+z+1} = 2\pi j \cdot \frac{1}{j\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

Now,

$$\oint_C = \int_\Gamma + \int_{-R}^{R}$$

and, as $R \to \infty, \int\limits_\Gamma \to 0$

On $\int\limits_{-R}^{R}, z = x =$ real.

Thus, letting $R \to \infty$ we find that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1} = \frac{2\pi}{\sqrt{3}}$$

65(b) This integral is done in precisely the same way as that of part (a). This time, the poles are at $\pm j$ but they are both double.

$$-\oint_C \frac{dz}{(z^2+1)^2} = 2\pi j \times \text{ residue at } j = 2\pi j \cdot \frac{1}{4j} = \frac{\pi}{2}$$

Thus

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

65(c) To evaluate $\int\limits_0^\infty \dfrac{dx}{(x^2+1)(x^2+4)^2}$ we use the same semicircular contour, except that we note

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$$

plus the fact that $\dfrac{1}{(z^2+1)(z^2+4)^2}$ has *two* poles inside C this time, the simple pole at $z = j$ and the double pole at $z = 2j$. Residue at $z = j$ is $\dfrac{1}{18j}$ and residue at $z = 2j$ is $-\dfrac{11}{288j}$.

Thus

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{1}{2}2\pi j\left(\frac{1}{18j} - \frac{11}{288j}\right) = \frac{5\pi}{288}$$

65(d) In order to evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$ we follow Example 4.39 and put $z = e^{j\theta}$ so that

$$\cos 3\theta = \frac{1}{2}(z^3 + z^{-3}) \quad \text{and} \quad \cos\theta = \frac{1}{2}(z + z^{-1})$$

With $dz = je^{j\theta}d\theta$. Hence we consider

$$\oint_C \frac{\frac{1}{2}(z^3 + z^{-3})}{jz(5 - 2(z + z^{-1}))} dz$$

The function under the integral can be written

$$\frac{1}{2j}\frac{1 + z^6}{5z^4 - 2z^5 - 2z^3} = \frac{1 + z^6}{2j(z - 2)(1 - 2z)z^3}$$

The poles inside $| z | = 1$ are a triple pole at $z = 0$ and a simple pole at $z = \frac{1}{2}$. Using the formula for the residue at $z = \frac{1}{2}$ gives $\frac{65}{48j}$. Using the Laurent expansion about $z = 0$ yields the residue $-\frac{21}{16j}$. The sum is $\frac{1}{24j}$. Hence

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = \frac{1}{2j}\oint_C \frac{1 + z^6}{(z - 2)(1 - 2z)z^3} dz = \frac{1}{24j} 2\pi j$$
$$= \frac{\pi}{12}$$

65(e)

$$\int_0^{2\pi} \frac{4d\theta}{5 + 4\sin\theta}$$

This follows in the same way as part (d).

Putting $z = e^{j\theta}$ yields $dz = je^{j\theta}d\theta$, that is, $d\theta = \frac{dz}{jz}$ and $\sin\theta = \frac{1}{2j}(z - z^{-1})$

Thus

$$\int_0^{2\pi} \frac{4d\theta}{5+4\sin\theta} = \oint_C \frac{4dz}{jz(5+\frac{4}{2j}(z-z^{-1}))}$$

Thus

$$\int_0^{2\pi} \frac{4d\theta}{5+4\sin\theta} = \oint_C \frac{4dz}{(2z+j)(z+2j)} = 2\pi j\cdot\frac{4}{3j} = \frac{8\pi}{3}$$

(C is the unit circle $\mid z \mid = 1$)

65(f)

$$\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)(x^2+2x+2)}$$

This follows along similar lines to parts (a) and (b). Consider the semicircular contour and centre the origin radius R on the upper half plane and labelled C

$$\oint_C \frac{z^2dz}{(z^2+1)(z^2+2z+2)} = 2\pi j\{\text{sum of residues inside } C\}$$

Double pole is at $z=j$, simple pole is at $z=-1+j$
Residue at $z=j$ is $\frac{3}{20}(-4+j3)$; residue at $-1+j$ is $\frac{1}{5}(3-j4)$
Sum $=-\frac{7}{20}j$ giving the integral as $\frac{7}{10}\pi$

65(g)

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$$

Once again let $z=e^{j\theta}$ and consider the integral around the unit circle

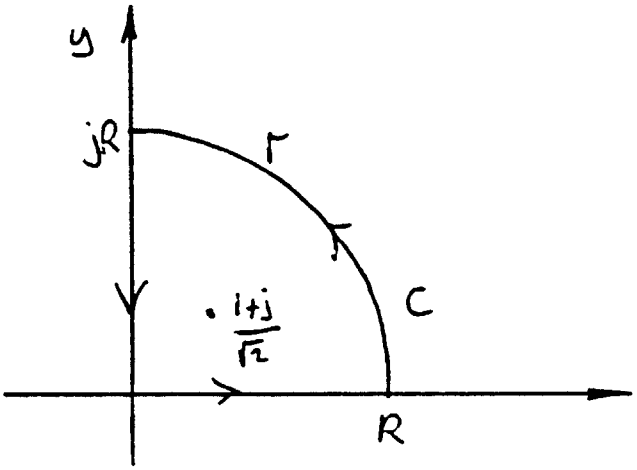
$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_C \frac{dz}{z^2(\frac{1}{2}-j)+3jz-j-\frac{1}{2}}$$

The poles are at $-\frac{j}{1-2j}$ and $-\frac{5}{1-2j}$. Only the first is inside $\mid z \mid = 1$.
The residue is $\frac{1}{2j}$ and so the value of the integral is $2\pi j\frac{1}{2j}=\pi$.

65(h)

$$\int_0^\infty \frac{dx}{x^4+1}$$

We have a choice here, let us choose a *quarter* circle contour as shown below.



Only the root of $z^4+1=0$ in the positive quadrant, that is $z=\dfrac{1+j}{\sqrt{2}}$, needs to be taken into account.

Residue at this point is $\dfrac{1}{4z^2}\Big|_{z=\frac{1+j}{\sqrt{2}}} = \dfrac{-1-j}{4\sqrt{2}}$

Hence

$$\oint_C \frac{dz}{1+z^4} = 2\pi j\frac{(-1-j)}{4\sqrt{2}} = \frac{\pi}{2\sqrt{2}} - \frac{j\pi}{2\sqrt{2}}$$

$$\oint_C \frac{dz}{z^4+1} = \int_{jR}^0 + \int_0^R + \int_\Gamma = \frac{\pi}{2\sqrt{2}} - j\frac{\pi}{2\sqrt{2}}$$

on the imaginary axis, $z=jy$, and on the real axis $z=x$. Therefore

$$\oint_C \frac{dz}{z^4+1} = \int_R^0 \frac{jdy}{y^4+1} + \int_0^R \frac{dx}{x^4+1} + \int_\Gamma \frac{dz}{z^4+1}$$

where we have used $(jy)^4 = y^4$. Letting $R\to\infty$, the last integral $\to 0$. Thus

$$\int_\infty^0 \frac{jdy}{y^4+1} + \int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}} - j\frac{\pi}{2\sqrt{2}}$$

Equating real (or indeed imaginary) parts gives

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$

65(i)

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4x+5)^2}$$

The semicircular contour is used and the poles of $\dfrac{1}{(z^2+4z+5)^2}$ are at $z=-2+j,-2-j$ both double. Only the residue at $z=-2+j$, which is $\dfrac{1}{4j}$, needs to be taken into account.

Hence

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4x+5)^2} = 2\pi j\frac{1}{4j} = \frac{\pi}{2}$$

65(j)

$$\int_{0}^{2\pi} \frac{\cos\theta d\theta}{3+2\cos\theta}$$

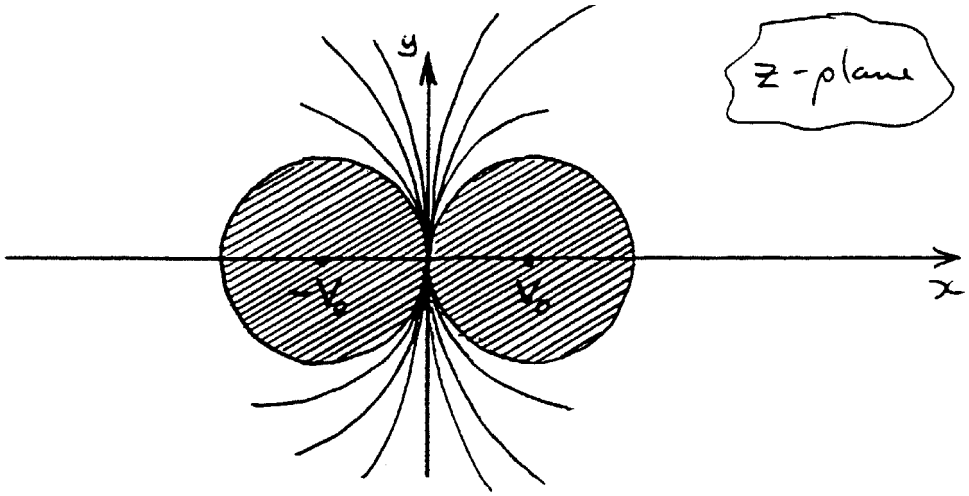
Again we use the unit circle on which $z=e^{j\theta}$. The integrand is $\dfrac{z^2+1}{2j(z^3+3z^2+z)}$ with simple poles at $z=0,-\dfrac{3}{2}+\dfrac{1}{2}\sqrt{5},-\dfrac{3}{2}-\dfrac{1}{2}\sqrt{5}$. Only the first two are inside C and residues are 1 and $-\dfrac{3}{\sqrt{5}}$. Hence the integral has the value $\pi\big(1-\dfrac{3}{\sqrt{5}}\big)$.

Exercises 4.8.3

■ **66** Since $w=\dfrac{1}{z},\ u+jv=\dfrac{1}{x+jy}=\dfrac{x-jy}{x^2+y^2}$

Thus $u=\dfrac{x}{x^2+y^2}$ if $u=\dfrac{1}{2a},\ x^2+y^2=2ax$.

For the two wires shown in Figure 4.41 potentials are centred at V_0 or $-V_0$ and are tangent to the imaginary (y) axis. They are thus circles of the form $x^2+y^2=2aV_0x$. The equipotential curves are shown



■ **67(a)**

$$z = -1 \Rightarrow w - 0,\ z = j \Rightarrow w = \frac{j+1}{1-j} = j$$

$$z = \frac{1}{25}(24 + j7) \Rightarrow w = \frac{49 + j7}{1 - j7} = 7\frac{(7+j)(1+j7)}{1+49} = j7$$

$$z = -\frac{3}{4} \Rightarrow w = \frac{\frac{3}{4}+1}{1-\frac{3}{4}} = \frac{\frac{7}{4}}{\frac{1}{4}} = 7$$

giving images as (0,0),(0,1),(0,7),(7,0).

67(b) If $w = \dfrac{z+1}{1-z}$ then $z = \dfrac{w-1}{w+1}$

$$\text{so that } x + yj = \frac{u - 1 + jv}{u + 1 + jv} = \frac{(u - 1 + jv)(u + 1 - jv)}{(u+1)^2 + v^2}$$

$$\text{or } x + yj = \frac{(u^2 - 1) + v^2 + j(vu + v - uv + v)}{(u+1)^2 + v^2}$$

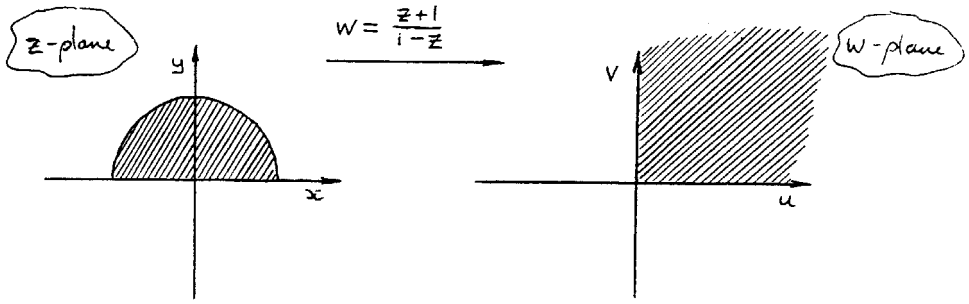
$$x + yj = \frac{u^2 + v^2 - 1 + 2vj}{(u+1)^2 + v^2}$$

Hence $y = 0$ corresponds to $v = 0$.

67(c) If $x^2 + y^2 = 1 \Rightarrow |\, z \,| = 1$

Since $z = \dfrac{w-1}{w+1}$ this means that $\left|\dfrac{w-1}{w+1}\right| = 1$

or $(u-1)^2 + v^2 = (u+1)^2 + v^2$ from which $u = 0$



In order to progress, note that the image of the semicircular conductor in the w plane is the positive quadrant $u \geq 0, v \geq 0$. Instead of temperature $T^{\circ}\mathrm{C}$ we consider $\dfrac{\pi T}{200}$ since this function has the value $\dfrac{\pi}{2}$ on $u = 0$ (where $T = 100^{\circ}\mathrm{C}$). The mapping $w = e^z$ (Example 4.14) provides a means of eliminating the singularity at $w = 0$. The complex variable z is already defined, therefore write $w = e^{\zeta}$ (complex variable ζ). The imaginary part of ζ is identified with the (scaled) temperature $\dfrac{\pi T}{200}$.

Thus $\dfrac{\pi T}{200} = \tan^{-1}\left(\dfrac{v}{u}\right) = \tan^{-1}\left(\dfrac{2y}{1 - x^2 - y^2}\right)$ as required.

■ **68(a)** $G(x, y) = 2x - 2xy$; thus $\dfrac{\partial H}{\partial y} = \dfrac{\partial G}{\partial x} = 2 - 2y$

and $\dfrac{\partial H}{\partial x} = -\dfrac{\partial G}{\partial y} = 2x$

Integrating this gives $H = x^2 - y^2 + 2y$

Hence $W = G + jH = 2z - jz^2$

68(b) If $w = lnz$, then $z = e^w$

Given $H(z) = 2z + jz^2 = 2e^w + je^{2w}$

equating real parts gives

$$G(x, y) = 2e^u \cos v - e^{2u} \sin 2v \text{ as required}$$

68(c) If $w = f(z)$ then the real and imaginary parts of $f(z)$ are harmonic functions. Hence if $\zeta = g(w)$, then the real and imaginary parts of g are harmonic. So $\zeta = g(w) = g(f(z))$ implies that harmonic functions (the real and imaginary

parts of $f(z)$) transform to harmonic functions (the real and imaginary parts of $g(w)$).

■ **69** If $w = \dfrac{z+3}{z-3}$ then $\mid w \mid = k$ transforms to

$$\left|\frac{z+3}{z-3}\right| = k$$

that is, $(x+3)^2 + y^2 = k^2(x-3)^2 + k^2y^2$
or $x^2 + y^2 + 6\dfrac{1+k^2}{1-k^2}x + 9 = 0$ as required.
If the centre of the circle is to be $(-5, 0)$, then

$$6\left(\frac{1+k^2}{1-k^2}\right) = -10 \text{ or } k = 2$$

We thus (following section 4.8) require the potential V to be a harmonic function which has a constant value on a circle $u^2 + v^2 = 4$. Hence V has the general form

$$V = A\ln(u^2 + v^2)$$

on $u^2 + v^2 = 4, V = V_0$ hence $A = \dfrac{V_0}{\ln 4}$
so that $V = \dfrac{V_0}{\ln 4}\ln(u^2 + v^2)$
Now $u^2 + v^2 = \mid w \mid^2 = \left|\dfrac{z+3}{z-3}\right|^2 = \dfrac{(x+3)^2 + y^2}{(x-3)^2 + y^2}$
Thus

$$V = \frac{V_0}{\ln 4}\{\ln[(x+3)^2 + y^2] - \ln[(x-3)^2 + y^2]\}$$

■ **70** This problem follows a similar pattern to Exercise 67.

70(a) The points $A, x = 1, y = 0$; $B, x = 0, y = 1$; $C, x = 0, y = -1$ under the mapping of $w = \dfrac{j(1-z)}{1+z}$ transform to

$$w = 0 \text{ that is } (0,0),\ w = \frac{j(1-j)}{1+j} = 1 \text{ that is } (1,0)$$

$$\text{and } w = \frac{j(1+j)}{1-j} = -1 \text{ that is } (-1,0) \text{ respectively}$$

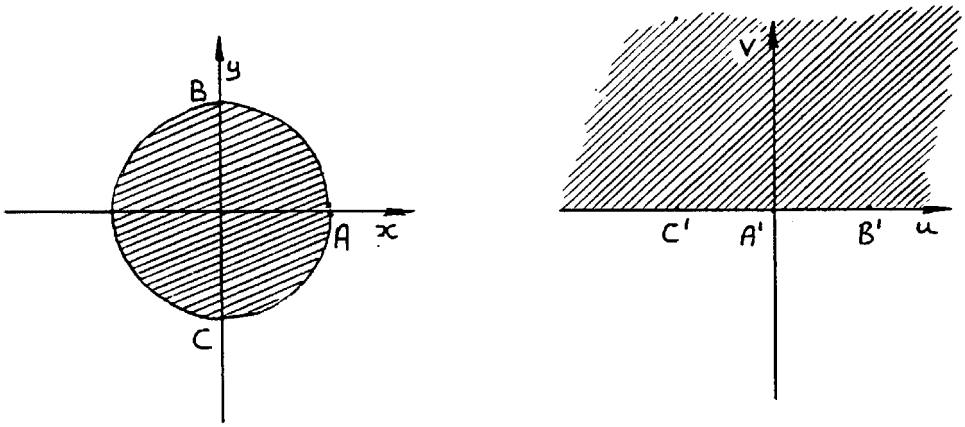
70(b) $w = \dfrac{j(1-z)}{1+z}$

If $z = x =$ real (i.e. $y = 0$) then $w = \dfrac{j(1-x)}{1+x}$ is purely imaginary. Since $\dfrac{1-x}{1+x}$ can take all real values, points on $y = 0$ correspond to points on $u = 0$, the imaginary axis.

70(c) If $w = \dfrac{j(1-z)}{1+z}$ then $z = \dfrac{j-w}{j+w}$

So that $\mid z \mid = 1 \Rightarrow \mid j - w \mid = \mid j + w \mid$ or $v = 0$

For the last part we note the following property of the mapping that is, $w = \dfrac{j(1-z)}{1+z}$ (from (a), (b), and (c))



In a way similar to Exercise 67, identify the function $\dfrac{\pi T}{100}$ which is (in the z plane), π on the negative real axis, and 0 on the positive real axis. The mapping of $w = e^{\zeta}$ (ζ - complex variable which has the values of $\dfrac{\pi T}{100}$ as imaginary part).

Thus

$$\frac{\pi T}{100} = \tan^{-1}\left(\frac{v}{u}\right) = \tan^{-1}\left(\frac{1-x^2-y^2}{2y}\right)$$

(using $w = \dfrac{j(1-z)}{1+z}$ to find u and v). This gives the result.

■ **71** This problem is similar to the last part of Exercise 69. The successive mappings are

$$z_1 = \frac{z+j4}{z-j4} w = \ln z$$

In order for the circle centre $5j$ to be mapped to a circle centred at the origin in the z plane we require

$$| z_1 |^2 = \left| \frac{z + j4}{z - j4} \right|^2 = k^2 \text{ for some constant } k$$

that is, $x^2 + y^2 + 8\frac{1+k^2}{1-k^2}\, y + 16 = 0$ needs a centre at $5j$

$$\text{Therefore, } 8(1 + k^2) = 10(1 - k)^2) \text{ or } h = \frac{1}{3}$$

In the w plane, $| w | = | \ln z_1 | = | \ln(x_1^2 + y_1^2) | = 2 \ln 3$ on the boundary of the circle on which $T = 100$.
Thus writing $T = A \mid \ln(x_1^2 + y_1^2) \mid$ gives $T = 100$ when

$$x_1^2 + y_1^2 = \frac{1}{9} \quad \text{if} \quad 100 = A \cdot 2 \ln 3 \text{ or } A = \frac{50}{\ln 3}$$

Thus

$$\begin{aligned} T &= \frac{50}{\ln 3} \ln(x_1^2 + y_1^2) \\ &= \frac{50}{\ln 3} \ln \left(\frac{x^2 + (4 + y)^2}{x^2 + (4 - y)^2} \right) \text{ as required} \end{aligned}$$

Note that $T = 0$ corresponds to $x_1^2 + y_1^2 = 1$ or $y = 0$ as is also required ($| z + j4 |^2 = | z - j4 |^2$ is $y = 0$).

■ **72** The mapping $w = z + \dfrac{1}{z}$ was studied in Example 4.13. Writing, as usual, $w = u + jv$ and $z = x + jv$ leads to

$$u = x + \frac{x}{x^2 + y^2} \quad \text{and} \quad v = y - \frac{y}{x^2 + y^2}$$

Hence the unit circle $x^2 + y^2 = 1$ in the z plane corresponds to $v = 0$ (the real axis) in the w plane.
Points $e^{j\pi/3}$ and $e^{2j\pi/3}$ (P and Q of this problem) correspond to $u = 2\cos\frac{\pi}{3}$ and $2\cos\frac{2\pi}{3}$ respectively. The arc PQ thus corresponds to $-1 \leq u \leq 1$ in the w plane. The further mapping $\zeta = \dfrac{w + 1}{w - 1}$ takes this portion of the real axis ($-1 \leq u \leq 1$) to $-\infty \leq \mathcal{R}e\{\zeta\} \leq 0$. This negative real axis corresponds to $T = 100$. Hence in a similar fashion to Exercise 70, we identify the variable $\frac{\pi T}{100}$ (which $= \pi$ on this

line) with the imaginary part of $\ln\zeta$ which is $arg\zeta$. We cannot use $\tan^{-1}$ here because the argument of the logarithm function is quadratic and so

$$\frac{\pi T}{100} = arg\zeta = arg\left(\frac{w+1}{w-1}\right) = arg\left(\frac{z+1/z+1}{z+1/z-1}\right)$$

$$\text{that is,} \quad T = \frac{100}{\pi}\left[arg(z^2+z+1) - arg(z^2-z+1)\right]$$

as required.

Review Exercises 4.9

■ **1(a)** $z = 1+j,\ w = (1+j)z+j = (1+j)^2+j = 1+2j-1+j = 3j$

1(b) $z = 1-j2,\ w = j3z+j+1 = j(1-j2)3+j+1 = 3j+6+j+1 = 4j+7$

1(c) $z = 1,\ w = \frac{1}{2}(1-j)z+\frac{1}{2}(1+j) = 1$

1(d) $z = j2,\ w = \frac{1}{2}(1-j)z+\frac{1}{2}(1+j) = (1-j)j+\frac{1}{2}(1+j) = \frac{3}{2}(1+j)$

■ **2(a)** $y = 2x$ **(b)** $x+y = 1$

For the mapping $w = (1+j)z+j$

$$u = x-y,\ v = x+y+1$$

so $y = 2x \Rightarrow v+3u = 1$

and $x+y = 1 \Rightarrow v = 2$

For the mapping $w = j3z+j+1$

$$u = -3y+1,\ v = 3x+1$$

so $y = 2x \Rightarrow u+2v = 3$

and $x+y = 1 \Rightarrow v-u = 3$

For the mapping $w = \frac{1}{2}(1-j)z+\frac{1}{2}(1+j)$

$$u = \frac{1}{2}(x+y+1),\ v = \frac{1}{2}(x-y+1)$$

so $y = 2x \Rightarrow 3v - u = 1$
and $x + y = 1 \Rightarrow u = 1$

■ **3** $w = \alpha z + \beta$
when $z = 2 - j,\ w = 1$, and when $z = 0, w = 3 + j$

3(a) Solving the simultaneous equations gives $\alpha = -\frac{1}{5}(3 + j4), \beta = 3 + j$.

3(b) Since

$$w = -\frac{1}{5}(3 + 4j)z + 3 + j$$
$$z = \frac{1}{5}(3 + j - w)(3 - j4)$$
$$\text{so} \quad x = 13 - 3u - 4v$$

and $\mathcal{R}e\{z\} \leq 0$ corresponds to $3u + 4v \geq 13$

3(c)

$$| z | = \frac{1}{5} | 3 + j - w | 5 \ \leq 1$$
$$\Rightarrow | w - 3 - j | \leq \frac{1}{5}$$

3(d) Fixed point is given by

$$z = \alpha z + \beta \text{ or } z = \frac{\beta}{1 - \alpha} = \frac{1}{4}(7 - j)$$

■ **4** $w = \frac{j}{z} \Rightarrow x = \frac{v}{u^2+v^2},\ y = \frac{u}{u^2+v^2}$

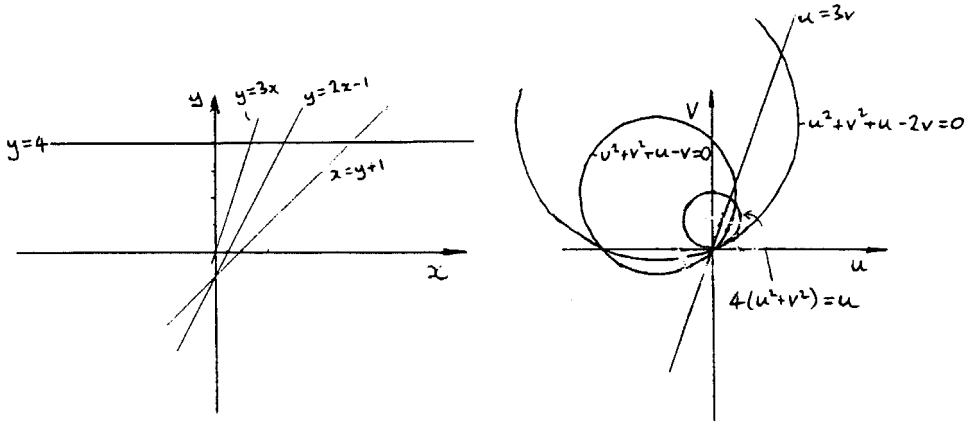
4(a) $x = y + 1 \Rightarrow \frac{v}{u^2+v^2} = \frac{u}{u^2+v^2} + 1$ or $u^2 + v^2 + u - v = 0$

4(b) $y = 3x \Rightarrow u = 3v$

4(c) Line joining $A(1 + j)$ to $B(z + j3)$ or $(1, 1)$ to $(2, 3)$ is $y = 2x - 1$ which transforms to $\dfrac{u}{u^2 + v^2} = \dfrac{2v}{u^2 + v^2} - 1$ or $u^2 + v^2 + u - 2v = 0$

4(d) $y = 4 \Rightarrow 4(u^2 + v^2) = u$

The following Argand diagram shows all these curves.



■ **5** $w = \dfrac{z+1}{z-1} \Rightarrow z = \dfrac{w+1}{w-1}$

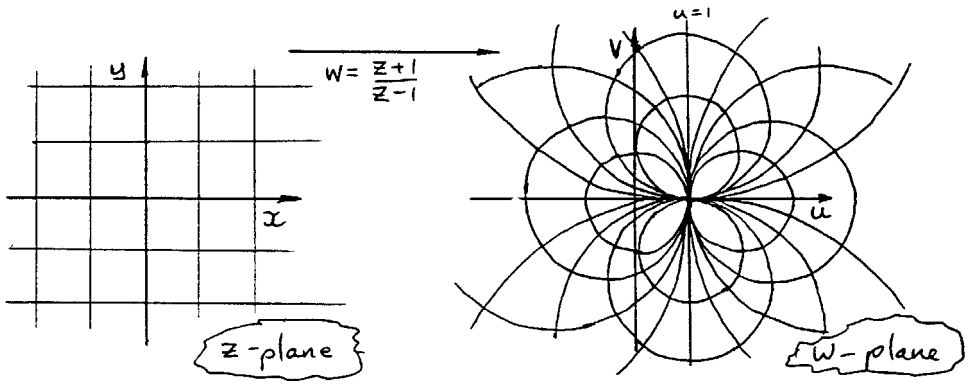
from which

$$x = \frac{u^2 + v^2 - 1}{(u-1)^2 + v^2},\ y = \frac{-2v}{(u-1)^2 + v^2}$$

lines $x = k$ and $y = l$ map to circles

$$u^2 + v^2 - \frac{2k}{k-1}u + \frac{k+1}{k-1} = 0 \ \text{ and } \ u^2 + v^2 - 2u + \frac{2v}{l} + 1 = 0$$

Fixed points are $z = \frac{z+1}{z-1}$ in $z^2 - 2z - 1 = 0$, that is, $z = 1 + \sqrt{2}, 1 - \sqrt{2}$ are the fixed points.



■ **6** $w = \dfrac{1-z^2}{z}$

Fixed points occur at $z = \dfrac{1-z^2}{2}$

$$\text{or} \quad z^2 = 1 - z^2$$
$$z^2 = 1/2$$
$$\text{Hence} \quad z = \pm\sqrt{2}/2$$

Writing $w = \dfrac{1}{z} - z = \dfrac{z^*}{|\, z \,|^2} - z$ ($z^* =$ complex conjugate).

Whence $u = \dfrac{x}{x^2+y^2} - x$, $v = -\dfrac{-y}{x^2+y^2} - y$ and $v = -y\big(\dfrac{1}{r^2} + 1\big)$

so $\dfrac{r^2 u}{r^2-1} = x$, $\dfrac{-r^2 v}{r^2+1} = y$

Squaring and adding gives

$$\left(\frac{ur^2}{r^2-1}\right)^2 + \left(\frac{vr^2}{r^2+1}\right)^2 = x^2 + y^2 = 1$$

the required ellipses.

Since $u = \frac{x}{x^2+y^2} - x$, if $x^2 + y^2 = 1$ then $u = 0$ (imaginary axis in the w plane).

■ **7** $w = z^3$

$= (x + jy)^3 = x^3 + 3jx^2y + 3j^2xy + j^3y^3$

so $u = x^3 - 3xy^2$

$v = 3x^2y - y^3$ are the real and imaginary parts.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \ \frac{\partial u}{\partial y} = -6xy$$
$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2, \ \frac{\partial v}{\partial x} = 6xy$$

hence verifying the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

■ **8** $u(x,y) = x\sin x\cosh y - y\cos x\sinh y$

$$\text{hence} \quad \frac{\partial u}{\partial x} = \sin x\cosh y + x\cos x\cosh y + \sin x\sinh y$$
$$\text{and} \quad \frac{\partial u}{\partial y} = x\sin x\sinh y - y\cos x\cosh y - \cos x\sinh y$$

By the Cauchy–Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hence, integrating $\dfrac{\partial u}{\partial x}$ with respect to y gives:

$$v = \sin x\sinh y + x\cos x\sinh y + y\sin x\cosh y - \sin x\sinh y + f_1(y)$$

Integrating $-\dfrac{\partial u}{\partial y}$ with respect to x gives:

$$v = (x\cos x - \sin x)\sinh y + y\sin x\cosh y + \sin x\sinh y + f_2(x)$$

where f_1 and f_2 are arbitrary functions.
Comparing these gives $v = y\sin x\cosh y + x\cos x\sinh y$ (ignoring the additive constant).
Thus

$$w = u + jv = x\sin x\cosh y - y\cos x\sinh y + j(y\sin x\cosh y + x\cos x\sinh y)$$

Since this is $f(z)$, we put $y = 0$ to find $f(x)$ which will give the functional form of f, namely

$$f(x) = x\sin x. \text{ Thus } f(z) = z\sin z.$$

■ **9** Writing $w = \dfrac{az+b}{cz+d}$ (the general bilinear mapping) since $z = 0 \Rightarrow w = \infty$ we must have $d = 0$, hence (relabelling the constants) $w = \dfrac{\alpha z + \beta}{z}$.
Writing this as $wz = \alpha z + \beta$ and inserting the pairs of values $z = j,\ w = -j$ and $z = \frac{1}{2}(1+j),\ w = 1-j$ gives

$$1 = \alpha j + \beta,\ 1 = \tfrac{1}{2}(1+j)\alpha + \beta$$

from which $\alpha = 0,\ \beta = 1$. Hence $w = \dfrac{1}{z}$.

9(a) Since $u = \dfrac{x}{x^2+y^2}$ and $v = -\dfrac{y}{x^2+y^2}$
the real axis (in the z plane) maps to the real axis (in the w plane).
If $y > 0$ then $v = -\frac{y}{x^2+y^2} < 0$ and vice versa.
Thus the lower half of the z plane maps to the upper half of the w plane.

9(b) The circle $\mid z - \frac{1}{2}j \mid = \frac{1}{2}$ is $x^2 + y^2 - y = 0$
or $v = -1\big(v = -\frac{y}{x^2+y^2}\big)$
If $\mid z - \frac{1}{2}j \mid < \frac{1}{2}$ then $x^2+y^2-y < 0$ or $v < -1$, that is, the interior of $\mid z - \frac{1}{2}j \mid = \frac{1}{2}$ maps to $Im(w) < -1$ as required.

■ **10** The mapping $z = \zeta + \dfrac{a^2}{4\zeta}$
maps $R =$ constant (where $\zeta = Re^{j\theta}$) to curves

$$z = Re^{j\theta} + \frac{a^2}{4R}e^{-j\theta}$$

which describe ellipses in the z plane as can be seen by writing

$$x = \big(R + \frac{a^2}{4R}\big)\cos\theta,\ y = \big(R - \frac{a^2}{4R}\big)\sin\theta$$

whence

$$\frac{x^2}{(R+\frac{a^2}{4R})^2} + \frac{y^2}{(R-\frac{a^2}{4R})^2} = 1$$

when $R = \frac{1}{2}a,\ y = 0$ (real axis). This mapping is used together with bilinear mappings to map an aerofoil shape onto the unit circle. This is useful in aeronautical engineering.

■ **11** $\dfrac{1}{1+z^3} = (1+z^3)^{-1}$
using the binomial expansion gives

$$\frac{1}{1+z^3} = 1 - z^3 + z^6 - z^9 + \cdots$$

Similarly

$$\frac{1}{(1+z^3)^2} = 1 - 2z^3 + 3z^6 - 4z^9 + \cdots$$

and both are valid in the disc $\mid z \mid < 1$.

■ **12(a)**

$$\frac{1-z}{1+z} = \frac{2-1-z}{1+z} = \frac{2}{1+z} - 1$$

Using the binomial series again gives

$$\frac{1-z}{1+z} = 1 - 2z + 2z^2 - 2z^3 + \dots$$

since the nearest singularity of the function to $z = 0$ is at $z = -1$, the radius of convergence is 1.

12(b) $f(z) = \dfrac{1}{z^2+1}$. This time we need to expand about the point $z = 1$. We use Taylor's series

$$f(z) = \frac{1}{z^2+1},\ f'(z) = -\frac{2z}{(z^2+1)^2},\ f''(z) = -\frac{2}{(z^2+1)^2} + \frac{8z^2}{(z^2+1)^3}$$
$$f''(z) = \frac{24z}{(z^2+1)^3} - \frac{48z^3}{(z^2+1)^4},\ f^{iv}(z) = \frac{24}{(z^2+1)^3} - \frac{288z^2}{(z^2+1)^4} + \frac{384z^4}{(z^2+1)^5}$$

At $z = 1$ these have values $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 0, -3$
giving the expansion

$$\frac{1}{z^2+1} = \frac{1}{2} - \frac{1}{2}(z-1) + \frac{1}{4}(z-1)^2 - \frac{1}{6}(z-1)^4 + \dots$$

The singularities of $\dfrac{1}{1+z^2}$ are at $z = \pm j$ which are a distance $\sqrt{2}$ from $z = 1$, hence the radius of convergence is $\sqrt{2}$.

12(c)

$$\frac{z}{1+z} = 1 - \frac{1}{1+z} = f(z)$$
$$f'(z) = \frac{1}{(1+z)^2},\ f'' = -\frac{2}{(1+z)^3},\ f'''(z) = +\frac{6}{(1+z)^4}$$

Thus $\dfrac{z}{1+z} = \dfrac{1}{2}(1+j) + \dfrac{1}{2}j(z-j) - \dfrac{1}{4}(1+j)(z-j)^2 - \dfrac{1}{8}(z-j)^3 + \dots$
The radius of convergence is again $\sqrt{2}$.

■ **13** The function $\dfrac{1}{z(z^2+1)}$ has singularities at $z = 0, j, -j$. The radius of convergence is the distance of the centre of the point of expansion from the nearest

of these singularities. These are found straightforwardly either by inspection or by using Pythagoras' theorem

$$z = 1;\ 1;\ z = -1;\ 1, z = 1 + j;\ 1, z = 1 + j\tfrac{1}{2};\ \tfrac{1}{2}\sqrt{5},\ z = 2 + j3;\ 2\sqrt{2}\ .$$

■ **14** $f(z) = \dfrac{1}{(z^2+1)z}$

14(a) The Laurent expansion is

$$\begin{aligned}&\frac{1}{z}(1+z^2)^{-1}\\ &= \frac{1}{z}(1 - z^2 + z^4 - z^6 + \ldots)\\ &= \frac{1}{z} - z + z^3 - z^5 + \ldots\end{aligned}$$

valid for $0 <\mid z \mid< 1$

14(b) Since $f(z)$ is regular at $z = 1$, $f(z)$ has a Taylor expansion :

$$\frac{1}{z(1+z^2)} = f(1) + (z-1)f'(1) + \frac{(z-1)^2}{2}f''(1) + \ldots$$

$$f(1) = \frac{1}{2}, f'(z) = -\frac{1+3z^2}{(z+z^3)^2},\ f'(1) = -1$$

$$f''(z) = -\frac{6z}{(z+z^3)^2} + \frac{2(1+3z^2)^2}{(z+z^3)^3}$$

$$\text{so } f''(1) = -\frac{4}{4} + \frac{2\times 16}{8} = \frac{5}{2}$$

so that

$$\frac{1}{z(1+z^2)} = \frac{1}{2} - (z-1) + \frac{5}{4}(z-1)^2 + \ldots$$

valid for $\mid z-1 \mid< 1$

■ **15** $f(z) = e^z \sin\left(\dfrac{1}{1-z}\right)$

15(a) At $z = 0$, $f(z)$ is regular. Thus the principal part is zero and $f(0) = \sin 1$, $f(z) = \sin 1 + q_1 z + q_2 z^2 + \ldots \mid z \mid< 1$, Taylor series.

15(b) At $z = 1$, $f(z)$ has an essential singularity.

15(c) At $z = \infty, e^z$ has an essential singularity. Hence for parts (b) and (c) the principal part has infinitely many terms.

■ **16(a)**

$$\begin{aligned} e^z \sinh z &= \frac{1}{2}e^z(e^z + e^{-z}) = \frac{1}{2}(e^{2z} + 1) \\ &= \frac{1}{2}(1 + e^{2x}(e^{2jy})) \\ &= \frac{1}{2}(1 + e^{2x}\cos 2y + je^{2x}\sin 2y) \\ \text{Real part} &= \frac{1}{2}(1 + e^{2x}\cos 2y) \\ \text{Imaginary part} &= \frac{1}{2}e^{2x}\sin 2y \end{aligned}$$

16(b)

$$\begin{aligned} \cos 2z &= \cos(2x + j2y) \\ &= \cos 2x \cosh 2y - j \sin 2x \sinh 2y \end{aligned}$$

16(c)

$$\begin{aligned} \frac{\sin z}{z} &= \frac{x - jy}{x^2 + y^2}\sin(x + jy) \\ &= \frac{(x - jy)(\sin x \cosh y + j \cos x \sinh y)}{x^2 + y^2} \\ &= \frac{x \sin x \cosh y + y \cos x \sinh y + j(x \cos x \sinh y - y \sin x \cosh y)}{x^2 + y^2} \end{aligned}$$

16(d)

$$\begin{aligned} \tan z = \tan(x + jy) &= \frac{\tan x + \tan jy}{1 - \tan x \tan jy} \\ &= \frac{\tan x + j \tanh y}{1 - j \tan x \tanh y} \cdot \frac{1 + j \tan x \tanh y}{1 + j \tan x \tanh y} \\ \text{from which } \tan z &= \frac{\tan x(1 - \tanh^2 y) + j \tanh y(1 + \tan^2 x)}{1 + \tan^2 x \tanh^2 y} \end{aligned}$$

■ **17(a)** Since $\dfrac{dw}{dz} = -\dfrac{2}{z^3} \neq 0$, this mapping is *conformal*.

17(b)

$$\begin{aligned}\frac{dw}{dz} &= 6z^2 + 6z + 6(1-j) \\ &= 0 \text{ when } z^2 + z + 1 - j = 0 \\ &\quad \text{or } (z-j)(z+j+1) = 0 \\ &\quad \text{so } z = j, -1-j\end{aligned}$$

are the points where the mapping fails to be conformal.

17(c) $w = 64z + \dfrac{1}{z^3}$

$$\begin{aligned}\frac{dw}{dz} &= 64 - \frac{3}{z^4} = 0 \\ &\text{where } z^4 = \frac{3}{64} \\ &\text{so } z = 0.465, -0.465, j0.465, -j0.465\end{aligned}$$

are the points where the mapping fails to be conformal.

■ **18** $w = \cos z$, $\dfrac{dw}{dz} = \sin z = 0$ when $z = n\pi, n =$ an integer.

$$u + jv = \cos(x + jy) = \cos x \cosh y - j \sin x \sinh y$$

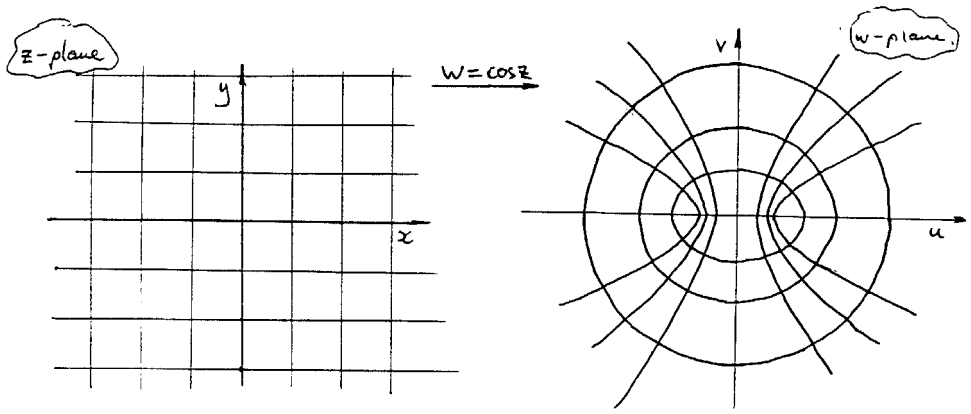
$$u = \cos x \cosh y$$
$$v = -\sin x \sinh y$$

Lines $x = k$ will thus transform to

$$\frac{u^2}{\cos^2 k} - \frac{v^2}{\sin^2 k} = 1 - \text{hyperpolae}$$

Lines $y = l$ will thus transform to

$$\frac{u^2}{\cosh^2 l} + \frac{v^2}{\sinh^2 l} = 1 - \text{ellipses}$$



■ **19(a)** $\dfrac{\sin z}{z^2} = \dfrac{\sin z}{z} \cdot \dfrac{1}{z}$. Since $\dfrac{\sin z}{z} \to 1$ is $z \to 0$ this function has a *simple* role at $z = 0$.

19(b) $\dfrac{1}{(z^3 - 8)^2}$ has double poles when $z^3 = 8$, that is, at 2, $2e^{2\pi j/3}$, $2e^{4\pi j/3}$.

19(c)

$$\frac{z+1}{z^4 - 1} = \frac{z+1}{(z^2-1)(z^2+1)} = \frac{1}{(z-1)(z^2+1)}$$

The singularity at $z = -1$ was removable, those at $z = 1, \pm j$ are simple poles.

19(d) $\operatorname{sech} z = \dfrac{1}{\cosh z}$ which has simple poles wherever $\cosh z = 0$ that is, $z = \frac{1}{2} j\pi(2n+1), n = 0, \pm 1, \pm 2, \ldots$

19(e) $\sinh z$ is entire (no singularities in the finite plane).

19(f) Essential singularity at $z = 0$.

19(g) $z^z = e^{z \ln z}$

which has an essential singularity at $z = 0$.

■ **20(a)** $\dfrac{e^{2z}}{(1+z)^2}$ double pole at $z = -1$ residue given by

$$\frac{d}{dz}(e^{2z})\Big|_{z=-1}$$
$$= 2e^{-2}$$

20(b) $\dfrac{\cos z}{2z-\pi}$ simple pole at $z=\pi/2$ residue is

$$\lim_{z\to\pi/2}(z-\pi/2)\frac{\cos z}{(2z-\pi)}=\frac{1}{2}\cos\frac{\pi}{2}=0$$

20(c) $\dfrac{\tan z}{2z-\pi}=\dfrac{\sin z}{\cos z(2z-\pi)}$ which has a *double* pole at $z=\dfrac{1}{2}\pi$

Writing $w=z-\pi/2$

$$\frac{\tan z}{2z-\pi}=-\frac{\cos w}{2w}=-\frac{\cos w}{2w\sin w}=-\frac{1-\frac{1}{2}w^2+\ldots}{2w(w-\frac{1}{6}w^3+\ldots)}$$
$$=-\frac{1}{2w^2}+\frac{1}{4}+\ldots$$

Hence residue is 0.

20(d) $\dfrac{z}{(z+8)^3}$ has a triple pole at $z=-8$

Writing $w=z+8$ and $z=w-8$ gives $\dfrac{z}{(z+8)^3}=\dfrac{1}{w^3}(w-8)=\dfrac{1}{w^2}-\dfrac{8}{w^3}$

Hence the residue is 0.

- **21**

$$f(z)=\frac{(z^2-1)(z^2+3z+5)}{z(z^4+1)}$$

Zeros are at $z=\pm1$ and at the roots of

$$z^2+3z+5=0$$
$$z=-\frac{3\pm\sqrt{9-20}}{2}=-\frac{3}{2}\pm j\frac{1}{2}\sqrt{11}$$

Simple poles are at $z=0$ and where $z^4=-1$

that is, $z=(\pm1\pm j)/\sqrt{2}$

The residue at $z=0$ is given by

$$\lim_{z\to0}\frac{(z^2-1)(z^2+3z+5)}{z^4+1}=-5$$

The residue at $z = z_0$ (where $z_0^4 + 1 = 0$) is

$$\lim_{z\to z_0}\left\{\frac{(z-z_0)(z^2-1)(z^2+3z+5)}{z(z^4+1)}\right\}$$
$$= \frac{(z_0^2-1)(z_0^2+3z_0+5)}{z_0}\lim_{z\to z_0}\left(\frac{z-z_0}{z^4+1}\right)$$
$$= \frac{(z_0^2-1)(z_0^2+3z_0+5)}{z_0}\frac{1}{4z_0^3} = -\frac{1}{4}(z_0^2-1)(z_0^2+3z_0+5)$$
(using $z_0^4 = -1$)

Putting $z_0 = \frac{1+j}{\sqrt{2}}, -\frac{1+j}{\sqrt{2}}, -\frac{1-j}{\sqrt{2}}, \frac{1-j}{\sqrt{2}}$ in turn gives the residues $\frac{3}{2}+\frac{3}{4}\sqrt{2}-j$, $\frac{3}{2}-\frac{3}{4}\sqrt{2}+j, \frac{3}{2}-\frac{3}{4}\sqrt{2}-j$ and $\frac{3}{2}+\frac{3}{4}\sqrt{2}+j$ respectively.

■ **22** $f(z) = \dfrac{z^7+6z^5-30z^4}{(z-1-j)^3}$ has a triple pole at $z = 1+j$

$$\text{Residue } = \frac{1}{2!}\frac{d^2}{dz^2}(z^7+6z^5-30z^4) \text{ is evaluated at } z = 1+j$$
$$= 21z^5+60z^3-180z^2\big|_{z=1+j}$$
$$= 21(-4-4j)+60(-2+2j)-360j$$
$$= -204-j324$$

■ **23(a)** The integrand $\dfrac{z}{z^2+7z+6}$ has poles at $z = -6$ and $z = -1$. Only the second is inside C. Residue $= \dfrac{z}{z+6}|_{z=-1} = -\dfrac{1}{5}$

Integral $= -\dfrac{2\pi j}{5}$

23(b) The integrand $\dfrac{(z^2+1)(z^2+3)}{(z^2+9)(z^2+4)}$ has four simple poles $\pm 2j, \pm 3j$ all inside C.
Residues are $-\dfrac{3}{20}j, \dfrac{3}{20}j, \dfrac{8}{5}j$ and $-\dfrac{8}{5}j$ the sum of which is 0.
The integral is thus 0.

23(c) The integrand $\dfrac{1}{z^2(1-z^2)^2}$ has double poles at $z = 0, 1$ and -1.
Residues are, at $z = 0$,

$$\frac{d}{dz}\frac{1}{(1-z^2)^2}\Big|_{z=0} = +\frac{2z}{(1-z^2)^3}\Big|_{z=0} = 0$$

$$\text{At } z = 1, \frac{d}{dz}\left[-\frac{1}{z^2(1+z)^2}\right] = \frac{2(2z+1)}{(z+z^2)^3}\Big|_{z=1} = \frac{3}{4}$$

$$\text{At } z = -1, \frac{d}{dz}\left[\frac{1}{z^2(1-z)^2}\right] = \frac{-2(1-2z)}{(z-z^2)^3}\Big|_{z=-1} = \frac{3}{4}$$

(i) If C is $|\, z \,| = \frac{1}{2}$ then only $z = 0$ is inside C, so integral $= 0$
(ii) If C is $|\, z \,| = 2$ then *all* poles are inside C, so integral $= 3\pi j$.

23(d) The integrand $\dfrac{1}{(2z-3j)(z+j)}$ has simple poles at $\dfrac{3j}{2}$ and $-j$.
Residues at $\dfrac{3j}{2}$ and $-j$ are, respectively, $-\dfrac{1}{5}j$ and $\dfrac{1}{5}j$.

(i) Both poles are inside $|\, z \,| = 2$, but since their sum is zero so is the integral
(ii) Inside $|\, z - 1 \,| = 1$ the function $\dfrac{1}{(2z-3j)(z+j)}$ is regular. By Cauchy's theorem the integral $= 0$.

23(e) The integrand $\dfrac{z^3}{(z^2+1)(z^2+z+1)}$ has simple poles at
$z = \pm j, -\frac{1}{2} \pm j\frac{1}{2}\sqrt{3}$.
C is the circle $|\, z - j \,| = \frac{1}{2}$ which contains the pole at $z = j$ but not the other three poles.
Residue at $z = j$ is $\frac{1}{2}j$ hence the integral $= -\pi$.

23(f) $\dfrac{z-1}{z(z-2)^2(z-3)}$ has simple poles at $z = 0, 3$ and a double pole at $z = 2$.
Residues at $z = 0, 2$ are respectively $\frac{1}{12}$ and $-\frac{3}{4}$.

(i) If C is $|\, z \,| = 1$ only the residue at $z = 0$ is considered : integral $= \frac{\pi j}{6}$
(ii) If C is $|\, z \,| = \frac{5}{2}$, residues at 0 at 2 are summed; integral $= \pi j\left(\frac{1}{12} - \frac{3}{4}\right) = -4\pi j/3$.

■ **24(a)** To evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)}$ we use the semicircular contour C on the upper half plane (see Exercise 65). The integral along the curved portion $\to 0$ as the radius of the semicircle $\to \infty$. The residues in the upper half plane are at the (double) pole at j and the (simple) pole at $-1+j$.

Residue of $\frac{z^2}{(z^2+1)^2(z^2+2z+2)}$ at $z=j$ is $\frac{d}{dz}\frac{z^2}{(z+j)^2(z^2+2z+2)}$ which is evaluated at $z=j$. This is (after some algebra) $\frac{9j-12}{100}$.

Residue at $z=-1+j$ is $\frac{z^2}{(z^2+1)(z+1+j)}$ evaluated at $z=-1+j$ which is $\frac{3-4j}{25}$. Sum of residues is $-\frac{7j}{100}$

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \int_C \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi j \frac{-7j}{100} = \frac{7\pi}{50}$$

24(b) To evaluate $\int_0^{\infty} \frac{x^2 dx}{x^4+16}$ one can either use a quarter circle contour (as in Exercise 65(h)) or note that, by symmetry, $\int_0^{\infty} \frac{x^2 dx}{x^4+16} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{x^4+16}$ and use the same semicircular contour as above.

The disadvantage of doing this is that there are *two* poles inside the semicircular contour, but only one in the quarter circle. However, this is compensated by the easier manipulation of the integral. We shall thus use the semicircle.

The poles inside C, both simple, are at

$$z = \sqrt{2}(-1+j) \quad \text{and} \quad z = \sqrt{2}(1+j)$$

The way to avoid unnecessary arithmetic/algebra is to determine the residue at $z=z_0$ where z_0 is one of the above poles. This is given by

$$\lim_{z \to z_0} \left[\frac{(z-z_0)z^2}{z^4+16} \right]$$

Since $z=z_0$ is a root of z^4+16, we can use L'Hôpital's rule to obtain

$$\text{Residue} = \left. \frac{3z^2 - 2zz_0}{4z^3} \right|_{z=z_0} = \frac{1}{4z_0}$$

Sum of residues is thus

$$\frac{1}{4\sqrt{2}(-1+j)}+\frac{1}{4\sqrt{2}(1+j)}=-\frac{j}{4\sqrt{2}}$$

Thus

$$\int_{-\infty}^{\infty}\frac{x^2dx}{x^4+16}=\int_0\frac{z^2dz}{z^4+16}=2\pi j\left(\frac{-j}{4\sqrt{2}}\right)=\frac{\pi}{2\sqrt{2}}=\frac{\pi}{4}\sqrt{2}$$

Hence

$$\int_0^{\infty}\frac{x^2dx}{x^4+16}=\frac{\pi}{8}\sqrt{2}$$

24(c) To evaluate

$$\int_0^{2\pi}\frac{\sin^2\theta d\theta}{5+4\cos\theta}$$

we follow Exercise 65(d) and put $z=e^{j\theta}$ so that $\dfrac{dz}{jz}=d\theta$ and $\dfrac{\sin^2\theta}{5+4\cos\theta}=\dfrac{z^2-z^4-1}{(2z^3+5z^2+2z)4}$

Hence

$$\int_0^{2\pi}\frac{\sin^2\theta d\theta}{5+4\cos\theta}=\frac{1}{4j}\oint_C\frac{z^2-z^4-1}{z^2(2z+1)(z+2)}dz$$

where C is the unit circle. Residues at $z=0$ and $z=-\frac{1}{2}$ (not that at $z=-2$) are summed.

Residue at $z=0$ is $\dfrac{d}{dz}\left(\dfrac{z^2-z^4-1}{2z^2+5z+2}\right)$ [evaluated at $z=0$] is $\dfrac{5}{4}$. Residue at $z=-\dfrac{1}{2}$ is $\left(\dfrac{z^2-z^4-1}{z^2(z+2)}\right)\Big|_{z=-\frac{1}{2}}=-\dfrac{13}{6}$

Hence

$$\int_0^{2\pi}\frac{\sin^2\theta d\theta}{5+4\cos\theta}=\frac{1}{4j}2\pi j\left(\frac{5}{4}-\frac{13}{6}\right)=-\frac{11\pi}{24}$$

24(d) The integral

$$\int_0^{2\pi}\frac{\cos 2\theta}{5-4\cos\theta}d\theta$$

is evaluated similarly

$$\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}d\theta = \frac{1}{2j}\oint_C \frac{z^4+1}{5z^3-2z^4-2z^2}dz = \frac{1}{2j}2\pi j\left(\frac{17}{6}-\frac{5}{4}\right) = \frac{19\pi}{12}$$

(In part (c) the negative sign arises from the choice of direction of the line integral. Since the integrand $\dfrac{\sin^2\theta}{5+4\cos\theta}$ is always positive it can be ignored.)

5

Laplace Transforms

Exercises 5.2.6

- **1(a)** $\mathcal{L}\{\cosh 2t\} = \mathcal{L}\{\frac{1}{2}(e^{2t} + e^{-2t}) = \frac{1}{2}\big[\frac{1}{s-2} + \frac{1}{s+2}\big] = \frac{s}{s^2-4},\ Re(s) > 2$

1(b) $\mathcal{L}\{t^2\} = \frac{2}{s^3},\ Re(s) > 0$

1(c) $\mathcal{L}\{3+t\} = \frac{3}{s} + \frac{1}{s^2} = \frac{3s+1}{s^2},\ Re(s) > 0$

1(d) $\mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2},\ Re(s) > -1$

- **2(a)** 5 **(b)** -3 **(c)** 0 **(d)** 3 **(e)** 2 **(f)** 0
  (g) 0 **(h)** 0 **(i)** 2 **(j)** 3

- **3(a)** $\mathcal{L}\{5-3t\} = \frac{5}{s} - \frac{3}{s^2} = \frac{5s-3}{s^2},\ Re(s) > 0$

3(b) $\mathcal{L}\{7t^3 - 2\sin 3t\} = 7.\frac{6}{s^4} - 2.\frac{3}{s^2+9} = \frac{42}{s^4} - \frac{6}{s^2+9},\ Re(s) > 0$

3(c) $\mathcal{L}\{3 - 2t + 4\cos 2t\} = \frac{3}{s} - \frac{2}{s^2} + 4.\frac{s}{s^2+4} = \frac{3s-2}{s^2} + \frac{4s}{s^2+4},\ Re(s) > 0$

3(d) $\mathcal{L}\{\cosh 3t\} = \frac{s}{s^2-9},\ Re(s) > 3$

3(e) $\mathcal{L}\{\sinh 2t\} = \frac{2}{s^2-4},\ Re(s) > 2$

3(f) $\mathcal{L}\{5e^{-2t}+3-2\cos 2t\} = \dfrac{5}{s+2}+\dfrac{3}{s}-2.\dfrac{s}{s^2+4},\ \mathrm{Re}(s)>0$

3(g) $\mathcal{L}\{4te^{-2t}\} = \dfrac{4}{(s+2)^2},\ Re(s)>-2$

3(h) $\mathcal{L}\{2e^{-3t}\sin 2t\} = \dfrac{4}{(s+3)^2+4} = \dfrac{4}{s^2+6s+13},\ Re(s)>-3$

3(i) $\mathcal{L}\{t^2e^{-4t}\} = \dfrac{2}{(s+4)^3},\ Re(s)>-4$

3(j)

$$\begin{aligned}\mathcal{L}\{6t^3-3t^2+4t-2\} &= \frac{36}{s^4}-\frac{6}{s^3}+\frac{4}{s^2}-\frac{2}{s}\\ &= \frac{36-6s+4s^2-2s^3}{s^4},\ Re(s)>0\end{aligned}$$

3(k) $\mathcal{L}\{2\cos 3t+5\sin 3t\} = 2.\dfrac{s}{s^2+9}+5.\dfrac{3}{s^2+9} = \dfrac{2s+15}{s^2+9},\ Re(s)>0$

3(l)

$$\begin{aligned}\mathcal{L}\{\cos 2t\} &= \frac{s}{s^2+4}\\ \mathcal{L}\{t\cos 2t\} &= -\frac{d}{ds}\Big[\frac{s}{s^2+4}\Big] = \frac{s^2-4}{(s^2+4)^2}, Re(s)>0\end{aligned}$$

3(m)

$$\begin{aligned}\mathcal{L}\{t\sin 3t\} &= -\frac{d}{ds}\Big[\frac{3}{s^2+9}\Big] = \frac{6s}{(s^2+9)^2}\\ \mathcal{L}\{t^2\sin 3t\} &= -\frac{d}{ds}\Big[\frac{6s}{(s^2+9)^2}\Big] = -\Big[\frac{(s^2+9)^2 6-6s(s^2+9)^2 4s}{(s^2+9)^4}\Big]\\ &= \frac{18s^2-54}{(s^2+9)^3},\ Re(s)>0\end{aligned}$$

3(n) $\mathcal{L}\{t^2-3\cos 4t\} = \dfrac{2}{s^3}-\dfrac{3s}{s^2+16},\ Re(s)>0$

3(o)

$$\mathcal{L}\{t^2e^{-2t} - e^{-t}\cos 2t + 3\} = \frac{2}{(s+2)^3} + \frac{(s+1)}{(s+1)^2+4} + \frac{3}{s}$$
$$= \frac{2}{(s+2)^3} + \frac{s+1}{s^2+2s+5} + \frac{3}{s}, \ Re(s) > 0$$

Exercises 5.2.10

4(a) $\mathcal{L}^{-1}\{\frac{1}{(s+3)(s+7)}\} = \mathcal{L}^{-1}\{\frac{\frac{1}{4}}{s+3} - \frac{\frac{1}{4}}{s+7}\} = \frac{1}{4}[e^{-3t} - e^{-7t}]$

4(b) $\mathcal{L}^{-1}\{\frac{s+5}{(s+1)(s-3)}\} = \mathcal{L}^{-1}\{\frac{-1}{s+1} + \frac{2}{s-3}\} = -e^{-t} + 2e^{3t}$

4(c) $\mathcal{L}^{-1}\{\frac{s-1}{s^2(s+3)}\} = \mathcal{L}^{-1}\{\frac{\frac{4}{9}}{s} - \frac{\frac{1}{3}}{s^2} - \frac{\frac{4}{9}}{s+3}\} = \frac{4}{9} - \frac{1}{3}t - \frac{4}{9}e^{-3t}$

4(d) $\mathcal{L}^{-1}\{\frac{2s+6}{s^2+4}\} = \mathcal{L}^{-1}\{2.\frac{s}{s^2+2^2} + 3.\frac{2}{s^2+2^2}\} = 2\cos 2t + 3\sin 2t$

4(e)

$$\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+16)}\} = \mathcal{L}^{-1}\{\frac{0}{s} + \frac{\frac{1}{16}}{s^2} - \frac{\frac{1}{16}}{s^2+16}\}$$
$$= \frac{1}{16}t - \frac{1}{64}\sin 4t = \frac{1}{64}[4t - \sin 4t]$$

4(f) $\mathcal{L}^{-1}\{\frac{s+8}{s^2+4s+5}\} = \mathcal{L}^{-1}\{\frac{(s+2)+6}{(s+2)^2+1}\} = e^{-2t}[\cos t + 6\sin t]$

4(g)

$$\mathcal{L}^{-1}\{\frac{s+1}{s^2(s^2+4s+8)}\} = \mathcal{L}^{-1}\{\frac{\frac{1}{8}}{s} + \frac{-\frac{1}{8}s + \frac{1}{2}}{(s+2)^2+2^2}\}$$
$$= \mathcal{L}^{-1}\{\frac{1}{8}.\frac{1}{s} - \frac{1}{8}\frac{(s+2)-3(2)}{(s+2)^2+2^2}\}$$
$$= \frac{1}{8}[1 - e^{-2t}\cos 2t + 3e^{-2t}\sin 2t]$$

4(h)

$$\mathcal{L}^{-1}\{\frac{4s}{(s-1)(s+1)^2}\} = \mathcal{L}^{-1}\{\frac{1}{s-1} - \frac{1}{(s+1)} + \frac{2}{(s+1)^2}\}$$
$$= e^t - e^{-t} + 2te^t$$

4(i) $\mathcal{L}^{-1}\{\frac{s+7}{s^2+2s+5}\} = \mathcal{L}^{-1}\{\frac{(s+1)+3(2)}{(s+1)^2+2^2}\} = e^{-t}[\cos 2t + 3\sin 2t]$

4(j)

$$\mathcal{L}^{-1}\{\frac{s^2-7s+5}{(s-1)(s-2)(s-3)}\} = \mathcal{L}^{-1}\{\frac{\frac{1}{2}}{s-1} - \frac{3}{s-2} + \frac{\frac{1}{2}}{s-3}\}$$
$$= \frac{1}{2}e^t - 3e^{2t} + \frac{11}{2}e^{3t}$$

4(k)

$$\mathcal{L}^{-1}\{\frac{5s-7}{(s+3)(s^2+2)}\} = \mathcal{L}^{-1}\{\frac{-2}{s+3} + \frac{2s-1}{s^2+2}\}$$
$$= -2e^{-3t} + 2\cos\sqrt{2}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t$$

4(l)

$$\mathcal{L}^{-1}\{\frac{s}{(s-1)(s^2+2s+2)}\} = \mathcal{L}^{-1}\{\frac{\frac{1}{5}}{s-1} - \frac{1}{5}\frac{s-2}{s^2+2s+2}\}$$
$$= \mathcal{L}^{-1}\{\frac{\frac{1}{5}}{s-1} - \frac{1}{5}\frac{(s+1)-3}{(s+1)^2+1}\}$$
$$= \frac{1}{5}e^t - \frac{1}{5}e^{-t}(\cos t - 3\sin t)$$

4(m) $\mathcal{L}^{-1}\{\frac{s-1}{s^2+2s+5}\} = \mathcal{L}^{-1}\{\frac{(s+1)-2}{(s+1)^2+2^2}\} = e^{-t}(\cos 2t - \sin 2t)$

4(n)

$$\mathcal{L}^{-1}\{\frac{s-1}{(s-2)(s-3)(s-4)}\} = \mathcal{L}^{-1}\{\frac{\frac{1}{2}}{s-2} - \frac{2}{s-3} + \frac{\frac{3}{2}}{s-4}\}$$
$$= \frac{1}{2}e^{2t} - 2e^{3t} + \frac{3}{2}e^{-4t}$$

4(o)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{3s}{(s-1)(s^2-4)}\} &= \mathcal{L}^{-1}\{\frac{3s}{(s-1)(s-2)(s+2)}\}\\ &= \mathcal{L}^{-1}\{\frac{-1}{s-1}+\frac{\frac{3}{2}}{s-2}-\frac{\frac{1}{2}}{s+2}\}\\ &= -e^t+\frac{3}{2}e^{2t}-\frac{1}{2}e^{-2t}\end{aligned}$$

4(p)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{36}{s(s^2+1)(s^2+9)}\} &= \mathcal{L}^{-1}\{\frac{4}{s}-\frac{\frac{9}{2}s}{s^2+1}+\frac{\frac{1}{2}s}{s^2+9}\}\\ &= 4-\frac{9}{2}\cos t+\frac{1}{2}\cos 3t\end{aligned}$$

4(q)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{2s^2+4s+9}{(s+2)(s^2+3s+3)}\} &= \mathcal{L}^{-1}\{\frac{9}{s+2}-\frac{7s+9}{(s+\frac{3}{2})^2+3/4}\}\\ &= \mathcal{L}^{-1}\{\frac{9}{s+2}-\frac{7(s+\frac{3}{2})-\sqrt{3}.\sqrt{3}/2}{(s+\frac{3}{2})^2+(\sqrt{3}/2)^2}\}\\ &= 9e^{-2t}-e^{-\frac{3}{2}t}\big[7\cos\frac{\sqrt{3}}{2}t-\sqrt{3}\sin\frac{\sqrt{3}}{2}t\big]\end{aligned}$$

4(r)

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{1}{(s+1)(s+2)(s^2+2s+10)}\} &= \mathcal{L}^{-1}\{\frac{\frac{1}{9}}{s+1}-\frac{\frac{1}{10}}{s+2}-\frac{\frac{1}{90}s+\frac{1}{9}}{s^2+2s+10}\}\\ &= \mathcal{L}^{-1}\Big\{\frac{\frac{1}{9}}{s+1}-\frac{\frac{1}{10}}{s+2}-\frac{1}{90}\big[\frac{s+10}{(s+1)^2+3^2}\big]\Big\}\\ &= \mathcal{L}^{-1}\Big\{\frac{\frac{1}{9}}{s+1}-\frac{\frac{1}{10}}{s+2}-\frac{1}{90}\big[\frac{(s+1)+3(3)}{(s+1)^2+3^2}\big]\Big\}\\ &= \frac{1}{9}e^{-t}-\frac{1}{10}e^{-2t}-\frac{1}{90}e^{-t}(\cos 3t+3\sin 3t)\end{aligned}$$

Exercises 5.3.5

- **5(a)**

$$(s+3)X(s) = 2 + \frac{1}{s+2} = \frac{2s+5}{s+2}$$
$$X(s) = \frac{2s+5}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{-2t} + e^{-3t}$$

5(b)

$$(3s-4)X(s) = 1 + \frac{2}{s^2+4} = \frac{s^2+6}{s^2+4}$$
$$X(s) = \frac{s^2+6}{(3s-4)(s^2+4)} = \frac{\frac{35}{26}}{3s-4} - \frac{\frac{3}{26}s + \frac{4}{26}}{s^2+4}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{35}{78}e^{\frac{4}{3}t} - \frac{3}{26}(\cos 2t + \frac{2}{3}\sin 2t)$$

5(c)

$$(s^2+2s+5)X(s) = \frac{1}{s}$$
$$X(s) = \frac{1}{s(s^2+2s+5)} = \frac{\frac{1}{5}}{s} - \frac{1}{5}\cdot\frac{s+2}{s^2+2s+5}$$
$$= \frac{\frac{1}{5}}{s} - \frac{1}{5}\frac{(s+1)+\frac{1}{2}(2)}{(s+1)^2+2^2}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{5}(1 - e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t)$$

5(d)

$$(s^2+2s+1)X(s) = 2 + \frac{4s}{s^2+4} = \frac{2s^2+4s+8}{s^2+4}$$
$$X(s) = \frac{2s^2+4s+8}{(s+1)^2(s^2+4)}$$
$$= \frac{\frac{12}{25}}{(s+1)} + \frac{\frac{6}{5}}{(s+1)^2} - \frac{1}{25}\left[\frac{12s-32}{s^2+4}\right]$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{12}{25}e^{-t} + \frac{6}{5}te^{-t} - \frac{12}{25}\cos 2t + \frac{16}{25}\sin 2t$$

5(e)

$$(s^2 - 3s + 2)X(s) = 1 + \frac{2}{s+4} = \frac{s+6}{s+4}$$

$$X(s) = \frac{s+6}{(s+4)(s-1)(s-2)} = \frac{\frac{1}{15}}{s+4} - \frac{\frac{7}{5}}{s-1} + \frac{\frac{4}{3}}{s-2}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{15}e^{-4t} - \frac{7}{5}e^{t} + \frac{4}{3}e^{2t}$$

5(f)

$$(s^2 + 4s + 5)X(s) = (4s - 7) + 16 + \frac{3}{s+2}$$

$$X(s) = \frac{4s^2 + 17s + 21}{(s+2)(s^2+4s+5)} = \frac{3}{s+2} + \frac{(s+2)+1}{(s+2)^2+1}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 3e^{-2t} + e^{-2t}\cos t + e^{-2t}\sin t$$

5(g)

$$(s^2 + s - 2)X(s) = s + 1 + \frac{5(2)}{(s+1)^2+4}$$

$$X(s) = \frac{s^3 + 3s^2 + 7s + 15}{(s+2)(s-1)(s^2+2s+5)}$$

$$= \frac{-\frac{1}{3}}{s+2} + \frac{\frac{13}{12}}{s-1} + \frac{\frac{1}{4}s - \frac{5}{4}}{s^2+2s+5}$$

$$= \frac{-\frac{1}{3}}{s+2} + \frac{\frac{13}{12}}{s-1} + \frac{1}{4}\left[\frac{(s+1) - 3(2)}{(s+1)^2 + 2^2}\right]$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = -\frac{1}{3}e^{-2t} + \frac{13}{12}e^{t} + \frac{1}{4}e^{-t}\cos 2t - \frac{3}{4}e^{-t}\sin 2t$$

5(h)

$$(s^2+2s+3)Y(s) = 1 + \frac{3}{s^2}$$

$$Y(s) = \frac{s^2+3}{s^2(s^2+2s+3)} = \frac{-\frac{2}{3}}{s} + \frac{1}{s^2} + \frac{\frac{2}{3}s+\frac{4}{3}}{s^2+2s+3}$$

$$= \frac{-\frac{2}{3}}{s} + \frac{1}{s^2} + \frac{2}{3}\Big[\frac{(s+1)-\frac{1}{\sqrt{2}}(\sqrt{2})}{(s+1)^2+(\sqrt{2})^2}\Big]$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{2}{3} + t + \frac{2}{3}e^{-t}(\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t)$$

5(i)

$$(s^2+4s+4)X(s) = \frac{1}{2}s + 2 + \frac{2}{s^3} + \frac{1}{s+2}$$

$$X(s) = \frac{s^5+6s^4+10s^3+4s+8}{2s^3(s+2)^3}$$

$$= \frac{\frac{3}{8}}{s} - \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{2}}{s^3} + \frac{\frac{1}{8}}{s+2} + \frac{\frac{3}{4}}{(s+2)^2} + \frac{1}{(s+2)^3}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{3}{8} - \frac{1}{2}t + \frac{1}{4}t^2 + \frac{1}{8}e^{-2t} + \frac{3}{4}te^{-2t} + \frac{1}{2}t^2e^{-2t}$$

5(j)

$$(9s^2+12s+5)X(s) = \frac{1}{s}$$

$$X(s) = \frac{1}{9s(s^2+\frac{4}{3}s+\frac{5}{9})} = \frac{\frac{1}{5}}{s} - \frac{\frac{1}{5}s+\frac{4}{15}}{(s+\frac{2}{3})^2+\frac{1}{9}}$$

$$= \frac{\frac{1}{5}}{s} - \frac{1}{5}\frac{[(s+\frac{2}{3})+\frac{2}{3}]}{(s+\frac{2}{3})^2+(\frac{1}{3})^2}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{5} - \frac{1}{5}e^{-\frac{2}{3}t}(\cos\frac{1}{3}t + 2\sin\frac{1}{3}t)$$

5(k)

$$(s^2+8s+16)X(s) = -\frac{1}{2}s + 1 - 4 + 16\cdot\frac{4}{s^2+16} = \frac{-s^3-6s^2-16s+32}{2(s^2+16)}$$

$$X(s) = \frac{-s^3-6s^2-16s+32}{2(s+4)^2(s^2+16)}$$

$$= \frac{0}{s+4} + \frac{1}{(s+4)^2} - \frac{\frac{1}{2}s}{s^2+16}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = te^{-4t} - \frac{1}{2}\cos 4t$$

5(l)

$$(9s^2+12s+4)Y(s) = 9(s+1) + 12 + \frac{1}{s+1}$$

$$Y(s) = \frac{9s^2+30s+22}{(3s+2)^2(s+1)}$$

$$= \frac{1}{s+1} + \frac{0}{3s+2} + \frac{18}{(3s+2)^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-t} + 2te^{-\frac{2}{3}t}$$

5(m)

$$(s^3-2s^2-s+2)X(s) = s - 2 + \frac{2}{s} + \frac{1}{s^2}$$

$$X(s) = \frac{s^3-2s^2+2s+1}{s^2(s-1)(s-2)(s+1)}$$

$$= \frac{\frac{5}{4}}{s} + \frac{\frac{1}{2}}{s^2} - \frac{1}{s-1} + \frac{\frac{5}{12}}{s-2} - \frac{\frac{2}{3}}{s+1}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{5}{4} + \frac{1}{2}t - e^t + \frac{5}{12}e^{2t} - \frac{2}{3}e^{-t}$$

5(n)

$$(s^3+s^2+s+1) = (s+1) + 1 + \frac{s}{s^2+9}$$

$$X(s) = \frac{s^3+2s^2+10s+18}{(s^2+9)(s+1)(s^2+1)} = \frac{\frac{9}{20}}{s+1} - \frac{1}{16}\frac{7s-25}{s^2+1} - \frac{1}{80}\frac{s+9}{s^2+9}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{9}{20}e^{-t} - \frac{7}{16}\cos t + \frac{25}{16}\sin t - \frac{1}{80}\cos 3t - \frac{3}{80}\sin 3t$$

- **6(a)**

$$2sX(s) - (2s+9)Y(s) = -\frac{1}{2} + \frac{1}{s+2}$$
$$(2s+4)X(s) + (4s-37)Y(s) = 1$$

Eliminating $X(s)$

$$[-(2s+9)(2s+4) - 2s(4s-37)]Y(s) = (-\frac{1}{2} + \frac{1}{s+2})(2s+4) - 2s = -3s$$
$$Y(s) = \frac{3s}{12s^2 - 48s + 36} = \frac{1}{4}\cdot\frac{s}{(s-3)(s-1)}$$
$$= \frac{1}{4}\big[\frac{\frac{3}{2}}{s-3} - \frac{\frac{1}{2}}{s-1}\big]$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4}\big[\frac{3}{2}e^{3t} - \frac{1}{2}e^{t}\big] = \frac{3}{8}e^{3t} - \frac{1}{8}e^{t}$$

Eliminating $\frac{dx}{dt}$ from the two equations

$$6\frac{dy}{dt} + 4x - 28y = -e^{-2t}$$
$$x(t) = \frac{1}{4}\big[-e^{-2t} + 28y - 6\frac{dy}{dt}\big] = \frac{1}{4}\big[-e^{-2t} + \frac{21}{4}e^{3t} - \frac{7}{2}e^{t} - \frac{27}{3}e^{3t} + \frac{3}{4}e^{t}\big]$$
$$\text{i.e. } x(t) = \frac{1}{4}\big(\frac{15}{4}e^{3t} - \frac{11}{4}e^{t} - e^{-2t}\big)\ , y(t) = \frac{1}{8}(3e^{3t} - e^{t})$$

6(b)

$$(s+1)X(s) + (2s-1)Y(s) = \frac{5}{s^2+1}$$
$$(2s+1)X(s) + (3s-1)Y(s) = \frac{1}{s-1}$$

Eliminating $X(s)$

$$[(2s-1)(2s+1) - (3s-1)(s+1)]Y(s) = \frac{5}{s^2+1}(2s+1) - \frac{s+1}{s-1}$$
$$Y(s) = \frac{10s+5}{s(s^2+1)(s-2)} - \frac{s+1}{s(s-1)(s-2)}$$
$$= \big[\frac{-\frac{5}{2}}{s} + \frac{\frac{5}{2}}{s-2} - \frac{5}{s^2+1}\big] - \big[\frac{\frac{1}{2}}{s} - \frac{2}{s-1} + \frac{\frac{3}{2}}{s-2}\big]$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{5}{2} + \frac{5}{2}e^{2t} - 5\sin t - \frac{1}{2} + 2e^{t} - \frac{3}{2}e^{2t}$$
$$= -3 + e^{2t} + 2e^{t} - 5\sin t$$

Eliminating $\frac{dx}{dt}$ from the original equations

$$\frac{dy}{dt} + x - y = 10\sin t - e^t$$
$$x(t) = 10\sin t - e^t - 3 + e^{2t} + 2e^t - 5\sin t - 2e^{2t} - 2e^t + 5\cos t$$
$$= 5\sin t + 5\cos t - 3 - e^t - e^{2t}$$

6(c)

$$(s+2)X(s) + (s+1)Y(s) = 3 + \frac{1}{s+3} = \frac{3s+10}{s+3}$$
$$5X(s) + (s+3)Y(s) = 4 + \frac{5}{s+2} = \frac{4s+13}{s+2}$$

Eliminating $X(s)$

$$[5(s+1) - (s+2)(s+3)]Y(s) = \frac{15s+50}{s+3} - (4s+13) = \frac{-4s^2 - 10s + 11}{s+3}$$
$$Y(s) = \frac{4s^2 + 10s - 11}{(s+3)(s^2+1)} = \frac{-\frac{1}{2}}{s+3} + \frac{\frac{9}{2}s - \frac{7}{2}}{s^2+1}$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{2}e^{-3t} + \frac{9}{2}\cos t - \frac{7}{2}\sin t$$

From the second differential equation

$$5x = 5e^{-2t} + \frac{3}{2}e^{-3t} - \frac{27}{2}\cos t + \frac{21}{2}\sin t - \frac{3}{2}e^{-3t}$$
$$+ \frac{9}{2}\sin t + \frac{7}{2}\cos t$$
$$x(t) = 3\sin t - 2\cos t + e^{-2t}$$

6(d)

$$(3s-2)X(s) + 3sY(s) = 6 + \frac{1}{s-1} = \frac{6s-5}{s-1}$$
$$sX(s) + (2s-1)Y(s) = 3 + \frac{1}{s} = \frac{3s+1}{s}$$

Eliminating $X(s)$

$$[3s^2-(3s-2)(2s-1)]Y(s)=\frac{s(6s-5)}{s-1}-\frac{(3s-2)(3s+1)}{s}$$

$$Y(s)=\frac{9s^2-3s-2}{s(3s-1)(s-2)}-\frac{6s^2-5s}{(s-1)(3s-1)(s-2)}$$

$$=\big[-\frac{1}{s}+\frac{\frac{18}{5}}{3s-1}+\frac{\frac{14}{5}}{s-2}\big]$$

$$-\big[\frac{-\frac{1}{2}}{s-1}-\frac{\frac{9}{10}}{3s-1}+\frac{\frac{14}{5}}{s-2}\big]$$

$$=-\frac{1}{s}+\frac{\frac{1}{2}}{s-1}+\frac{\frac{9}{2}}{3s-1}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=-1+\frac{1}{2}e^t+\frac{3}{2}e^{\frac{t}{3}}$$

Eliminating $\dfrac{dx}{dt}$ from the original equations

$$x(t)=\frac{1}{2}\big[3-e^t-3+\frac{3}{2}e^t+\frac{9}{2}e^{\frac{t}{3}}-\frac{3}{2}e^t-\frac{3}{2}e^{\frac{t}{3}}\big]$$

$$=\frac{3}{2}e^{\frac{t}{3}}-\frac{1}{2}e^t$$

6(e)

$$(3s-2)X(s)+sY(s)=-1+\frac{3}{s^2+1}+\frac{5s}{s^2+1}=\frac{-s^2+5s+2}{s^2+1}$$

$$2sX(s)+(s+1)Y(s)=-1+\frac{1}{s^2+1}+\frac{s}{s^2+1}=\frac{-s^2+s}{s^2+1}$$

Eliminating $Y(s)$

$$[(3s-2)(s+1)-2s^2]X(s)=\frac{1}{s^2+1}[(-s^2+5s+2)(s+1)-(-s^2+s)s]$$

$$X(s)=\frac{3s^2+7s+2}{(s+2)(s-1)(s^2+1)}=\frac{3s+1}{(s-1)(s^2+1)}$$

$$=\frac{2}{s-1}-\frac{2s-1}{s^2+1}$$

$$x(t)=\mathcal{L}^{-1}\{X(s)\}=2e^t-2\cos t+\sin t$$

Eliminating $\dfrac{dy}{dt}$ from the original equation

$$\begin{aligned} y(t) &= -2\sin t - 4\cos t - 2x + \frac{dx}{dt} \\ &= -2\sin t - 4\cos t - 4e^t + 4\cos t - 2\sin t + 2e^t + 2\sin t + \cos t \end{aligned}$$

$$\text{that is, } y(t) = -2e^t - 2\sin t + \cos t,\ x(t) = 2e^t - 2\cos t + \sin t$$

6(f)

$$sX(s) + (s+1)Y(s) = 1 + \frac{1}{s^2} = \frac{s^2+1}{s^2}$$
$$(s+1)X(s) + 4sY(s) = 1 + \frac{1}{s} = \frac{s+1}{s}$$

Eliminating $Y(s)$

$$[4s^2 - (s+1)^2]X(s) = 4s\big(\frac{s^2+1}{s^2}\big) - \frac{(s+1)^2}{s} = \frac{3s^2 - 2s + 3}{s}$$
$$X(s) = \frac{3s^2 - 2s + 3}{s(s-1)(3s+1)} = \frac{-3}{s} - \frac{1}{s-1} + \frac{9}{3s+1}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = -3 + e^t + 3e^{-\frac{t}{3}}$$

Eliminating $\dfrac{dy}{dt}$ from the original equation

$$\begin{aligned} y &= \frac{1}{4}\big[4t - 1 + x + 3\frac{dx}{dt}\big] \\ &= \frac{1}{4}\big[4t - 1 - 3 + e^t + 3e^{-\frac{t}{3}} - 3e^t + 3e^{-\frac{t}{3}}\big] \end{aligned}$$

$$\text{that is, } y(t) = t - 1 - \frac{1}{2}e^t + \frac{3}{2}e^{-\frac{t}{3}},\ x(t) = -3 + e^t + 3e^{-\frac{t}{3}}$$

6(g)

$$(2s+7)X(s) + 3sY(s) = \frac{12}{s^2} + \frac{7}{s} = \frac{14+7s}{s^2}$$
$$(5s+4)X(s) - (3s-6)Y(s) = \frac{14}{s^2} - \frac{14}{s} = \frac{14-14s}{s^2}$$

Eliminating $Y(s)$

$$[(2s+7)(3s-6)+(5s+4)(3s)]X(s) = \frac{1}{s^2}[(3s-6)(14+7s)+3s(14-14s)]$$
$$21(s^2+s-2)X(s) = 21(s+2)(s-1)X(s) = \frac{21}{s^2}(-s^2+2s-4)$$
$$X(s) = \frac{-s^2+2s-4}{s^2(s+2)(s-1)}$$
$$= \frac{-1}{s-1}+\frac{1}{s+2}+\frac{0}{s}+\frac{2}{s^2}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = -e^t+e^{-2t}+2t$$

Eliminating $\dfrac{dy}{dt}$ from the original equations

$$6y = 28t-7-11x-7\frac{dx}{dt}$$
$$= 28t-7+7e^t+14e^{-2t}-14+11e^t-11e^{-2t}-22t$$
$$\text{giving } y(t) = t-\frac{7}{2}+3e^t+\frac{1}{2}e^{-2t},\ x(t) = -e^t+e^{-2t}+2t.$$

6(h)

$$(s^2+2)X(s)-Y(s) = 4s$$
$$-X(s)+(s^2+2)Y(s) = 2s$$

Eliminating $Y(s)$

$$[(s^2+2)^2-1]X(s) = 4s(s^2+2)+2s$$
$$(s^4+4s^2+3)X(s) = 4s^3+10s$$
$$X(s) = \frac{4s^3+10s}{(s^2+1)(s^2+3)} = \frac{3s}{s^2+1}+\frac{s}{s^2+3}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 3\cos t+\cos\sqrt{3}t$$

From the first of the given equations

$$y(t) = 2x+\frac{d^2x}{dt^2} = 6\cos t+2\cos\sqrt{3}t-3\cos t-3\cos\sqrt{3}t$$
$$\text{that is, } y(t) = 3\cos t-\cos\sqrt{3}t,\ x(t) = 3\cos t-\cos\sqrt{3}t$$

6(i)

$$(5s^2+6)X(s)+12s^2Y = s\big[\frac{35}{4}+12\big] = \frac{83}{4}s$$
$$5s^2X(s)+(16s^2+6)Y(s) = s\big[\frac{35}{4}+16\big] = \frac{99}{4}s$$

Eliminating $X(s)$

$$[60s^4-(5s^2+6)(16s^2+6)]Y(s) = \frac{s}{4}[83(5s^2)-99(5s^2+6)]$$
$$[-20s^4-126s^2-36]Y(s) = \frac{s}{4}[-80s^2-594]$$

$$Y(s) = \frac{s(40s^2+297)}{4(s^2+6)(10s^2+3)}$$
$$= \frac{-\frac{1}{4}s}{s^2+6}+\frac{\frac{25}{2}s}{10s^2+3}$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{4}\cos\sqrt{6}t+\frac{5}{4}\cos\sqrt{\frac{3}{10}}t$$

Eliminating $\dfrac{d^2x}{dt^2}$ from the original equations

$$3x = 3y+3\frac{d^2y}{dt^2} = \big(\frac{15}{4}-\frac{3}{4}\big)\cos\sqrt{\frac{3}{10}}t+\big(-\frac{3}{4}+3\big)\cos\sqrt{6}t$$
$$\text{i.e. } x(t) = \cos\sqrt{\frac{3}{10}}t+\frac{3}{4}\cos\sqrt{6}t,\ x(t) = \frac{5}{4}\cos\sqrt{\frac{3}{10}}t-\frac{1}{4}\cos\sqrt{6}t.$$

6(j)

$$(2s^2-s+9)X(s)-(s^2+s+3)Y(s) = 2(s+1)-1 = 2s+1$$
$$(2s^2+s+7)X(s)-(s^2-s+5)Y(s) = 2(s+1)+1 = 2s+3$$

Subtract

$$(-2s+2)X(s)-(2s-2)Y(s) = -2 \ \Rightarrow\ X(s)+Y(s) = \frac{1}{s-1}$$
$$\Rightarrow\ x(t)+y(t) = e^t \qquad \text{(i)}$$

Add

$$(4s^2+16)X(s)-(2s+8)Y(s) = 4(s+1)$$
$$2X(s)-Y(s) = \frac{2(s+1)}{s^2+4} \Rightarrow 2x(t)-y(t)$$
$$= 2\cos 2t+\sin 2t \qquad \text{(ii)}$$

Then from (i) and (ii)

$$x(t) = \frac{1}{3}e^t + \frac{2}{3}\cos 2t + \frac{1}{3}\sin 2t,\ y(t) = \frac{2}{3}e^t - \frac{2}{3}\cos 2t - \frac{1}{3}\sin 2t$$

Exercises 5.4.3

■ **7** $1\mu\text{F} = 10^{-6}\text{F}$ so $50\mu = 5.10^5\text{F}$

Applying Kirchhoff's second law to the left hand loop

$$\frac{1}{5.10^5}\int i_1 dt + 2\big(\frac{di_1}{dt} - \frac{di_2}{dt}\big) = E.\sin 100t$$

Taking Laplace transforms

$$\frac{2.10^4}{s}I_1(s) + 2s[I_1(s) - I_2(s)] = E.\frac{100}{s^2 + 10^4}$$
$$(10^4 + s^2)I_1(s) - s^2 I_2(s) = E.\frac{50s}{s^2 + 10^4} \qquad \text{(i)}$$

Applying Kirchhoff's law to the right hand loop

$$100i_2(t) - 2\big(\frac{di_1}{dt} - \frac{di_2}{dt}\big) = 0$$

which on taking Laplace transforms gives

$$sI_1(s) = (50 + s)I_2(s) \qquad \text{(ii)}$$

Substituting in (i)

$$(10^4 + s^2)(50 + s)I_2(s) - s^2 I_2(s) = E.\frac{50s^2}{s^2 + 10^4}$$
$$(s^2 + 200s + 10^4)I_2(s) = \frac{Es^2}{s^2 + 10^4}$$
$$I_2(s) = E\big[\frac{s^2}{(s^2 + 10^4)(s + 100)^2}\big]$$
$$\text{then from (ii) } I_1(s) = E\big[\frac{s(50 + s)}{(s^2 + 10^4)(s + 100)^2}\big]$$

Expanding in partial functions

$$I_2(s) = E\big[\frac{-\frac{1}{200}}{s+100} + \frac{\frac{1}{2}}{(s+100)^2} + \frac{\frac{1}{200}s}{s^2+10^4}\big]$$

$$i_2(t) = \mathcal{L}^{-1}\{I_2(s)\} = E\big[-\frac{1}{200}e^{-100t} + \frac{1}{2}te^{-100t} + \frac{1}{200}\cos 100t\big]$$

■ **8** Applying Kirchhoff's second law to the primary and secondary circuits respectively gives

$$2i_1 + \frac{di_1}{dt} + 1\frac{di_2}{dt} = 10\sin t$$

$$2i_2 + 2\frac{di_2}{dt} + \frac{di_1}{dt} = 0$$

Taking Laplace transforms

$$(s+2)I_1(s) + sI_2(s) = \frac{10}{s^2+1}$$

$$sI_1(s) + 2(s+1)I_2(s) = 0$$

Eliminating $I_1(s)$

$$[s^2 - 2(s+1)(s+2)]I_2(s) = \frac{10s}{s^2+1}$$

$$I_2(s) = -\frac{10s}{(s^2+1)(s^2+7s+6)} = -\frac{10s}{(s^2+1)(s+6)(s+1)}$$

$$= -\big[\frac{-1}{s+1} + \frac{\frac{12}{37}}{s+6} + \frac{\frac{25}{37}s + \frac{35}{37}}{s^2+1}\big]$$

$$i_2(t) = \mathcal{L}^{-1}\{I_2(s)\} = e^{-t} - \frac{12}{37}e^{-6t} - \frac{25}{37}\cos t - \frac{35}{37}\sin t$$

■ **9** Applying Kirchhoff's law to the left and right hand loops gives

$$(i_1 + i_2) + \frac{d}{dt}(i_1 + i_2) + 1\int i_1 dt = E_0 = 10$$

$$i_2 + \frac{di_2}{dt} - 1\int i_1 dt = 0$$

Applying Laplace transforms

$$(s+1)I_1(s) + (s+1)I_2(s) + \frac{1}{s}I_1(s) = \frac{10}{s}$$
$$(s+1)I_2(s) - \frac{1}{s}I_1(s) = 0 \;\Rightarrow\; I_1(s) = s(s+1)I_2(s) \qquad \text{(i)}$$

Substituting back in the first equation

$$s(s+1)^2 I_2(s) + (s+1)I_2(s) + (s+1)I_2(s) = \frac{10}{s}$$
$$(s^2+s+2)I_2(s) = \frac{10}{s(s+1)}$$
$$I_2(s) = \frac{10}{s(s+1)(s^2+s+2)}$$

Then from (i)

$$I_1(s) = \frac{10}{s^2+s+2} = \frac{10}{(s+\frac{1}{2})^2 + \frac{7}{4}}$$
$$i_1(t) = \mathcal{L}^{-1}\{I_1(s)\} = \frac{20}{\sqrt{7}} e^{-\frac{1}{2}t} \sin\frac{\sqrt{7}}{2}t$$

■ **10** Applying Newton's law to the motion of each mass

$$\ddot{x}_1 = 3(x_2 - x_1) - x_1 = 3x_2 - 4x_1$$
$$\ddot{x}_2 = -9x_2 - 3(x_2 - x_1) = -12x_2 + 3x_1$$

giving

$$\ddot{x}_1 + 4x_1 - 3x_2 = 0,\; x_1(0) = -1,\; x_2(0) = 2$$
$$\ddot{x}_2 + 12x_2 - 3x_1 = 0$$

Taking Laplace transforms

$$(s^2+4)X_1(s) - 3X_2(s) = -s$$
$$-3X_1(s) + (s^2+12)X_2(s) = 2s$$

Eliminating $X_2(s)$

$$\begin{aligned}[(s^2+4)(s^2+12)-9]X_1(s) &= -s(s^2+12)+6s \\ (s^2+13)(s^2+3)X_1(s) &= -s^3-6s \\ X_1(s) &= \frac{-s^3-6s}{(s^2+13)(s^2+3)} = \frac{-\frac{3}{10}s}{s^2+3} - \frac{\frac{7}{10}s}{s^2+13} \\ x_1(t) = \mathcal{L}^{-1}\{X_1(s)\} &= -\frac{3}{10}\cos\sqrt{3}t - \frac{7}{10}\cos\sqrt{13}t\end{aligned}$$

From the first differential equation

$$\begin{aligned}3x_2 &= 4x_1 + \ddot{x}_1 \\ &= -\frac{6}{5}\cos\sqrt{3}t - \frac{14}{5}\cos\sqrt{13}t + \frac{9}{10}\cos\sqrt{3}t + \frac{91}{10}\cos\sqrt{13}t \\ x_2(t) &= \frac{1}{10}[21\cos\sqrt{13}t - \cos\sqrt{3}t]\end{aligned}$$

Thus, $x_1(t) = -\frac{1}{10}(3\cos\sqrt{3}t + 7\cos\sqrt{13}t)$, $x_2(t) = \frac{1}{10}[21\cos\sqrt{13}t - \cos\sqrt{3}t]$
Natural frequencies are $\sqrt{13}$ and $\sqrt{3}$.

■ **11** The equation of motion is

$$M\ddot{x} + b\dot{x} + Kx = Mg \;;\; x(0) = 0 \;,\; \dot{x}(0) = \sqrt{2gh}$$

The problem is then an investigative one where students are required to investigate for different h values either analytically or by simulation.

■ **12** By Newton's second law of motion

$$\begin{aligned}M_2\ddot{x}_2 &= -K_2x_2 - B_1(\dot{x}_2 - \dot{x}_1) + u_2 \\ M_1\ddot{x}_1 &= B_1(\dot{x}_2 - \dot{x}_1) - K_1x_1 + u_1\end{aligned}$$

Taking Laplace transforms and assuming quiescent initial state

$$\begin{aligned}(M_2s^2 + B_1s + K_2)X_2(s) - B_1sX_1(s) &= U_2(s) \\ -B_1sX_2(s) + (M_1s^2 + B_1s + K_1)X_1(s) &= U_1(s)\end{aligned}$$

Eliminating $X_1(s)$

$$[(M_1s^2 + B_1s + K_1)(M_2s^2 + B_1s + K_2) - B_1^2s^2]X_2(s)$$
$$= (M_1s^2 + B_1s + K_1)U_2(s) + B_1sU_1(s)$$
$$\text{i.e. } X_2(s) = \frac{B_1s}{\Delta}U_1(s) + \frac{(M_1s^2 + B_1s + K_1)}{\Delta}U_2(s)$$
$$\text{and } x_2(t) = \mathcal{L}^{-1}\{X_2(s)\} = \mathcal{L}^{-1}\Big\{\frac{B_1s}{\Delta}U_1(s) + \frac{(M_1s^2 + B_1s + K_1)}{\Delta}U_2(s)\Big\}$$

Likewise eliminating $X_2(t)$ from the original equation gives

$$x_1(t) = \mathcal{L}^{-1}\{X_1(s)\} = \mathcal{L}^{-1}\Big\{\frac{(M_1s + B_1s + K_2)}{\Delta}U_1(s) + \frac{B_1s}{\Delta}U_2(s)\Big\}$$

Exercises 5.5.7

- **13**

$$f(t) = tH(t) - tH(t-1)$$
$$= tH(t) - (t-1)H(t-1) - 1H(t-1)$$

Thus, using theorem 2.4

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-s} = \frac{1}{s^2}(1 - e^{-s}) - \frac{1}{s}e^{-s}$$

- **14(a)**

$$f(t) = 3t^2H(t) - (3t^2 - 2t + 3)H(t-4) - (2t-8)H(t-6)$$
$$= 3t^2H(t) - [3(t-4)^2 + 22(t-4) + 43]H(t-4) - [2(t-6) + 4]H(t-6)$$

Thus,

$$\mathcal{L}\{f(t)\} = \frac{6}{s^3} - e^{-4s}\mathcal{L}[3t^2 + 22t + 43] - e^{-6s}\mathcal{L}[2t+4]$$
$$= \frac{6}{s^3} - \Big[\frac{6}{s^3} + \frac{22}{s^2} + \frac{43}{s}\Big]e^{-4s} - \Big[\frac{2}{s^2} + \frac{4}{s}\Big]e^{-6s}$$

14(b)

$$\begin{aligned} f(t) &= tH(t) + (2-2t)H(t-1) - (2-t)H(t-2) \\ &= tH(t) - 2(t-1)H(t-1) - (t-2)H(t-2) \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \frac{1}{s^2} - 2e^{-s}\mathcal{L}\{t\} + e^{-2s}\mathcal{L}\{t\} \\ &= \frac{1}{s^2}[1 - 2e^{-s} + e^{-2s}] \end{aligned}$$

15(a) $\mathcal{L}^{-1}\{\dfrac{e^{-5s}}{(s-2)^4}\} = \mathcal{L}^{-1}\{e^{-5s}F(s)\}$ where $F(s) = \dfrac{1}{(s-2)^4}$ and by the first shift theorem $f(t) = \mathcal{L}^{-1}\{F(s)\} = \dfrac{1}{6}t^3e^{2t}$.
Thus, by the second shift theorem

$$\begin{aligned} \mathcal{L}^{-1}\{\frac{e^{-5s}}{(s-2)^4}\} &= f(t-5)H(t-5) \\ &= \frac{1}{6}(t-5)^3e^{2(t-5)}H(t-5) \end{aligned}$$

15(b) $\mathcal{L}^{-1}\{\dfrac{3e^{-2s}}{(s+3)(s+1)}\} = \mathcal{L}^{-1}\{e^{-2s}F(s)\}$ where

$$\begin{aligned} F(s) &= \frac{3}{(s+3)(s+1)} = \frac{-\frac{3}{2}}{s+3} + \frac{\frac{3}{2}}{s+1} \\ f(t) = \mathcal{L}^{-1}\{F(s)\} &= \frac{3}{2}e^{-t} - \frac{3}{2}e^{-3t} \end{aligned}$$

$$\text{so} \quad \begin{aligned} \mathcal{L}^{-1}\{\frac{3e^{-2s}}{(s+3)(s+1)}\} &= f(t-2)H(t-2) \\ &= \frac{3}{2}[e^{-(t-2)} - e^{-3(t-2)}]H(t-2) \end{aligned}$$

15(c) $\mathcal{L}^{-1}\{\dfrac{s+1}{s^2(s^2+1)}e^{-s}\} = \mathcal{L}^{-1}\{e^{-s}F(s)\}$ where

$$\begin{aligned} F(s) &= \frac{s+1}{s^2(s^2+1)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s+1}{s^2+1} \\ f(t) = \mathcal{L}^{-1}\{F(s)\} &= 1 + t - \cos t - \sin t \end{aligned}$$

so

$$\mathcal{L}^{-1}\{\frac{s+1}{s^2(s^2+1)}e^{-s}\} = f(t-1)H(t-1)$$
$$= [1+(t-1)-\cos(t-1)-\sin(t-1)]H(t-1)$$
$$= [t-\cos(t-1)-\sin(t-1)]H(t-1)$$

15(d) $\mathcal{L}^{-1}\{\frac{s+1}{s^2+s+1}e^{-\pi s}\} = \mathcal{L}^{-1}\{e^{-\pi s}F(s)\}$ where

$$F(s) = \frac{s+1}{(s^2+s+1)} = \frac{(s+\frac{1}{2})+\frac{1}{\sqrt{3}}(\frac{\sqrt{3}}{2})}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$
$$f(t) = e^{-\frac{1}{2}t}\{\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\}$$

so

$$\mathcal{L}^{-1}\{\frac{s+1}{s^2+s+1}e^{-\pi s}\} = \frac{1}{\sqrt{3}}e^{-\frac{1}{2}(t-\pi)}\big[\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi)+\sin\frac{\sqrt{3}}{2}(t-\pi)\big].H(t-\pi)$$

15(e) $\mathcal{L}^{-1}\{\frac{s}{s^2+25}e^{-4\pi s/5}\} = \mathcal{L}^{-1}\{e^{-4\pi s/5}F(s)\}$ where

$$F(s) = \frac{s}{s^2+25} \;\Rightarrow\; f(t) = \mathcal{L}^{-1}\{F(s)\} = \cos 5t$$

so

$$\mathcal{L}^{-1}\{\frac{s}{s^2+25}e^{-4\pi s/5}\} = f\big(t-\frac{4\pi}{5}\big)H\big(t-\frac{4\pi}{5}\big)$$
$$= \cos(5t-4\pi)H\big(t-\frac{4\pi}{5}\big)$$
$$= \cos 5t\; H\big(t-\frac{4\pi}{5}\big)$$

15(f) $\mathcal{L}^{-1}\{\frac{e^{-s}(1-e^{-s})}{s^2(s^2+1)}\} = \mathcal{L}^{-1}\{(e^{-s}-e^{-2s})F(s)\}$ where

$$F(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t - \sin t$$

so

$$\begin{aligned}\mathcal{L}^{-1}\{(e^{-s}-e^{-2s})F(s)\} =& f(t-1)H(t-1) - f(t-2)H(t-2)\\ =& [(t-1)-\sin(t-1)]H(t-1)\\ & -[(t-2)-\sin(t-2)]H(t-2)\end{aligned}$$

■ **16** $\dfrac{dx}{dt} + x = f(t),\ \mathcal{L}\{f(t)\} = \dfrac{1}{s^2}(1 - e^{-s} - se^{-s})$

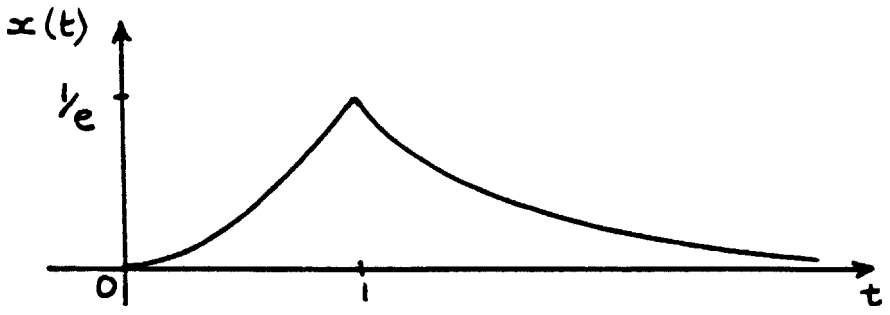
Taking Laplace transforms with $x(0) = 0$

$$\begin{aligned}(s+1)X(s) &= \frac{1}{s^2} - e^{-s}\frac{(1+s)}{s^2}\\ X(s) &= \frac{1}{s^2(s+1)} - e^{-s}\frac{1}{s^2}\\ &= -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} - e^{-s}\mathcal{L}\{t\}\end{aligned}$$

Taking inverse transforms

$$\begin{aligned}x(t) &= -1 + e^{-t} + t - (t-1)H(t-1)\\ &= e^{-t} + (t-1)[1 - H(t-1)]\\ \text{or } x(t) &= e^{-t} + (t-1) \text{ for } t \le 1\\ x(t) &= e^{-t} \text{ for } t \ge 1\end{aligned}$$

Sketch of response is



■ **17** $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x = g(t),\ x(0) = 1,\ \dot{x}(0) = 0$

with $\mathcal{L}\{g(t)\} = \dfrac{1}{s^2}(1 - 2e^{-s} + e^{-2s})$

Taking Laplace transforms

$$(s^2 + s + 1)X(s) = s + 1 + \frac{1}{s^2}(1 - 2e^{-s} + e^{-2s})$$

$$\begin{aligned} X(s) &= \frac{s+1}{(s^2+s+1)} + \frac{1}{s^2(s^2+s+1)}(1 - 2e^{-s} + e^{-2s}) \\ &= \frac{(s+1)}{(s^2+s+1)} + \big[-\frac{1}{s} + \frac{1}{s^2} + \frac{s}{s^2+s+1}\big][1 - 2e^{-s} + e^{-2s}] \\ &= \frac{(s+\frac{1}{2}) + \frac{1}{\sqrt{3}}(\frac{\sqrt{3}}{2})}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + \left[-\frac{1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{2}) - \frac{1}{\sqrt{3}}(\frac{\sqrt{3}}{2})}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right][1 - 2e^{-s} + e^{-2s}] \end{aligned}$$

$$\begin{aligned} x(t) = \mathcal{L}^{-1}\{X(s)\} = {} & e^{-\frac{1}{2}t}\big(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\big) \\ & + t - 1 + e^{-\frac{1}{2}t}\big(\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\big) \\ & - 2H(t-1)\Big[t - 2 + e^{-\frac{1}{2}(t-1)}\big\{\cos\frac{\sqrt{3}}{2}(t-1) \\ & - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-1)\big\}\Big] \\ & + H(t-2)\Big[t - 3 + e^{-\frac{1}{2}(t-2)}\big\{\cos\frac{\sqrt{3}}{2}(t-2) \\ & - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-2)\big\}\Big] \end{aligned}$$

that is,

$$\begin{aligned} x(t) = {} & 2e^{-\frac{1}{2}t}\cos\tfrac{\sqrt{3}}{2}t + t - 1 \\ & - 2H(t-1)\left[t - 2 + e^{-\frac{1}{2}(t-1)}\big\{\cos\frac{\sqrt{3}}{2}(t-1) - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-1)\big\}\right] \\ & + H(t-2)\left[t - 3 + e^{-\frac{1}{2}(t-2)}\big\{\cos\frac{\sqrt{3}}{2}(t-2) - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-2)\big\}\right] \end{aligned}$$

■ **18**

$$f(t) = \sin t H\big(t - \frac{\pi}{2}\big) = \cos\big(t - \frac{\pi}{2}\big) H\big(t - \frac{\pi}{2}\big)$$
$$\text{since } \cos\big(t - \frac{\pi}{2}\big) = \sin t.$$

Taking Laplace transforms with $x(0) = 1, \dot{x}(0) = -1$

$$\begin{aligned}
(s^2 + 3s + 2)X(s) &= s + 2 + \mathcal{L}\big\{\cos\big(t - \tfrac{\pi}{2}\big) H\big(t - \tfrac{\pi}{2}\big)\big\} \\
&= s + 2 + e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos t\} \\
&= s + 2 + e^{-\frac{\pi}{2}s} \cdot \frac{s}{s^2 + 1} \\
X(s) &= \frac{1}{s+1} + e^{-\frac{\pi}{2}s}\big[\frac{s}{(s+1)(s+2)(s^2+1)}\big] \\
&= \frac{1}{s+1} + e^{-\frac{\pi}{2}s}\big[\frac{-\frac{1}{2}}{s+1} + \frac{\frac{2}{5}}{s+2} + \frac{1}{10} \cdot \frac{s+3}{s^2+1}\big] \\
&= \frac{1}{s+1} + e^{-\frac{\pi}{2}s}\mathcal{L}\big\{-\frac{1}{2}e^{-t} + \frac{2}{5}e^{-2t} + \frac{1}{10}(\cos t + 3\sin t)\big\} \\
\text{so } x(t) = \mathcal{L}^{-1}\{X(s)\} &= e^{-t} + \Big[-\frac{1}{2}e^{-(t-\frac{\pi}{2})} + \frac{2}{5}e^{-2(t-\frac{\pi}{2})} + \frac{1}{10}(\cos\big(t - \frac{\pi}{2}\big) \\
&\qquad + 3\sin\big(t - \frac{\pi}{2}\big))\Big] H\big(t - \frac{\pi}{2}\big) \\
&= e^{-t} + \frac{1}{10}\big[\sin t - 3\cos t + 4e^{\pi}e^{-2t} - 5e^{\frac{\pi}{2}}e^{-t}\big] H\big(t - \frac{\pi}{2}\big)
\end{aligned}$$

■ **19**

$$\begin{aligned}
f(t) &= 3H(t) - (8 - 2t)H(t-4) \\
&= 3H(t) + 2(t-4)H(t-4) \\
\mathcal{L}\{f(t)\} &= \frac{3}{s} + 2e^{-4s}\mathcal{L}\{t\} = \frac{3}{s} + \frac{2}{s^2}e^{-4s}
\end{aligned}$$

Taking Laplace transforms with $x(0) = 1,\ \dot{x}(0) = 0$

$$\begin{aligned}
(s^2 + 1)X(s) &= s + \frac{3}{s} + \frac{2}{s^2}e^{-4s} \\
X(s) &= \frac{s}{s^2+1} + \frac{3}{s(s^2+1)} + \frac{2}{s^2(s^2+1)}e^{-4s} \\
&= \frac{s}{s^2+1} + \frac{3}{5} - \frac{3}{s^2+1} + 2\big[\frac{1}{s^2} - \frac{1}{s^2+1}\big]e^{-4s} \\
&= \frac{3}{5} - \frac{2}{s^2+1} + 2e^{-4s}\mathcal{L}\{t - \sin t\}
\end{aligned}$$

Thus, taking inverse transforms

$$x(t) = 3 - 2\cos t + 2(t - 4 - \sin(t-4))H(t-4)$$

■ **20**

$$\ddot{\theta}_0 + 6\dot{\theta}_0 + 10\theta_0 = \theta_i \tag{1}$$

$$\theta_i(t) = 3H(t) - 3H(t-a)$$

$$\text{so } \mathcal{L}\{\theta_i\} = \frac{3}{s} - \frac{3}{s}e^{-as} = \frac{3}{s}(1 - e^{-as})$$

Taking Laplace transforms in (1) with $\theta_0 = \dot{\theta}_0 = 0$ at $t = 0$

$$(s^2 + 6s + 10)\Phi_0(s) = \frac{3}{s}(1 - e^{-as})$$

$$\Phi_0(s) = 3(1 - e^{-as})\Big[\frac{1}{s(s^2 + 6s + 10)}\Big]$$

$$= \frac{3}{10}(1 - e^{-as})\Big[\frac{1}{s} - \frac{(s+3)+3}{(s+3)^2 + 1}\Big]$$

$$= \frac{3}{10}(1 - e^{-as})\mathcal{L}\big[1 - e^{-3t}\cos t - 3e^{-3t}\sin t\big]$$

Thus, taking inverse transforms

$$\theta_0(t) = \frac{3}{10}[1 - e^{-3t}\cos t - 3e^{-3t}\sin t]H(t)$$
$$- \frac{3}{10}[1 - e^{-3(t-a)}\cos(t-a) - 3e^{-3(t-a)}\sin(t-a)]H(t-a)$$

If $T > a$ then $H(T) = 1,\ H(T-a) = 1$ giving

$$\theta_0(T) = -\frac{3}{10}[e^{-3T}\cos T - e^{-3(T-a)}\cos(T-a)]$$
$$- \frac{3}{10}[3e^{-3T}\sin T - 3e^{-3(T-a)}\sin(T-a)]$$
$$= -\frac{3}{10}e^{-3T}\{\cos T + 3\sin T - e^{3a}[\cos(T-a) + 3\sin(T-a)]\}$$

■ **21**

$$\theta_i(t) = f(t) = (1-t)H(t) - (1-t)H(t-1)$$
$$= (1-t)H(t) + (t-1)H(t-1)$$

so

$$\mathcal{L}\{\theta_i(t)\} = \frac{1}{s} - \frac{1}{s^2} + e^{-s}\mathcal{L}\{t\}$$
$$= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-s} = \frac{s-1}{s^2} + \frac{1}{s^2}e^{-s}$$

Then taking Laplace transforms, using $\theta_0(0) = \dot{\theta}_0(0) = 0$

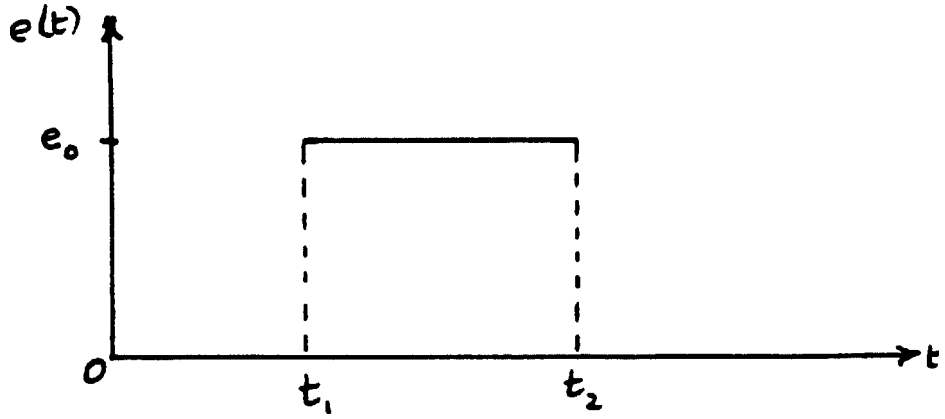
$$(s^2 + 8s + 16)\Phi_0(s) = \frac{s-1}{s^2} + \frac{1}{s^2}e^{-s}$$

$$\Phi_0(s) = \frac{s-1}{s^2(s+4)^2} + e^{-s}\left[\frac{1}{s^2(s+4)^2}\right]$$
$$= \frac{1}{s^2}\left[\frac{3}{s} - \frac{2}{s^2} - \frac{3}{s+4} - \frac{10}{(s+4)^2}\right] + \frac{e^{-s}}{32}\left[\frac{3}{s} + \frac{2}{s^2} + \frac{1}{s+4} + \frac{2}{(s+4)^2}\right]$$

which on taking inverse transforms gives

$$\theta_0(t) = \mathcal{L}^{-1}\{\Phi_0(s)\} = \frac{1}{32}[3 - 2t - 3e^{-4t} - 10te^{-4t}]$$
$$+ \frac{1}{32}[-1 + 2(t-1) + e^{-4(t-1)} + 2(t-1)e^{-4(t-1)}]H(t-1)$$
$$= \frac{1}{32}[3 - 2t - 3e^{-4t} - 10te^{-4t}]$$
$$+ \frac{1}{32}[2t - 3 + (2t-1)e^{-4(t-1)}]H(t-1)$$

■ **22**



$$e(t) = e_0H(t-t_1) - e_0H(t-t_2)$$
$$\mathcal{L}\{e(t)\} = \frac{e_0}{s}(e^{-st_1} - e^{-st_2})$$

By Kirchhoff's second law current in the circuit is given by

$$Ri + \frac{1}{C}\int idt = e$$

which on taking Laplace transforms

$$RI(s) + \frac{1}{Cs}I(s) = \frac{e_0}{s}(e^{-st_1} - e^{-st_2})$$

$$I(s) = \frac{e_0 C}{RCs+1}(e^{-st_1} - e^{-st_2})$$

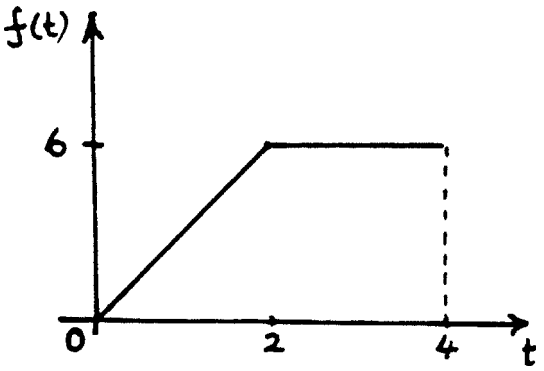
$$= \frac{e_0/R}{s+\frac{1}{RC}}(e^{-st_1} - e^{-st_2})$$

$$= \frac{e_0/R}{s+\frac{1}{RC}}e^{-st_1} - \frac{e_0/R}{s+\frac{1}{RC}}e^{-st_2}$$

then

$$\begin{aligned} i(t) &= \mathcal{L}^{-1}\{I(s)\} \\ &= \frac{e_0}{R}\left[e^{-(t-t_1)/RC}H(t-t_1) - e^{-(t-t_2)/RC}H(t-t_2)\right] \end{aligned}$$

■ **23**



Sketch over one period as shown and readily extended to $0 \le t < 12$.

$$\begin{aligned} f_1(t) &= 3tH(t) - (3t-6)H(t-2) - 6H(t-4) \\ &= 3tH(t) - 3(t-2)H(t-2) - 6H(t-4) \end{aligned}$$

$$\mathcal{L}\{f_1(t)\} = F_1(s) = \frac{3}{s^2} - \frac{3}{s^2}e^{-2s} - \frac{6}{s}e^{-4s}$$

Then by theorem 5.5

$$\begin{aligned} \mathcal{L}\{f(t)\} = F(s) &= \frac{1}{1-e^{-4s}}F_1(s) \\ &= \frac{1}{s^2(1-e^{-4s})}(3 - 3e^{-2s} - 6se^{-4s}) \end{aligned}$$

■ **24** Take

$$\begin{aligned} f_1(t) &= \frac{K}{T}t, \quad 0 < t < T \\ &= 0, \qquad t > T \end{aligned}$$

then $f_1(t) = \dfrac{K}{T}tH(t) - \dfrac{Kt}{T}H(t-T) = \dfrac{K}{T}tH(t) - \dfrac{K}{T}(t-T)H(t-T) - KH(t-T)$

$$\mathcal{L}\{f_1(t)\} = F_1(s) = \frac{K}{Ts^2} - e^{-sT}\frac{K}{Ts^2} - e^{-sT}\frac{K}{s} = \frac{K}{Ts^2}(1-e^{-sT}) - \frac{K}{s}e^{-sT}$$

Then by theorem 5.5

$$\mathcal{L}\{f(t)\} = F(s) = \frac{1}{1-e^{-sT}}F_1(s) = \frac{K}{Ts^2} - \frac{K}{s}\frac{e^{-sT}}{1-e^{-sT}}$$

Exercises 5.5.12

■ **25(a)**

$$\frac{2s^2+1}{(s+2)(s+3)} = 2 - \frac{10s+11}{(s+2)(s+3)} = 2 + \frac{9}{s+2} - \frac{19}{s+3}$$

$$\mathcal{L}^{-1}\Big\{\frac{2s^2+1}{(s+2)(s+3)}\Big\} = 2\delta(t) + 9e^{-2t} - 19e^{-3t}$$

25(b)

$$\frac{s^2-1}{s^2+4} = 1 - \frac{5}{s^2+4}$$
$$\mathcal{L}^{-1}\{\frac{s^2-1}{s^2+4}\} = \delta(t) - \frac{5}{2}\sin 2t$$

25(c)

$$\frac{s^2+2}{s^2+2s+5} = 1 - \frac{2s+3}{s^2+2s+5}$$
$$= 1 - \big[\frac{2(s+1)+\frac{1}{2}(2)}{(s+1)^2+s^2}\big]$$
$$\mathcal{L}^{-1}\{\frac{s^2+2}{s^2+2s+5}\} = \delta(t) - e^{-t}\big(2\cos 2t + \frac{1}{2}\sin 2t\big)$$

■ **26(a)** $(s^2+7s+12)X(s) = \dfrac{2}{s} + e^{-2s}$

$$X(s) = \frac{2}{s(s+4)(s+3)} + \big[\frac{1}{(s+4)(s+3)}\big]e^{-2s}$$
$$= \frac{\frac{1}{6}}{s} - \frac{\frac{2}{3}}{s+3} + \frac{\frac{1}{2}}{s+4} + \big[\frac{1}{s+3} - \frac{1}{s+4}\big]e^{-2s}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \big(\frac{1}{6} - \frac{2}{3}e^{-3t} + \frac{1}{2}e^{-4t}\big) + \big(e^{-3(t-2)} - e^{-4(t-2)}\big)H(t-2)$$

26(b)

$$(s^2+6s+13)X(s) = e^{-2\pi s}$$
$$X(s) = \frac{1}{(s+3)^2+2^2}e^{-2\pi s}$$
$$= e^{-2\pi s}\mathcal{L}\{\frac{1}{2}e^{-3t}\sin 2t\}$$
$$\text{so } x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2}e^{-3(t-2\pi)}\sin 2(t-2\pi).H(t-2\pi)$$
$$= \frac{1}{2}e^{6\pi}e^{-3t}\sin 2t.H(t-2\pi)$$

26(c)

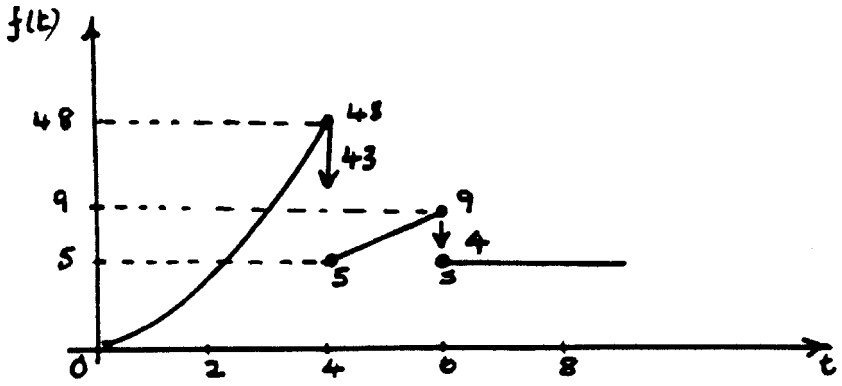
$$(s^2+7s+12)X(s) = s+8+e^{-3s}$$

$$X(s) = \frac{s+8}{(s+4)(s+3)} + \left[\frac{1}{(s+4)(s+3)}\right]e^{-3s}$$

$$= \left[\frac{5}{s+3} - \frac{4}{s+4}\right] + \left[\frac{1}{s+3} - \frac{1}{s+4}\right]e^{-3s}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 5e^{-3t} - 4e^{-4t} + [e^{-3(t-3)} - e^{-4(t-3)}]H(t-3)$$

- **27(a)**



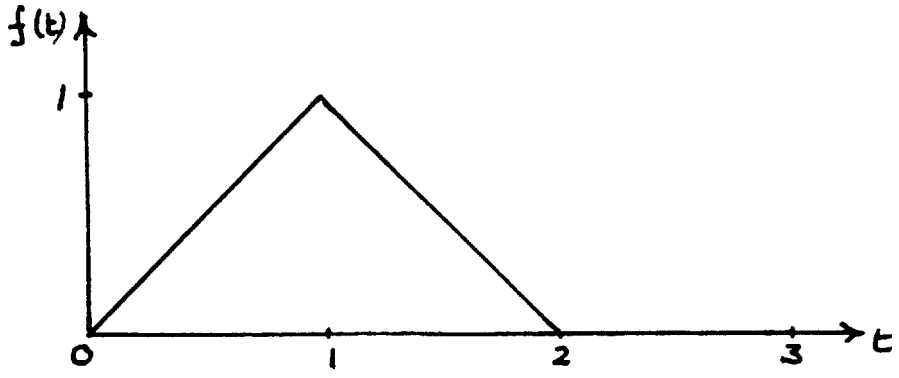
Generalized derivative is

$$f'(t) = g'(t) - 43\delta(t-4) - 4\delta(t-6)$$

where

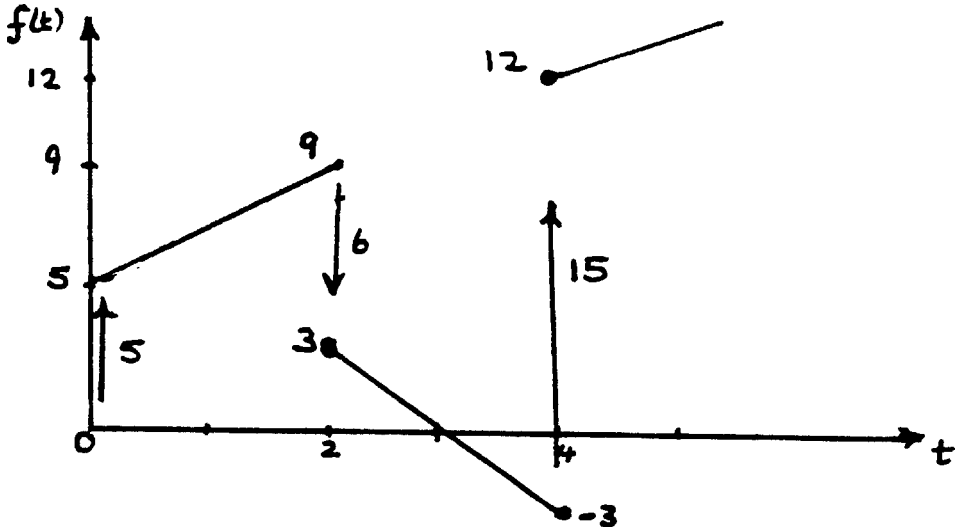
$$g'(t) = \begin{cases} 6t, & 0 \le t < 4 \\ 2, & 4 \le t < 6 \\ 0, & t \ge 6 \end{cases}$$

27(b)



$$f'(t) = g'(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$$

27(c)



$$f'(t) = g'(t) + 5\delta(t) - 6\delta(t-2) + 15\delta(t-4)$$

where

$$g'(t) = \begin{cases} 2, & 0 \le t < 2 \\ -3, & 2 \le t < 4 \\ 2t-1, & t \ge 4 \end{cases}$$

■ **28**

$$(s^2+7s+10)X(s) = 2+(3s+2)U(s)$$

$$= 2+(3s+2)\frac{1}{s+2} = \frac{5s+6}{s+2}$$

$$X(s) = \frac{5s+6}{(s+2)^2(s+5)}$$

$$= \frac{\frac{19}{9}}{s+2} - \frac{\frac{4}{3}}{(s+2)^2} - \frac{\frac{19}{9}}{(s+5)}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{19}{9}e^{-2t} - \frac{4}{3}te^{-2t} - \frac{19}{9}e^{-5t}$$

■ **29** $f(t) = \sum_{n=0}^{\infty} \delta(t-nT)$

Thus,

$$F(s) = \mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} \mathcal{L}\{\delta(t-nT)\} = \sum_{n=0}^{\infty} e^{-snT}$$

This is an infinite GP with first term 1 and common ratio e^{-sT} and therefore having sum $(1-e^{-sT})^{-1}$. Hence,

$$F(s) = \frac{1}{1-e^{-sT}}$$

Assuming zero initial conditions and taking Laplace transforms the response of the harmonic oscillator is given by

$$(s^2+w^2)X(s) = F(s) = \frac{1}{1-e^{-sT}}$$

$$X(s) = \Big(\sum_{n=0}^{\infty} e^{-snT}\Big)\Big(\frac{1}{s^2+w^2}\Big)$$

$$= [1+e^{-sT}+e^{-2sT}+\ldots]\mathcal{L}\Big\{\frac{1}{w}\sin wt\Big\}$$

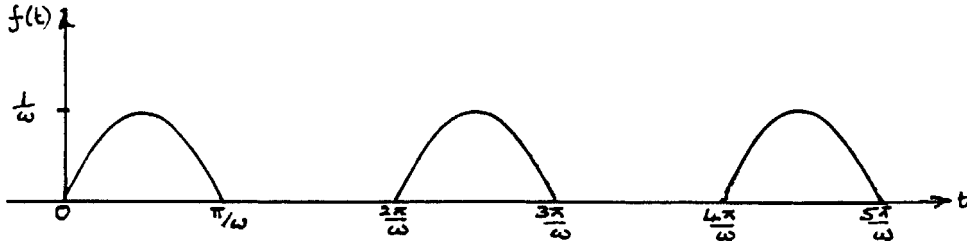
giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{w}[\sin wt + H(t-T).\sin w(t-T) + H(t-2T).\sin w(t-2T)+\ldots]$

or $x(t) = \frac{1}{w}\sum_{n=0}^{\infty} H(t-nT)\sin w(t-nT)\,.$

29(a)

$$T = \frac{\pi}{w}\ ; \quad x(t) = \frac{1}{w}\sum_{n=0}^{\infty} H\big(t - \frac{n\pi}{w}\big)\sin(wt - n\pi)$$
$$= \frac{1}{w}\left[\sin wt - \sin wt.\, H\big(t - \frac{\pi}{w}\big) + \sin wt.\, H\big(t - \frac{2\pi}{w}\big) + \ldots\right]$$

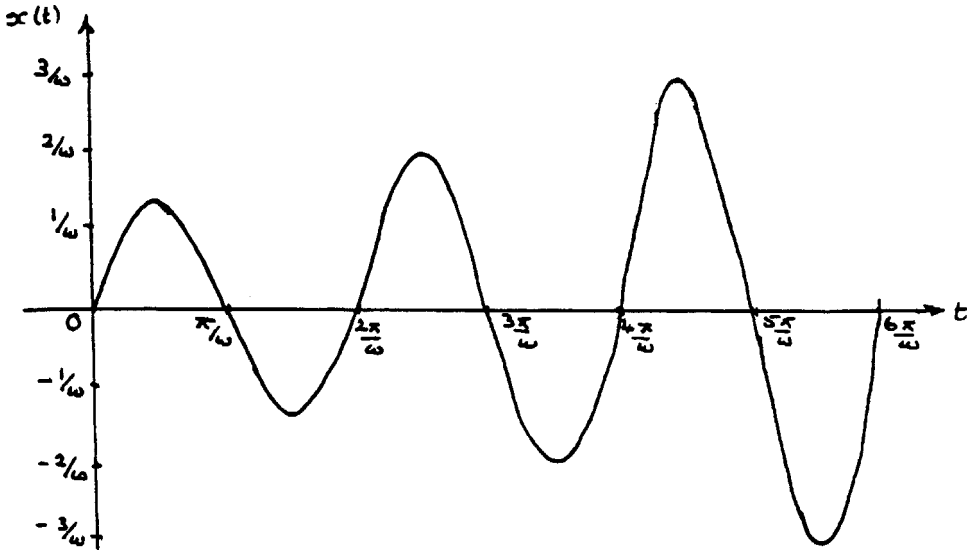
and a sketch of the response is as follows



29(b)

$$T = \frac{2\pi}{w}\ ; \quad x(t) = \frac{1}{w}\sum_{n=0}^{\infty} H\big(t - \frac{2\pi n}{w}\big)\sin(wt - 2\pi n)$$
$$= \frac{1}{w}\left[\sin wt + \sin wt.H\big(t - \frac{2\pi}{w}\big) + \sin wt.H\big(t - \frac{4\pi}{w}\big) + \ldots\right]$$

and the sketch of the response is as follows



■ **30** The charge q on the LCR circuit is determined by

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t)$$

where $e(t) = E\delta(t), q(0) = \dot{q}(0) = 0$.
Taking Laplace transforms

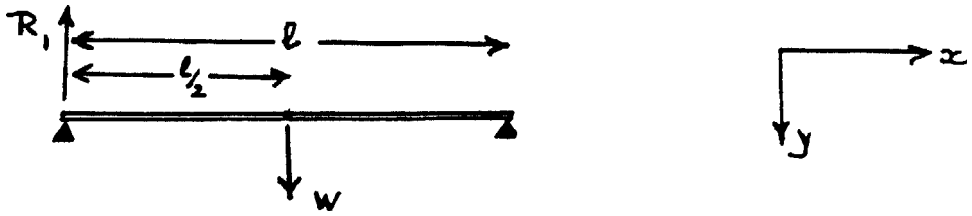
$$\big(Ls^2 + Rs + \frac{1}{C}\big)Q(s) = \mathcal{L}\{E\delta(t)\} = E$$

$$Q(s) = \frac{E/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{E/L}{(s + \frac{R}{2L})^2 + (\frac{1}{LC} - \frac{R^2}{4L^2})}$$
$$= \frac{E/L}{(s+\mu)^2 + \eta^2}, \mu = \frac{R}{2L}, \eta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Thus, $q(t) = \dfrac{E}{L\eta}e^{-\mu t}\sin\eta t$
and current $i(t) = \dot{q}(t) = \dfrac{E}{L\eta}e^{-\mu t}(\eta\cos\eta t - \mu\sin\eta t)$

Exercises 5.5.14

■ **31**



Load $W(x) = \dfrac{M}{\ell}H(x) + W\delta\big(x - \dfrac{\ell}{2}\big) - R_1\delta(x)$, where $R_1 = \dfrac{1}{2}(M+W)$
so the force function is

$$W(x) = \frac{M}{\ell}H(x) + W\delta\big(x - \frac{\ell}{2}\big) - \big(\frac{M+W}{2}\big)\delta(x)$$

having Laplace transform

$$W(s) = \frac{M}{\ell s} + We^{-\ell s/2} - \frac{(M+W)}{2}$$

Since the beam is freely supported at both ends

$$y(0) = y_2(0) = y(\ell) = y_2(\ell) = 0$$

and the transformed equation (2.64) of the text becomes

$$Y(s) = \frac{1}{EI}\left[\frac{M}{\ell s^5} + \frac{W}{s^4}e^{-\ell s/2} - \big(\frac{M+W}{2}\big)\frac{1}{s^4}\right] + \frac{y_1(0)}{s^2} + \frac{y_3(0)}{s^4}$$

Taking inverse transforms gives

$$y(x) = \frac{1}{EI}\left[\frac{1}{24}\frac{M}{\ell}x^4 + \frac{1}{6}W(x-\frac{\ell}{2})^3 \cdot H\big(x-\frac{\ell}{2}\big) - \frac{1}{12}(M+W)x^3\right]$$
$$+ y_1(0)x + \frac{1}{6}y_3(0)x^3$$

for $x > \frac{\ell}{2}$

$$y(x) = \frac{1}{EI}\left[\frac{1}{24}\frac{M}{\ell}x^4 + \frac{1}{6}W\big(x-\frac{\ell}{2}\big)^3 - \frac{1}{12}(M+W)x^3\right] + y_1(0)x + \frac{1}{6}y_3(0)x^3$$

$$y_2(x) = \frac{1}{EI}\left[\frac{1}{2}\frac{M}{\ell}x^2 + W\big(x-\frac{\ell}{2}\big) - \frac{1}{2}(M+W)x\right] + y_3(0)x$$

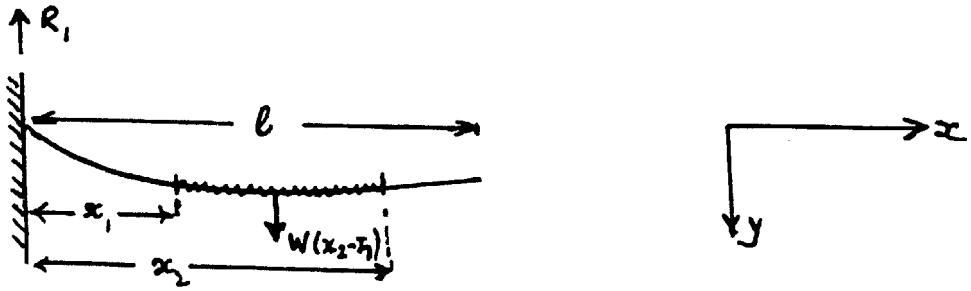
$y_2(\ell) = 0$ then gives $y_3(0) = 0$ and $y(\ell) = 0$ gives

$$0 = \frac{1}{EI}\left[\frac{M\ell^3}{24} + \frac{W\ell^3}{24} - \frac{1}{12}M\ell^3 - \frac{1}{2}W\ell^3\right] + y_1(0)\ell$$

$$y_1(0) = \frac{1}{EI}\left[\frac{1}{24}M\ell^2 + \frac{1}{16}W\ell^2\right]$$

$$\text{so } y(x) = \frac{1}{48EI}\left[\frac{2}{\ell}Mx^4 + 8W(x-\frac{\ell}{2})^3 H\big(x-\frac{\ell}{2}\big) - 4(M+W)x^3 + (2M+3W)\ell^2 x\right]$$

■ **32**



Load $W(x) = w(H(x-x_1) - H(x-x_2)) - R_1\delta(x),\ R_1 = w(x_2 - x_1)$
so the force function is

$$W(x) = w(H(x-x_1) - H(x-x_2)) - w(x_2-x_1)\delta(x)$$

having Laplace transform

$$W(s) = w\big(\frac{1}{s}e^{-x_1 s} - \frac{1}{s}e^{-x_2 s}\big) - w(x_2 - x_1)$$

with corresponding boundary conditions

$$y(0) = y_1(0) = 0,\ y_2(\ell) = y_3(\ell) = 0$$

The transformed equation (2.64) of the text becomes

$$Y(s) = \frac{w}{EI}\left[\frac{1}{s^5}e^{-x_1 s} - \frac{1}{s^5}e^{-x_2 s} - \frac{(x_2 - x_1)}{s^4}\right] + \frac{y_2(0)}{s^3} + \frac{y_3(0)}{s^4}$$

which on taking inverse transforms gives

$$y(x) = \frac{w}{EI}\big[\frac{1}{24}(x-x_1)^4 H(x-x_1) - \frac{1}{24}(x-x_2)^4 H(x-x_2) - \frac{1}{6}(x_2-x_1)x^3\big] + y_2(0)\frac{x^2}{2} + y_3(0)\frac{x^3}{6}$$

For $x > x_2$

$$y(x) = \frac{w}{EI}\big[\frac{1}{24}(x-x_1)^4 - \frac{1}{24}(x-x_2)^4 - \frac{1}{6}(x_2-x_1)x^3\big] + y_2(0)\frac{x^2}{2} + y_3(0)\frac{x^3}{6}$$

$$y_2(x) = \frac{w}{EI}\big[\frac{1}{24}(x-x_1)^2 - \frac{1}{2}(x-x_2)^2 - (x_2-x_1)x\big] + y_2(0) + y_3(0)x$$

$$y_3(x) = \frac{w}{EI}\big[(x-x_1) - (x-x_2) - (x_2-x_1)\big] + y_3(0) \ \Rightarrow\ y_3(0) = 0$$

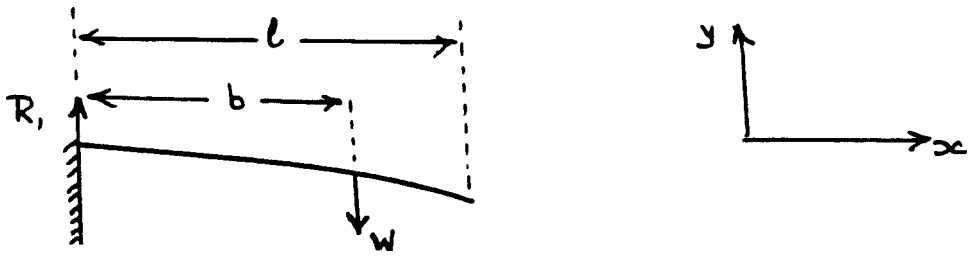
The boundary condition $y_2(\ell) = 0$ then gives

$$0 = \frac{w}{EI}\big[\frac{1}{2}(\ell^2 - 2\ell x_1 + x_1^2) - \frac{1}{2}(\ell^2 - 2\ell x_2 + x_2^2) - x_2\ell + x_1\ell\big] + y_2(0)$$
$$\Rightarrow\ y_2(0) = \frac{w}{2EI}(x_2^2 - x_1^2)$$
$$y(x) = \frac{w}{24EI}\big[(x-x_1)^4 H(x-x_1) - (x-x_2)^4 H(x-x_2) - 4(x_2-x_1)x^3 + 6(x_2^2 - x_1^2)x^2\big]$$

When $x_1 = 0, x_2 = \ell$, max deflection at $x = \ell$

$$y_{\max} = \frac{w}{24EI}\{\ell^4 - 4\ell^4 + 6\ell^4\} = \frac{w\ell^4}{8EI}$$

■ **33**



Load $W(x) = W\delta(x-b) - R_1\delta(x),\ R_1 = W$ so the force function is

$$W(x) = W\delta(x-b) - W\delta(x)$$

having Laplace transform

$$W(s) = We^{-bs} - W$$

with corresponding boundary conditions

$$y(0) = y_1(0) = 0,\ y_2(\ell) = y_3(\ell) = 0$$

The transformed equation (2.64) of the text becomes

$$Y(s) = -\frac{1}{EI}\big[\frac{W}{s^4}e^{-bs} - \frac{W}{s^4}\big] + \frac{y_2(0)}{s^3} + \frac{y_3(0)}{s^4}$$

which on taking inverse transforms gives

$$y(x) = -\frac{W}{EI}\big[\frac{1}{6}(x-b)^3 H(x-b) - \frac{1}{6}x^3\big] + y_2(0)\frac{x^2}{2} + y_3(0)\frac{x^3}{6}$$

For $x > b$

$$y(x) = -\frac{W}{EI}\big[\frac{1}{6}(x-b)^3 - \frac{1}{6}x^3\big] + y_2(0)\frac{x^2}{2} + y_3(0)\frac{x^3}{6}$$

$$y_2(x) = -\frac{W}{EI}\big[(x-b) - x\big] + y_2(0) + y_3(0)x$$

$$y_3(x) = -\frac{W}{EI}\big[1-1\big] + y_3(0) \;\Rightarrow\; y_3(0) = 0$$

Using the boundary condition $y_2(\ell) = 0$

$$0 = -\frac{W}{EI}(-h) + y_2(0) \;\Rightarrow\; y_2(0) = -\frac{Wb}{EI}$$

giving

$$y(x) = \frac{W}{EI}\big[\frac{x^3}{6} - \frac{(x-b)^3}{6}H(x-b) - \frac{bx^2}{2}\big]$$
$$= \begin{cases} -\dfrac{Wx^2}{6EI}(3b-x), & 0 < x \le b \\ -\dfrac{Wb^2}{6EI}(3x-b), & b < x \le \ell \end{cases}$$

Exercises 5.6.5

- **34(a)** Assuming all the initial conditions are zero taking Laplace transforms gives

$$(s^2 + 2s + 5)X(s) = (3s+2)U(s)$$

so that the system transfer function is given by

$$G(s) = \frac{X(s)}{U(s)} = \frac{3s+2}{s^2+2s+5}$$

34(b) The characteristic equation of the system is

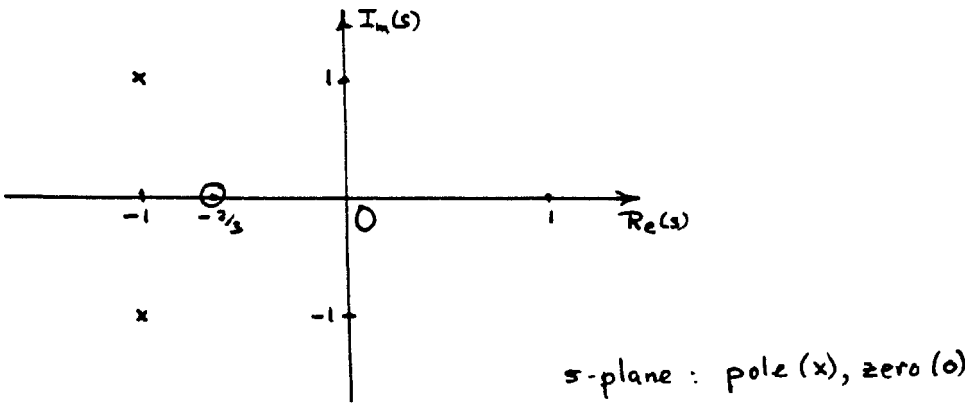
$$s^2 + 2s + 5 = 0$$

and the system is of order 2.

34(c) The transfer function poles are the roots of the characteristic equation

$$s^2 + 2s + 5 = 0$$

which are $s = -1 \pm j$. That is, the transfer function has single poles at $s = -1 + j$ and $s = -1 - j$.
The transfer function zeros are determined by equating the numerator polynomial to zero; that is, a single zero at $s = -\dfrac{2}{3}$.



■ **35** Following the same procedure as for Exercise 34

35(a) The transfer function characterizing the system is

$$G(s) = \frac{s^3 + 5s + 6}{s^3 + 5s^2 + 17s + 13}$$

35(b) The characteristic equation of the system is

$$s^3 + 5s^2 + 17s + 13 = 0$$

and the system is of order 3.

35(c) The transfer function poles are given by

$$s^3 + 5s^2 + 17s + 13 = 0$$
$$\text{that is, } (s+1)(s^2 + 4s + 13) = 0$$

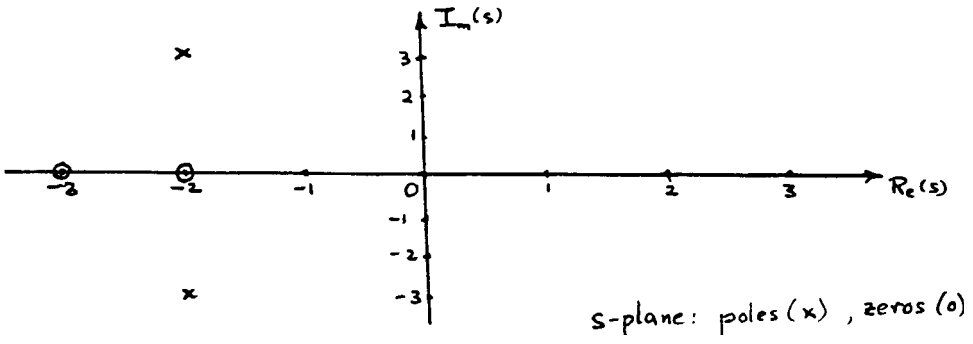
That is, the transfer function has simple poles at

$$s = -1,\ s = -2 + j3,\ s = -2 - j3$$

The transfer function zeros are given by

$$s^2 + 5s + 6 = 0$$
$$(s+3)(s+2) = 0$$

that is, zeros at $s = -3$ and $s = -2$.



■ **36(a)** Poles at $(s+2)(s^2+4)=0$; that is, $s=-2,\ s=+2j,\ s=-j$.
Since we have poles on the imaginary axis in the s-plane, system is marginally stable.

36(b) Poles at $(s+1)(s-1)(s+4)=0$; that is, $s=-1,\ s=1,\ s=-4$.
Since we have the pole $s=1$ in the right hand half of the s-plane, the system is unstable.

36(c) Poles at $(s+2)(s+4)=0$; that is, $s=-2,\ s=-4$.
Both the poles are in the left hand half of the plane so the system is stable.

36(d) Poles at $(s^2+s+1)(s+1)^2=0$; that is, $s=-1$ (repeated), $s=-\dfrac{1}{2}\pm j\dfrac{\sqrt{3}}{2}$.
Since all the poles are in the left hand half of the s-plane the system is stable.

36(e) Poles at $(s+5)(s^2-s+10)=0$; that is, $s=-5,\ s=\dfrac{1}{2}\pm j\dfrac{\sqrt{39}}{2}$.
Since both the complex poles are in the right hand half of the s-plane the system is unstable.

■ **37(a)** $s^2-4s+13=0\ \Rightarrow\ s=2\pm j3$.
Thus, the poles are in the right hand half s-plane and the system is unstable.

37(b)

$$\begin{array}{cccc} 5s^3 & +13s^2 & +31s & +15=0 \\ a_3 & a_2 & a_1 & a_0 \end{array}$$

Routh–Hurwitz (R-H) determinants are:

$$\Delta_1=13>0,\ \Delta_2=\begin{vmatrix}13 & 5\\ 15 & 31\end{vmatrix}>0,\ \Delta_3=15\Delta_2>0$$

so the system is stable.

37(c) $s^3 + s^2 + s + 1 = 0$

R–H determinants are

$$\Delta_1 = 1 > 0,\ \Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0,\ \Delta_3 = 1\Delta_2 = 0$$

Thus, system is marginally stable. This is readily confirmed since the poles are at $s = -1, s = \pm j$

37(d) $24s^4 + 11s^3 + 26s^2 + 45s + 36 = 0$

R–H determinants are

$$\Delta_1 = 11 > 0,\ \Delta_2 = \begin{vmatrix} 11 & 24 \\ 45 & 26 \end{vmatrix} < 0$$

so the system is unstable.

37(e) $s^3 + 2s^2 + 2s + 1 = 0$

R–H determinants are

$$\Delta_1 = 2 > 0,\ \Delta_2 = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1 > 0, \Delta_3 = 1\Delta_2 > 0$$

and the system is stable. The poles are at $s = -1,\ s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ confirming the result.

■ **38** $m\dfrac{d^3x}{dt^3} + c\dfrac{d^2x}{dt^2} + K\dfrac{dx}{dt} + Krx = 0;\ m, K, r, c > 0$

R–H determinants are

$$\Delta_1 = c > 0$$

$$\Delta_2 = \begin{vmatrix} c & m \\ Kr & K \end{vmatrix} = cK - mKr > 0 \text{ provided } r < \frac{c}{m}$$

$$\Delta_3 = Kr\Delta_2 > 0 \text{ provided } \Delta_2 > 0$$

Thus, system stable provided $r < \dfrac{c}{m}$

■ **39**

$$\begin{array}{cccccc} s^4 + & 2s^2 + & (K+2)s^2 + & 7s + & K = 0 \\ a_3 & & a_2 & a_1 & a_0 \end{array}$$

R–H determinants are

$$\Delta_1 = |\, a_3 \,| = 9 > 0$$

$$\Delta_2 = \begin{vmatrix} a_3 & a_4 \\ a_1 & a_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 7 & K+2 \end{vmatrix} = 2K - 3 > 0 \text{ provided } K > \frac{3}{2}$$

$$\Delta_3 = \begin{vmatrix} a_3 & a_4 & 0 \\ a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 7 & K+2 & 2 \\ 0 & K & 7 \end{vmatrix} = 10K - 21 > 0 \text{ provided } K > 2$$

$$\Delta_4 = K\Delta_3 > 0 \text{ provided } \Delta_3 > 0$$

Thus, the system is stable provided $K > 2.1$.

■ **40** $s^2 + 15Ks^2 + (2K-1)s + 5K = 0,\ K > 0$

R–H determinants are

$$\Delta_1 = 15K > 0$$

$$\Delta_2 = \begin{vmatrix} 15K & 1 \\ 5K & (2K-1) \end{vmatrix} = 30K^2 - 20K$$

$$\Delta_3 = 5K\Delta_2 > 0 \text{ provided } \Delta_2 > 0$$

Thus, system stable provided $K(3K-2) > 0$ that is $K > \frac{2}{3}$, since $K > 0$.

■ **41(a)** Impulse response $h(t)$ is given by the solution of

$$\frac{d^2h}{dt^2} + 15\frac{dh}{dt} + 56h = 3\delta(t)$$

with zero initial conditions. Taking Laplace transforms

$$(s^2 + 15s + 56)H(s) = 3$$

$$H(s) = \frac{3}{(s+7)(s+8)} = \frac{3}{s+7} - \frac{3}{s+8}$$

$$\text{so } h(t) = \mathcal{L}^{-1}\{H(s)\} = 3e^{-7t} - 3e^{-8t}$$

Since $h(t) \to 0$ as $t \to \infty$ the system is stable.

41(b) Following (a) impulse response is given by

$$(s^2+8s+25)H(s)=1$$
$$H(s)=\frac{1}{(s+4)^2+3^2}$$
$$\text{so } h(t)=\mathcal{L}^{-1}\{H(s)\}=\frac{1}{3}e^{-4t}\sin 3t$$

Since $h(t)\to 0$ as $t\to\infty$ the system is stable.

41(c) Following (a) impulse response is given by

$$(s^2-2s-8)H(s)=4$$
$$H(s)=\frac{4}{(s-4)(s+2)}=\frac{2}{3}\frac{1}{s-4}-\frac{2}{3}\frac{1}{s+2}$$
$$\text{so } h(t)=\mathcal{L}^{-1}\{H(s)\}=\frac{2}{3}(e^{4t}-e^{-2t})$$

Since $h(t)\to\infty$ as $t\to\infty$ system is unstable.

41(d) Following (a) impulse response is given by

$$(s^2-4s+13)H(s)=1$$
$$H(s)=\frac{1}{s^2-4s+13}=\frac{1}{(s-2)^2+3^2}$$
$$\text{so } h(t)=\mathcal{L}^{-1}\{H(s)\}=\frac{1}{3}e^{2t}\sin 3t$$

Since $h(t)\to\infty$ as $t\to\infty$ system is unstable.

■ **42** Impulse response $h(t)=\dfrac{dx}{dt}=\dfrac{7}{3}e^{-t}-3e^{-2t}+\dfrac{2}{3}e^{-4t}$
System transfer function $G(s)=\mathcal{L}\{h(t)\}$; that is,

$$G(s)=\frac{7}{3(s+1)}-\frac{3}{s+2}+\frac{2}{3(s+4)}$$
$$=\frac{s+8}{(s+1)(s+2)(s+4)}$$

Note: The original unit step response can be reconstructed by evaluating $\mathcal{L}^{-1}\{G(s)\frac{1}{s}\}$.

■ **43(a)** $f(t) = 2 - 3\cos t\ ,\ F(s) = \dfrac{2}{s} - \dfrac{3s}{s^2+1}$

$$sF(s) = 2 - \frac{3s^2}{s^2+1} = 2 - \frac{3}{1+\frac{1}{s^2}}$$

Thus, $\lim\limits_{t\to 0+}(2 - 3\cos t) = 2 - 3 = -1$

and $\lim\limits_{s\to\infty} sF(s) = 2 - \dfrac{3}{1} = -1$ so confirming the i.v. theorem.

43(b)

$$f(t) = (3t-1)^2 = 9t^2 - 6t + 1,\ \lim_{t\to 0+} f(t) = 1$$

$$F(s) = \frac{18}{s^3} - \frac{6}{s^2} + \frac{1}{s} \text{ so } \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{18}{s^2} - \frac{6}{s} + 1\right] = 1$$

thus, confirming the i.v. theorem.

43(c)

$$f(t) = t + 3\sin 2t\ ,\ \lim_{t\to 0+} = 0$$

$$F(s) = \frac{1}{s^2} - \frac{6}{s^2+4} \text{ so } \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{1}{s} + \frac{6}{s+\frac{4}{s}}\right] = 0$$

thus, confirming the i.v. theorem.

■ **44(a)**

$$f(t) = 1 + 3e^{-t}\sin 2t\ ,\ \lim_{t\to\infty} f(t) = 1$$

$$F(s) = \frac{1}{s} + \frac{6}{(s+1)^2+4} \text{ and } \lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1 + \frac{6s}{(s+1)^2+4}\right] = 1$$

thus confirming the f.v. theorem. Note that, $sF(s)$ has its poles in the left half of the s-plane so the theorem is applicable.

44(b)

$$f(t) = t^2 3e^{-2t} \ , \ \lim_{t\to\infty} f(t) = 0$$

$$F(s) = \frac{2}{(s+2)^3} \text{ and } \lim_{s\to 0} sF(s) = \lim_{s\to 0} \big[\frac{2s}{(s+2)^3}\big] = 0$$

thus confirming the f.v. theorem. Again note that $sF(s)$ has its poles in the left half of the s-plane.

44(c)

$$f(t) = 3 - 2e^{-3t} + e^{-t}\cos 2t \ , \ \lim_{t\to\infty} f(t) = 3$$

$$F(s) = \frac{3}{s} - \frac{2}{s+3} + \frac{(s+1)}{(s+1)^2+4}$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \big[3 - \frac{2s}{s+3} + \frac{s(s+1)}{(s+1)^2+4}\big] = 3$$

confirming the f.v. theorem. Again $sF(s)$ has its poles in the left half of the s-plane.

■ **45** For the circuit of Example 5.28

$$I_2(s) = \frac{3.64}{s} + \frac{1.22}{s+59.1} - \frac{4.86}{s+14.9}$$

Then by the f.v. theorem

$$\lim_{t\to\infty} i_2(t) = \lim_{s\to 0} sI_2(s) = \lim_{s\to 0} \big[3.64 + \frac{1.22s}{s+59.1} - \frac{4.86s}{s+14.9}\big]$$
$$= 3.64$$

which confirms the answer obtained in Example 5.28. Note that, $sI_2(s)$ has all its poles in the left half of the s-plane.

■ **46** For the circuit of Example 5.29

$$sI_2(s) = \frac{28s^2}{(3s+10)(s+1)(s^2+4)}$$

and since it has poles at $s = \pm j2$ not in the left hand half of the s-plane the f.v. theorem is not applicable.

■ **47** Assuming quiescent initial state taking Laplace transforms gives

$$\begin{aligned}(7s+5)Y(s) &= \frac{4}{s} + \frac{1}{s+3} + 2\\ Y(s) &= \frac{4}{s(7s+5)} + \frac{1}{(s+3)(7s+5)} + \frac{2}{7s+5}\\ sY(s) &= \frac{4}{7s+5} + \frac{s}{(s+3)(7s+5)} + \frac{2s}{7s+5}\end{aligned}$$

By the f.v. theorem,

$$\begin{aligned}\lim_{t\to\infty} y(t) = \lim_{s\to 0} sF(s) &= \lim_{s\to 0}\big[\frac{4}{7s+5} + \frac{s}{(s+3)(7s+5)} + \frac{2s}{7s+5}\big]\\ &= \frac{4}{5}\end{aligned}$$

By the i.v. theorem,

$$\begin{aligned}\lim_{t\to 0^+} y(t) = y(0+) = \lim_{s\to\infty} sF(s) &= \lim_{s\to\infty}\big[\frac{4}{7s+5} + \frac{s}{(1+\frac{3}{s})(7s+5)} + \frac{2}{7+\frac{5}{s}}\big]\\ &= \frac{2}{7}\end{aligned}$$

Thus, jump at $t = 0 = y(0+) - y(0-) = 1\frac{2}{7}$.

Exercises 5.6.8

■ **48(a)**

$$\begin{aligned}f * g(t) &= \int_0^t \tau\cos(3t-3\tau)d\tau\\ &= \big[-\frac{1}{3}\tau\sin(3t-3\tau) + \frac{1}{9}\cos(3t-3\tau)\big]_0^t\\ &= \frac{1}{9}(1-\cos 3t)\\ g * f(t) &= \int_0^t (t-\tau)\cos 3\tau d\tau\\ &= \big[\frac{t}{3}\sin 3\tau - \frac{\tau}{3}\sin 3\tau - \frac{1}{9}\cos 3\tau\big]_0^t = \frac{1}{9}(1-\cos 3t)\end{aligned}$$

48(b)

$$\begin{aligned} f*g(t) &= \int_0^t (\tau+1)e^{-2(t-\tau)}d\tau \\ &= \big[\frac{1}{2}(\tau+1)e^{-2(t-\tau)} - \frac{1}{4}e^{-2(t-\tau)}\big]_0^t \\ &= \frac{1}{2}t + \frac{1}{4} - \frac{1}{4}e^{-2t} \\ g*f(t) &= \int_0^t (t-\tau+1)e^{-2\tau}d\tau \\ &= \big[-\frac{1}{2}(t-\tau+1)e^{-2\tau} + \frac{1}{4}e^{-2\tau}\big]_0^t \\ &= \frac{1}{2}t + \frac{1}{4} - \frac{1}{4}e^{-2t} \end{aligned}$$

48(c) Integration by parts gives

$$\begin{aligned} \int_0^t \tau^2 \sin 2(t-\tau)d\tau &= \int_0^t (t-\tau)^2 \sin 2\tau d\tau \\ &= \frac{1}{4}\cos 2t + \frac{1}{2}t^2 - \frac{1}{4} \end{aligned}$$

48(d) Integration by parts gives

$$\begin{aligned} \int_0^t e^{-\tau}\sin(t-\tau)d\tau &= \int_0^t e^{-(t-\tau)}\sin\tau d\tau \\ &= \frac{1}{2}(\sin t - \cos t + e^{-t}) \end{aligned}$$

■ **49(a)** Since $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1 = f(t)$ and $\mathcal{L}^{-1}\{\frac{1}{(s+3)^3}\} = \frac{1}{2}t^2e^{-3t}$

$$\begin{aligned} \mathcal{L}^{-1}\{\frac{1}{s}\cdot\frac{1}{(s+3)^3}\} &= \int_0^t f(t-\tau)g(\tau)d\tau \\ &= \int_0^t 1.\frac{1}{2}\tau^2e^{-3\tau}d\tau \\ &= \frac{1}{4}\big[-\tau^2e^{-3\tau} - \frac{2}{3}\tau e^{-3\tau} - \frac{2}{9}e^{-3\tau}\big]_0^t \\ &= \frac{1}{54}[2 - e^{-3t}(9t^2+6t+2)] \end{aligned}$$

Directly

$$\mathcal{L}^{-1}\{\frac{1}{s}\cdot\frac{1}{(s+3)^3}\} = \mathcal{L}^{-1}\frac{1}{54}\{\frac{2}{s} - \frac{18}{(s+3)^3} - \frac{6}{(s+3)^2} - \frac{2}{(s+3)}\}$$
$$= \frac{1}{54}[2 - e^{-3t}(9t^2 + 6t + 2)]$$

49(b) $\mathcal{L}^{-1}\{\frac{1}{(s-2)^2}\} = te^{2t} = f(t),\ \mathcal{L}^{-1}\{\frac{1}{(s+3)^2}\} = te^{-3t} = g(t)$

$$\mathcal{L}^{-1}\{\frac{1}{(s-2)^2}\cdot\frac{1}{(s+3)^2}\} = \int_0^t (t-\tau)e^{2(t-\tau)}.\tau e^{-3\tau}d\tau$$
$$= e^{-2t}\int_0^t (t\tau - \tau^2)e^{-5\tau}d\tau$$
$$= e^{2t}\big[-\frac{1}{5}(t\tau - \tau^2)e^{-5\tau} - \frac{1}{25}(t-2\tau)e^{-5\tau} + \frac{2}{125}e^{-5\tau}\big]_0^t$$
$$= e^{2t}\big[\frac{t}{25}e^{-5t} + \frac{2}{125}e^{-5t} + \frac{t}{25} - \frac{2}{125}\big]$$
$$= \frac{1}{125}\big[e^{2t}(5t-2) + e^{-3t}(5t+2)\big]$$

Directly

$$\frac{1}{(s-2)^2(s+3)^2} = \frac{\frac{-2}{125}}{s-2} + \frac{\frac{1}{25}}{(s-2)^2} + \frac{\frac{2}{125}}{(s+3)} + \frac{\frac{1}{25}}{(s+3)^2}$$
$$\therefore \mathcal{L}^{-1}\{\frac{1}{(s-2)^2(s+3)^2}\} = \frac{-2}{125}e^{2t} + \frac{1}{25}te^{2t} + \frac{2}{125}e^{-3t} + \frac{1}{25}te^{-3t}$$
$$= \frac{1}{125}[e^{2t}(5t-2) + e^{-3t}(5t+2)]$$

49(c) $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t = f(t),\ \mathcal{L}^{-1}\{\frac{1}{(s+4)}\} = e^{-4t} = g(t)$

$$\mathcal{L}^{-1}\{\frac{1}{s^2}\cdot\frac{1}{s+4}\} = \int_0^t (t-\tau)e^{-4t}d\tau$$
$$= \big[-\frac{1}{4}(t-\tau)e^{-4\tau} + \frac{1}{16}e^{-4\tau}\big]_0^t$$
$$= \frac{1}{16}e^{-4t} + \frac{1}{4}t - \frac{1}{16}$$

Directly

$$\mathcal{L}^{-1}\{\frac{1}{s^2(s+4)}\} = \mathcal{L}^{-1}\{\frac{1}{16}\cdot\frac{1}{s+4} - \frac{1}{16}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s^2}\}$$
$$= \frac{1}{16}e^{-4t} - \frac{1}{16} + \frac{1}{4}t$$

■ **50** Let $f(\lambda) = \lambda$ and $g(\lambda) = e^{-\lambda}$ so

$$F(s) = \frac{1}{s^2} \text{ and } G(s) = \frac{1}{s+1}$$

Considering the integral equation

$$y(t) = \int_0^t \lambda e^{-(t-\lambda)} d\lambda$$

By (5.80) in the text

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\lambda)g(t-\lambda)d\lambda$$
$$= \int_0^t \lambda e^{-(t-\lambda)} d\lambda = y(t)$$

so

$$y(t) = \mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{\frac{1}{s^2(s+1)}\}$$
$$= \mathcal{L}^{-1}\{-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\}$$
$$= (t-1) + e^{-t}$$

■ **51** Impulse response $h(t)$ is given by the solution of

$$\frac{d^2h}{dt^2} + \frac{7dh}{dt} + 12h = \delta(t)$$

subject to zero initial conditions. Taking Laplace transforms

$$(s^2 + 7s + 12)H(s) = 1$$
$$H(s) = \frac{1}{(s+3)(s+4)} = \frac{1}{s+3} - \frac{1}{s+4}$$
giving $h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-3t} - e^{-4t}$

Response to pulse input is

$$\begin{aligned}
x(t) &= A\Big\{\int_0^t [e^{-3(t-\tau)} - e^{-4(t-\tau)}]d\tau\Big\}H(t) \\
&\quad - A\Big\{\int_T^t [e^{-3(t-\tau)} - e^{-4(t-\tau)}]d\tau\Big\}H(t-T) \\
&= A\Big\{\big[\frac{1}{3} - \frac{1}{4} - \frac{1}{3}e^{-3t} + \frac{1}{4}e^{-4t}\big]H(t) \\
&\quad - \big[\frac{1}{3} - \frac{1}{4} - \frac{1}{3}e^{-3(t-T)} - \frac{1}{4}e^{-4(t-T)}\big]H(t-T)\Big\} \\
&= \frac{1}{12}A\big[1 - 4e^{-3t} + 3e^{-4t} - (1 - 4e^{-3(t-T)} + 3e^{-4(t-T)})H(t-T)\big]
\end{aligned}$$

or directly

$$u(t) = A[H(t) - H(t-T)] \text{ so } U(s) = \mathcal{L}\{u(t)\} = \frac{A}{s}[1 - e^{-sT}]$$

Thus, taking Laplace transforms with initial quiescent state

$$\begin{aligned}
(s^2 + 7s + 12)X(s) &= \frac{A}{s}[1 - e^{-sT}] \\
X(s) &= A\big[\frac{1}{12}\cdot\frac{1}{s} - \frac{1}{3}\cdot\frac{1}{s+3} + \frac{1}{4}\cdot\frac{1}{s+4}\big](1 - e^{-sT}) \\
x(t) = \mathcal{L}^{-1}\{X(s)\} &= \frac{A}{12}[1 - 4e^{-3t} + 3e^{-4t} - (1 - 4e^{-3(t-T)} + 3e^{-4(t-T)})H(t-T)]
\end{aligned}$$

■ **52** Impulse response $h(t)$ is the solution of

$$\frac{d^2h}{dt^2} + 4\frac{dh}{dt} + 5h = \delta(t),\ h(0) = \dot{h}(0) = 0$$

Taking Laplace transforms

$$\begin{aligned}
(s^2 + 4s + 5)H(s) &= 1 \\
H(s) &= \frac{1}{s^2 + 4s + 5} = \frac{1}{(s+2)^2 + 1} \\
\text{so } h(t) = \mathcal{L}^{-1}\{H(s)\} &= e^{-2t}\sin t.
\end{aligned}$$

By the convolution integral response to unit step is

$$\begin{aligned}\theta_0(t) &= \int_0^t e^{-2(t-\tau)} \sin(t-\tau).1 d\tau \\ &= e^{-2t} \int_0^t e^{2\tau} \sin(t-\tau) d\tau\end{aligned}$$

which using integration by parts gives

$$\begin{aligned}\theta_0(t) &= \frac{e^{-2t}}{5} \left[e^{2\tau}[2\sin(t-\tau) + \cos(t-\tau)]\right]_0^t \\ &= \frac{1}{5} - \frac{1}{5} e^{-2t}(2\sin t + \cos t)\end{aligned}$$

Check

Solving

$$\frac{d^2\theta_0}{dt^2} + 4\frac{d\theta_0}{dt} + 5\theta_0 = 1 \ , \ \dot{\theta}_0(0) = \theta_0(0) = 0$$

gives

$$\begin{aligned}(s^2 + 4s + 5)\Phi_0(s) &= \frac{1}{s} \\ \Phi_0(s) = \frac{1}{s(s^2+4s+5)} &= \frac{1}{5s} - \frac{1}{5} \cdot \frac{s+4}{(s+2)^2+1} \\ \text{so } \theta_0(t) = \mathcal{L}^{-1}\{\Phi_0(s)\} &= \frac{1}{5} - \frac{1}{5}[\cos t + 2\sin t]e^{-2t}.\end{aligned}$$

Exercises 5.7.2

- **53** State–space form of model is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{u} \Rightarrow \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-5 & -1 \\ 3 & -1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}2 \\ 5\end{bmatrix} u$$

$$y = \mathbf{c}^T\mathbf{x} \Rightarrow y = [\,1 \quad 2\,]\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

System transfer function is

$$G(s) = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = [\,1 \quad 2\,]\begin{bmatrix}s+5 & 1 \\ -3 & s+1\end{bmatrix}^{-1}\begin{bmatrix}2 \\ 5\end{bmatrix}$$

$$\det\begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix} = \Delta = (s+5)(s+1)+3 = (s+2)(s+4),$$

$$\begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} = \frac{1}{\Delta}\begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}$$

Thus, $G(s) = \frac{1}{\Delta}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \frac{1}{\Delta}(12s+59)$
so the system transfer function is

$$G(s) = \frac{12s+59}{(s+2)(s+4)}$$

■ **54** In this case, the denominator can be factorized

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+1}{(s+1)(s+6)}$$

and care must be taken not to cancel the common factor, to avoid the system being mistaken for a first order system. To proceed it is best to model the system by the differential equation

$$\ddot{y} + 7\dot{y} + 6y = \dot{u} + u$$

from which

$$\dot{x}_1 = -7x_1 + x_2 + u$$
$$\dot{x}_2 = -6x_1 + u$$
$$y = x_1$$

so that a state–space model is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ -6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$\mathbf{y} = \mathbf{c}^T\mathbf{x} \Rightarrow y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Check

$$G(s) = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$
$$\det(s\mathbf{I} - \mathbf{A}) = \Delta = \begin{vmatrix} s+7 & -1 \\ 6 & s \end{vmatrix} = s^2 + 7s + 6$$
$$G(s) = \frac{1}{\Delta}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & 1 \\ -6 & s+7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{s+1}{\Delta} = \frac{s+1}{s^2+7s+6}$$

■ **55(a)** Taking $\mathbf{A}$ to be the companion matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -5 & -6 \end{bmatrix}$$

$$\text{then} \quad \mathbf{b} = [0\ 0\ 1]^T$$

$$\mathbf{c} = [5\ 3\ 1]^T$$

Then from equation (5.84) in the text the state space form of the dynamic model is

$$\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{c}^T\mathbf{x}$$

55(b) Taking $\mathbf{A}$ to be the companion matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

$$\text{then} \quad \mathbf{b} = [0\ 0\ 1]^T$$

$$\mathbf{c} = [2\ 3\ 1]^T$$

And state space model is

$$\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x} + \mathbf{b}u,\ y = \mathbf{c}^T\mathbf{x}$$

■ **56** We are required to express the transfer function in the state space form

$$\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x} + \mathbf{b}u$$

$$\mathbf{y} = \mathbf{c}^T\mathbf{x}$$

where $\mathbf{A}$ is the companion matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ and $y = [1\ 0\ 0]\mathbf{x}$. To determine $\mathbf{b}$, we divide the denominator into the numerator as follows

$$\begin{array}{r} s^3 + 6s^2 + 11s + 6 \overline{)\ 5s^{-1} - 29s^{-2} + 120s^{-3}} \\ \qquad\qquad\qquad\qquad 5s^2 + s + 1 \\ \qquad\qquad\qquad\qquad \underline{5s^2 + 30s + 55}\ | \\ \qquad\qquad\qquad\qquad\quad -29s - 54\ | \\ \qquad\qquad\qquad\qquad \underline{-29s - 174}\ | \\ \qquad\qquad\qquad\qquad\quad 120\ | \end{array} \quad \text{neglect these terms}$$

giving $\mathbf{b} = [5 \ -29 \ -120]^T$. Thus, state space form is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 5 \\ -29 \\ 120 \end{bmatrix} u(t)$$
$$y = [1\ 0\ 0]\mathbf{x}(t)$$

It is readily checked that this is a true representation of the given transfer function.

■ **57** $[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s-3 & -4 \\ -2 & s-1 \end{bmatrix}$, $\det[s\mathbf{I} - \mathbf{A}] = (s-5)(s+1)$

Thus,

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{(s-5)(s+1)} \begin{bmatrix} s-1 & 4 \\ 2 & s-3 \end{bmatrix} = \begin{bmatrix} \frac{2/3}{s-5} + \frac{1/3}{s+1} & \frac{2/3}{s-5} - \frac{2/3}{s+1} \\ -\frac{1/3}{s-5} - \frac{1/3}{s+1} & \frac{1/3}{s-5} + \frac{2/3}{s+1} \end{bmatrix}$$
$$\therefore \mathcal{L}^{-1}[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \frac{2}{3}e^{5t} + \frac{1}{3}e^{-t} & \frac{2}{3}e^{5t} - \frac{2}{3}e^{-t} \\ \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} \end{bmatrix} = \mathbf{\Phi_1}$$
$$[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\ \mathbf{U}(\mathbf{s}) = \frac{1}{(s-5)(s+1)} \begin{bmatrix} s-1 & 4 \\ 2 & 5-3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{s} \\ \frac{7}{s} \end{bmatrix}$$
$$= \frac{1}{s(s-5)(s+1)} \begin{bmatrix} 3s+25 \\ 7s-15 \end{bmatrix} = \begin{bmatrix} -\frac{5}{s} + \frac{11/3}{s+1} + \frac{4/3}{s-5} \\ \frac{3}{s} - \frac{11/3}{s+1} + \frac{2/3}{s-5} \end{bmatrix}$$

$$\therefore \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\ \mathbf{U}(\mathbf{s})\} = \begin{bmatrix} -5 + \frac{11}{3}e^{-t} + \frac{4}{3}e^{5t} \\ 3 - \frac{11}{3}e^{-t} + \frac{2}{3}e^{5t} \end{bmatrix} = \mathbf{\Phi}_2$$

Thus, solution is

$$\mathbf{x}(t) = \Phi_1\mathbf{x}(0) + \Phi_2 = \begin{bmatrix} -5 + \frac{8}{3}e^{-t} + \frac{10}{3}e^{5t} \\ 3 - \frac{8}{3}e^{-t} + \frac{5}{3}e^{5t} \end{bmatrix}$$

which confirms the answer obtained in Exercise 61 in Chapter 1.

■ **58** $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}$, $s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s-1 & 3 \\ -2 & s+4 \end{bmatrix}$

$$| s\mathbf{I} - \mathbf{A} | = \Delta = (s-1)(s+4) + 6 = (s+2)(s+1)$$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{\Delta} \begin{bmatrix} s+4 & -3 \\ 2 & s-1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{s+1} - \frac{2}{s+2} & -\frac{3}{s+1} + \frac{3}{s+2} \\ \frac{2}{s+1} - \frac{2}{s+2} & -\frac{2}{s+1} + \frac{3}{s+2} \end{bmatrix}$$

$$\text{giving } \mathcal{L}^{-1}[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} 3e^{-t} - 2e^{-2t} & -3e^{-t} + 3e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -2e^{-t} + 3e^{-2t} \end{bmatrix}$$

$$\text{so } \mathcal{L}^{-1}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) = \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix}$$

$$[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}U(s) = \frac{1}{\Delta} \begin{bmatrix} s+4 & -3 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{s+3}$$

$$= \begin{bmatrix} \frac{1}{(s+2)(s+3)} \\ \frac{1}{(s+2)(s+3)} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+2} - \frac{1}{s+3} \\ \frac{1}{s+2} - \frac{1}{s+3} \end{bmatrix}$$

$$\text{so } \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}U(s)\} = \begin{bmatrix} e^{-2t} - e^{-3t} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

Thus,

$$\mathbf{x}(t) = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{-2t} - e^{-3t} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

giving

$$x_1(t) = x_2(t) = 2e^{-2t} - e^{-3t}$$

■ **59** $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $u(t) = e^{-t}H(t)$, $\mathbf{x}_0 = [1\ 0]^T$

with $X(s)$ and the solution $x(t)$ given by equations (5.88) and (5.89) in the text

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \text{ giving } (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & 5 \end{bmatrix}$$

so that,

$$(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} \\ -\frac{2}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

and

$$(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 2(s+3) \\ -4 \end{bmatrix} \frac{1}{s+1} = \begin{bmatrix} \frac{2}{s+2} + \frac{4}{(s+1)^2} - \frac{2}{s+1} \\ \frac{4}{s+1} - \frac{4}{(s+1)^2} - \frac{4}{s+2} \end{bmatrix}$$

Thus, $X(s) = \begin{bmatrix} \frac{4}{(s+1)^2} + \frac{1}{(s+2)} \\ \frac{2}{(s+1)} - \frac{2}{(s+2)} - \frac{4}{(s+1)^2} \end{bmatrix}$

giving $x(t) = \begin{bmatrix} 4te^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} - 4te^{-t} \end{bmatrix}$

■ **60** Taking $\mathbf{A}$ as the companion matrix following the procedure of Example 5.61 we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix},\ \mathbf{b} = [0\ 0\ 1]^T,\ \mathbf{c} = [1\ 2\ 3]^T$$

Eigenvalue of $\mathbf{A}$ given by

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -6 & -11 & -6-\lambda \end{vmatrix} = -(\lambda^3 + 6\lambda^2 + 11\lambda + 6) = -(\lambda+1)(\lambda+2)(\lambda+3) = 0$$

so the eigenvalues are $\lambda_1 = -3, \lambda_2 = -2, \lambda_3 = -1$. The eigenvectors are given by the corresponding solutions of

$$\begin{aligned} -\lambda_i e_{i1} + e_{i2} + 0e_{i3} &= 0 \\ 0e_{i1} - \lambda_i e_{i2} + e_{i3} &= 0 \\ -6e_{i1} - 11e_{i2} - (6+\lambda_i)e_{i3} &= 0 \end{aligned}$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [1\ -3\ 9]^T,\ \mathbf{e}_2 = [1\ -2\ 4]^T,\ \mathbf{e}_3 = [1\ -1\ 1]^T$$

Taking $\mathbf{M}$ to be the modal matrix $\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & -2 & -1 \\ 9 & 4 & 1 \end{bmatrix}$ then the transformation $\mathbf{X} = \mathbf{M}\ \boldsymbol{\xi}$ will reduce the system to the canonical form

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\Lambda}\ \boldsymbol{\xi} + \mathbf{M}^{-1}\mathbf{b}u\ ,\ y = \mathbf{c}^T\mathbf{M}\ \boldsymbol{\xi}$$

$$\mathbf{M}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 3 & 1 \\ -6 & -8 & -2 \\ 6 & 5 & 1 \end{bmatrix},\ \mathbf{M}^{-1}\mathbf{b} = \frac{1}{2}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},\ \mathbf{c}^T\mathbf{M} = [22\ 9\ 2]$$

Thus, canonical form is

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\end{bmatrix} = \begin{bmatrix}-3 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\\ -1\\ \frac{1}{2}\end{bmatrix} u$$
$$y = [22\ 9\ 2]\ [\xi_1\ \xi_2\ \xi_3]^T$$

Since the eigenvalues of $\mathbf{A}$ are negative the system is stable. Since vector $\mathbf{M}^{-1}\mathbf{b}$ has no zero elements the system is controllable and since $\mathbf{c}^T\mathbf{M}$ has no zero elements the system is also observable.

$$\mathbf{b} = [0\ 0\ 1]^T,\ \mathbf{A}\ \mathbf{b} = [0\ 1\ -6]^T,\ \mathbf{A}^2\mathbf{b} = [1\ 6\ 25]^T$$

$$[\mathbf{b}\ \ \mathbf{A}\ \mathbf{b}\ \ \mathbf{A}^2\mathbf{b}] = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 6\\ 1 & -6 & 25\end{bmatrix} \sim \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} \text{ which is of full rank 3}$$

so the system is controllable.

$$\mathbf{c} = [1\ 2\ 3]^T,\ \mathbf{A}^T\ \mathbf{c} = [-18\ -32\ -16],\ (\mathbf{A}^T)^2\mathbf{c} = [96\ 158\ 63]^T$$

$$[\mathbf{c}\ \ \mathbf{A}^T\ \mathbf{c}\ \ (\mathbf{A}^T)^2\mathbf{c}] = \begin{bmatrix}1 & -18 & 96\\ 2 & -32 & 158\\ 3 & -16 & 63\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ which is of full rank 3}$$

so the system is observable.

■ **61** $\mathbf{A} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -5 & -6\end{bmatrix}$, $\mathbf{b} = [0\ 0\ 1]^T$, $\mathbf{c} = [5\ 3\ 1]^T$

Eigenvalues of $\mathbf{A}$ given by

$$0 = \begin{vmatrix}-\lambda & 1 & 0\\ 0 & -\lambda & 1\\ 0 & -5 & -6-\lambda\end{vmatrix} = -\lambda(\lambda+5)(\lambda+1) = 0$$

so eigenvalues are $\lambda_1 = -5, \lambda_2 = -1, \lambda_3 = 0$. The corresponding eigenvectors are determined as

$$\mathbf{e}_1 = [1\ -5\ 25]^T,\ \mathbf{e}_2 = [1\ -1\ 1]\ ,\ \mathbf{e}_3 = [1\ 0\ 0]^T$$

Take $\mathbf{M}$ to be the modal matrix $\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ -5 & -1 & 0 \\ 25 & 1 & 0 \end{bmatrix}$ then the transformation $\mathbf{x} = \mathbf{M}\boldsymbol{\xi}$ will reduce the system to the canonical form

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\Lambda}\ \boldsymbol{\xi} + \mathbf{M}^{-1}\mathbf{b}u\ ,\ y = \mathbf{c}^T\ \mathbf{M}\ \boldsymbol{\xi}$$

$$\mathbf{M}^{-1} = \frac{1}{20}\begin{bmatrix} 0 & 1 & 1 \\ 0 & -25 & -5 \\ 20 & 24 & 4 \end{bmatrix},\ \mathbf{M}^{-1}\mathbf{b} = \frac{1}{20}\begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix},\ \mathbf{c}^T\mathbf{M} = [15\ 3\ 5]$$

Thus, canonical form is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{20} \\ -\frac{1}{4} \\ \frac{1}{5} \end{bmatrix} u$$

$$y = [15\ 3\ 5]\ [\xi_1\ \xi_2\ \xi_3]^T$$

Since $\mathbf{A}$ has zero eigenvalues the system is marginally stable, since $\mathbf{M}^{-1}\mathbf{b}$ has no zero elements the system is controllable and since $\mathbf{c}^T\mathbf{M}$ has no zero elements the system is observable. Again as in Exercise 60 these results can be confirmed by using the Kalman matrices.

Exercises 5.7.4

■ **62** $\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\ \mathbf{u} = \begin{bmatrix} 1 \\ t \end{bmatrix},\ \mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The solution $x(t)$ is given by equation (5.97) in the text.

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix} \text{ giving } (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}$$

so that,

$$(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 = \frac{1}{(s+1)(s+2)}\begin{bmatrix} -2 \\ s \end{bmatrix} = \begin{bmatrix} \frac{2}{s+2} - \frac{2}{s+1} \\ \frac{2}{s+2} - \frac{1}{s+1} \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\ \mathbf{U}(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{s} \\ \\ \frac{1}{s^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s(s+2)} - \frac{(s+5)}{s^2(s+1)(s+2)} \\ \frac{1}{s(s+2)} + \frac{s-1}{s^2(s+1)(s+2)} \end{bmatrix}$$

Thus,

$$X_1(s) = \mathcal{L}\{x_1(t)\} = \frac{2}{s+2} - \frac{2}{s+1} + \frac{1}{s(s+2)} - \frac{(s+5)}{s^2(s+1)(s+2)}$$
$$= \frac{9/4}{s+2} - \frac{6}{s+1} + \frac{15/4}{s} - \frac{5/2}{s^2}$$

giving $x_1(t) = \frac{9}{4}e^{-2t} - 6e^{-t} + \frac{15}{4} - \frac{5}{2}t$

■ **63** $\ddot{y}_1 + \dot{y}_1 - \dot{y}_2 + y_1 = u_1 \qquad$ (i)

$\ddot{y}_2 + \dot{y}_2 - \dot{y}_1 + y_2 = u_2 \qquad$ (ii)

Let $\mathbf{x} = [\, x_1 \quad x_2 \quad x_3 \quad x_4 \,]^T = [\, y_1 \quad \dot{y}_1 \quad y_2 \quad \dot{y}_2 \,]^T$ then

$$\dot{x}_1 = \dot{y}_1 = x_2$$
$$(i) \Rightarrow \ddot{y}_1 = \dot{x}_2 = -\dot{y}_1 + \dot{y}_2 - y_1 + u_1 \Rightarrow \dot{x}_2 = -x_2 + x_4 - x_1 + u_1$$
$$\dot{x}_3 = \dot{y}_2 = x_4$$
$$(ii) \Rightarrow \ddot{y}_2 = \dot{x}_4 = xx - \dot{y}_2 + \dot{y}_1 - y_2 + u_2 \Rightarrow \dot{x}_4 = -x_4 + x_2 - x_3 + u_2$$

giving the state–space representation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Transfer function $= G(s) = \frac{\mathbf{C}adj(s\mathbf{I}-\mathbf{A})\mathbf{B}}{\det(s\mathbf{I}-\mathbf{A})}$

$$\det(s\mathbf{I} - \mathbf{A}) = \Delta = \begin{vmatrix} s & -1 & 0 & 0 \\ 1 & s+1 & 0 & -1 \\ 0 & 0 & s & -1 \\ 0 & -1 & 1 & s+1 \end{vmatrix} = s^4 + 2s^3 + 2s^2 + 2s + 1$$
$$\Rightarrow \Delta = (s+1)^2(s^2+1)$$

so that

$$G(s) = \frac{1}{\Delta} \begin{bmatrix} s^2+s+1 & s \\ s & s^2+s+1 \end{bmatrix} = \frac{1}{(s+1)^2(s^2+1)} \begin{bmatrix} s^2+s+1 & s \\ s & s^2+s+1 \end{bmatrix}$$

Poles given by $\Delta = (s+1)^2(s^2+1) = 0$
Eigenvalues associated matrix $\boldsymbol{A}$ given by $\det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda+1)^2(\lambda^2+1) = 0$
Thus, poles and eigenvalues are identical.

■ **64** (a) $u_2 = R_1(u_1 + x_1 + x_2 + x_3) + L_1\dot{x}_1; R_1 = 1, L_1 = 1$

$$\Rightarrow \dot{x}_1 = -x_1 - x_2 - x_3 - u_1 + u_2$$
$$u_2 = R_1(u_1 + x_1 + x_2 + x_3) + R_2(x_2 + x_3) + L_2\dot{x}_2; R_2 = 2, L_2 = 1$$
$$\Rightarrow \dot{x}_2 = -x_1 - 3x_2 - 3x_3 - u_1 + u_2$$
$$u_2 = R_1(u_1 + x_1 + x_2 + x_3) + R_2(x_2 + x_3) + R_3x_3 + L_3\dot{x}_3; R_3 = 3, L_3 = 1$$
$$\Rightarrow \dot{x}_3 = -x_1 - 3x_2 - 6x_3 - u_1 + u_2$$

giving the state–space model

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & -1 & -1\\ -1 & -3 & -3\\ -1 & -3 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}-1 & 1\\ -1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\begin{bmatrix}\mathrm{y}_1\\ \mathrm{y}_2\end{bmatrix} = \begin{bmatrix}0 & 2 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

(b) Transfer matrix $= \mathbf{G}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \mathbf{C}\frac{adj(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}\mathbf{B}$

$$\det(s\mathbf{I} - \mathbf{A}) = \Delta = \begin{vmatrix} s+1 & 1 & 1\\ 1 & s+3 & 3\\ 1 & 3 & s+6\end{vmatrix} = s^3 + 10s^2 + 16s + 6$$

$$adj(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s^2+9s+9 & -(s+3) & -s\\ -(s+3) & s^2+7s+5 & -(3s+2)\\ -s & -(3s+2) & s^2+4s+2\end{bmatrix}$$

$$\mathbf{G}(s) = \frac{1}{\Delta}\begin{bmatrix}0 & 2 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} s^2+9s+9 & -(s+3) & -s\\ -(s+3) & s^2+7s+5 & -(3s+2)\\ -s & -(3s+2) & s^2+4s+2\end{bmatrix}\begin{bmatrix}-1 & 1\\ -1 & 1\\ -1 & 1\end{bmatrix}$$
$$= \frac{1}{\Delta}\begin{bmatrix}-2s(2s+3) & 2s(2s+3)\\ -s^2 & s^2\end{bmatrix} \text{ with } \Delta = s^3 + 10s^2 + 16s + 6$$

(c) $\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$ where $\mathbf{U}(s) = \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s^2} \end{bmatrix}$

$$= \frac{1}{\Delta}\begin{bmatrix} -2s(2s+3) & 2s(2s+3) \\ -s^2 & s^2 \end{bmatrix}\begin{bmatrix} \frac{1}{s} \\ \frac{1}{s^2} \end{bmatrix}$$

$$= \frac{1}{\Delta}\begin{bmatrix} \frac{-4s^2-2s+6}{s} \\ -s+1 \end{bmatrix}$$

To obtain response express in partial fractions and take inverse Laplace transforms
Factorizing Δ gives $\Delta = (s+8.12)(s+0.56)(s+1.32)$ so

$$Y_1(s) = \frac{-4s^2-2s+6}{s(s+8.12)(s+0.56)(s+1.32)} = \frac{1}{s} + \frac{0.578}{s+8.12} - \frac{1.824}{s+0.56} + \frac{0.246}{s+1.32}$$

$$\Rightarrow y_1(t) = 1 + 0.578e^{-8.12t} - 1.824e^{-0.56t} + 0.246e^{-1.32t}$$

$$Y_2(s) = \frac{-s+1}{(s+8.12)(s+0.56)(s+1.32)} = \frac{0.177}{s+8.12} + \frac{0.272}{s+0.56} - \frac{0.449}{s+1.32}$$

$$\Rightarrow y_2(t) = 0.177e^{-8.12t} + 0.272e^{-0.56t} - 0.449e^{-1.32t}$$

Exercises 5.9.3

■ **65** Choose $x_1(t) = y(t),\ x_2(t) = \dot{x}_1(t) = \dfrac{dy}{dt}$ then

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t),\ y(t) = [1\ 0]\mathbf{x}(t)$$

Taking $u(t) = K_1x_1(t) + K_2x_2(t) + u_{\text{ext}}(t)$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ K_1 + \frac{1}{2} & K_2 + \frac{1}{2} \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_{\text{ext}}$$

The eigenvalues of the matrix are given by

$$\begin{vmatrix} 0-\lambda & 1 \\ K_1 + \frac{1}{2} & K_2 + \frac{1}{2} - \lambda \end{vmatrix} = 0$$

or $\lambda^2 - (K_2 + \frac{1}{2})\lambda - (K_1 + \frac{1}{2}) = 0$
If the poles are to be at $\lambda = -4$ then we require the characteristic equation to be

$$\lambda^2 + 8\lambda + 16 = 0$$

By comparison, we have $-K_2 - \frac{1}{2} = 8$ and $-K_1 - \frac{1}{2} = 18$ giving $K_1 = -\frac{33}{2}$, $K_2 = -\frac{17}{2}$ so $u(t) = \big[-\dfrac{33}{2} - \dfrac{17}{2}\big]\mathbf{x} + u_{\text{ext}}$

■ **66**

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & -\frac{5}{4} \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)\ ,\ y(t) = [0\ 2]\mathbf{x}(t)$$

Setting $u = \mathbf{K}^T\mathbf{x} + u_{\text{ext}},\ \mathbf{K} = [K_1\ K_2]^T$ gives the system matrix

$$A = \begin{bmatrix} 0 & 1 \\ K_1 - \frac{1}{4} & K_2 - \frac{5}{4} \end{bmatrix}$$

whose eigenvalues are given by $\lambda^2 - (K_2 - \frac{5}{4})\lambda - (K_1 - \frac{5}{4}) = 0$. Comparing with the desired characteristic equation

$$(\lambda + 5)^2 = \lambda^2 + 10\lambda + 25 = 0$$

gives $K_1 = -\frac{99}{4},\ K_2 = -\frac{35}{4}$. Thus,

$$u(t) = \big[-\frac{99}{4} - \frac{35}{4}\big]\mathbf{x}(t) + u_{\text{ext}}$$

■ **67** With $u_1 = [K_1\ K_2]\mathbf{x}(t)$ the system matrix $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} K_1 & 1 + K_2 \\ 6 + K_1 & 1 + K_2 \end{bmatrix}$$

having characteristic equation

$$\lambda^2 - \lambda(1 + K_1 + K_2) - 6(1 - K_2) = 0$$

which on comparing with

$$\lambda^2 + 10\lambda + 25 = 0$$

gives $K_1 = -\dfrac{35}{6},\ K_2 = -\dfrac{31}{6}$ so that $u_1(t) = \big[-\dfrac{35}{6} - \dfrac{31}{6}\big]\mathbf{x}(t)$

Using $u_2(t)$ the matrix $\mathbf{A}$ becomes $\begin{bmatrix} 0 & 1 \\ 6 + K_1 & 1 + K_2 \end{bmatrix}$ where $\mathbf{K}^T = [-31\ \ -11]$

■ **68** See p. 472 in the text.

■ **69** For the matrix of Exercise 68 the Kalman matrix is

$$\mathbf{M} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which is of rank 1. Thus, the system is uncontrollable.
For the matrix of Exercise 65 the Kalman matrix is

$$\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is of full rank 2. Thus, the system is controllable.

■ **70**

$$\mathbf{M} = (\mathbf{b} \;\; \mathbf{A}\,\mathbf{b}) = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix},\ \mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix},\ \mathbf{v}^T = [4 \;\; -1]$$

$$\mathbf{v}^T\mathbf{A} = [4 \;\; -1] \begin{bmatrix} 8 & -2 \\ 35 & -9 \end{bmatrix} = [-3 \;\; 1] \text{ so } \mathbf{T} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix},\ \mathbf{T}^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

Taking $\mathbf{z}(t) = T\mathbf{x}$ or $\mathbf{x} = \mathbf{T}^{-1}z(t)$ then equation reduces to

$$\begin{aligned} \mathbf{T}^{-1}\dot{\mathbf{z}}(t) &= \begin{bmatrix} 8 & -2 \\ 35 & -9 \end{bmatrix} \mathbf{T}^{-1}\mathbf{z}(t) + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u \\ \text{or } \dot{\mathbf{z}}(t) &= I\begin{bmatrix} 8 & -2 \\ 35 & -9 \end{bmatrix} \mathbf{T}^{-1}\mathbf{z}(t) + \mathbf{T}\begin{bmatrix} 1 \\ 4 \end{bmatrix} u \\ &= \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 8 & -2 \\ 35 & -9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \mathbf{z}(t) + \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} u \\ &= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \mathbf{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \end{aligned}$$

Clearly both system matrices have the same eigenvalues $\lambda = -2, \lambda = 1$. This will always be so since we have carried out a singularity transformation.

Review Exercises 5.10

■ **1(a)** $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 5x = 8\cos t,\ x(0) = \dot{x}(0) = 0$ Taking Laplace transforms

$$(s^2 + 4s + 5)X(s) = \frac{8s}{s^2 + 1}$$

$$X(s) = \frac{8s}{(s^2+1)(s^2+4s+5)}$$
$$= \frac{s+1}{s^2+1} - \frac{s+5}{s^2+4s+5}$$
$$= \frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{(s+2)+3}{(s+2)^2+1}$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = \cos t + \sin t - e^{-2t}[\cos t + 3\sin t]$

1(b) $5\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} - 2x = 6,\ x(0) = \dot{x}(0) = 1$

Taking Laplace transforms

$$(5s^2 - 3s - 2)X(s) = 5(s+1) - 3(1) + \frac{6}{s} = \frac{5s^2+2s+6}{s}$$
$$X(s) = \frac{5s^2+2s+6}{5s(s+\frac{2}{5})(s+1)}$$
$$= -\frac{3}{5} + \frac{\frac{13}{7}}{s-1} + \frac{\frac{15}{7}}{s+\frac{2}{5}}$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = -3 + \dfrac{13}{7}e^{t} + \dfrac{15}{7}e^{-\frac{2}{5}t}$

- **2(a)**

$$\frac{1}{(s+1)(s+2)(s^2+2s+2)} = \frac{1}{s+1} - \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\cdot\frac{s+2}{s^2+2s+2}$$
$$= \frac{1}{s+1} - \frac{1}{2}\cdot\frac{1}{s+2} - \frac{1}{2}\cdot\frac{(s+1)+1}{(s+1)^2+1}$$

Thus , $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s+2)(s^2+2s+2)}\right\} = e^{-t} - \dfrac{1}{2}e^{-2t} - \dfrac{1}{2}e^{-t}(\cos t + \sin t)$

2(b) From equation (5.26) in the text the equation is readily deduced.

Taking Laplace transforms

$$(s^2+3s+2)I(s) = s + 2 + 3 + V.\frac{1}{(s+1)^2+1}$$

$$I(s) = \frac{s+5}{(s+2)(s+1)} + V\Big[\frac{1}{(s+2)(s+1)(s^2+2s+2)}\Big]$$
$$= \frac{4}{s+1} - \frac{3}{s+2} + V\big[\text{extended as in (a)}\big]$$

Thus, using the result of (a) above

$$i(t) = \mathcal{L}^{-1}\{I(s)\} = 4e^{-t} - 3e^{-2t} + V\big[e^{-t} - \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-t}(\cos t + \sin t)\big]$$

■ **3** Taking Laplace transforms

$$(s^2-1)X(s) + 5sY(s) = \frac{1}{s^2}$$
$$-2sX(s) + (s^2-4)Y(s) = -\frac{2}{s}$$

Eliminating $Y(s)$

$$[(s^2-1)(s^2-4) + 2s(5s)]X(s) = \frac{s^2-4}{s^2} + 10 = \frac{11s^2-4}{s^2}$$

$$X(s) = \frac{11s^2-4}{s^2(s^2+1)(s^2+4)}$$
$$= -\frac{1}{s^2} + \frac{5}{s^2+1} - \frac{4}{s^2+4}$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = -t + 5\sin t - 2\sin 2t$
From the first differential equation

$$\frac{dy}{dt} = \frac{1}{5}\big[t + x - \frac{d^2x}{dt^2}\big]$$
$$= \frac{1}{5}[t - t + 5\sin t - 2\sin 2t + 5\sin t - 8\sin 2t]$$
$$= (2\sin t - 2\sin 2t)$$

then $y = -2\cos t + \cos 2t + \text{const.}$

and since $y(0) = 0$, constant $= 1$ giving

$$y(t) = 1 - 2\cos t + \cos 2t$$
$$x(t) = -t + 5\sin t - 2\sin 2t$$

■ **4** Taking Laplace transforms

$$(s^2+2s+2)X(s) = sx_0 + x_1 + 2x_0 + \frac{s}{s^2+1}$$

$$X(s) = \frac{sx_0 + x_1 + 2x_0}{s^2 + 2s + 2} + \frac{s}{(s^2+1)(s^2+2s+2)}$$
$$= \frac{x_0(s+1) + (x_1 + x_0)}{(s+1)^2 + 1} + \frac{1}{5}\cdot\frac{s+2}{s^2+1} - \frac{1}{5}\cdot\frac{s+4}{(s+1)^2+1}$$

giving

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$
$$= e^{-t}(x_0\cos t + (x_1 + x_0)\sin t) + \frac{1}{5}(\cos t + 2\sin t)$$
$$- \frac{1}{5}e^{-t}(\cos t + 3\sin t)$$

that is,

$$x(t) = \underset{\text{steady state}}{\underbrace{\frac{1}{5}(\cos t + 2\sin t)}} + \underset{\text{transient}}{\underbrace{e^{-t}\big[(x_0 - \frac{1}{5})\cos t + (x_1 + x_0 - \frac{3}{5})\sin t\big]}}$$

Steady state solution is $x_s(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t \equiv A\cos(t-\alpha)$
having amplitude $A = \sqrt{(\frac{1}{5})^2 + (\frac{2}{5})^2} = \frac{1}{\sqrt{5}}$
and phase lag $\alpha = \tan^{-1} 2 = 63.4°$.

■ **5** Denoting the currents in the primary and secondary circuits by $i_1(t)$ and $i_2(t)$ respectively Kirchoff's second law gives

$$5i_1 + 2\frac{di_1}{dt} + \frac{di_2}{dt} = 100$$
$$20i_2 + 3\frac{di_2}{dt} + \frac{di_1}{dt} = 0$$

Taking Laplace transforms

$$(5+2s)I_1(s) + sI_2(s) = \frac{100}{s}$$
$$sI_1(s) + (3s+20)I_2(s) = 0$$

Eliminating $I_1(s)$

$$[s^2 - (3s+20)(2s+5)]I_2(s) = 100$$

$$I_2(s) = \frac{-100}{5s^2+55s+100} = -\frac{20}{s^2+11s+20}$$
$$= -\frac{20}{(s+\frac{11}{2})^2-\frac{41}{4}} = -\frac{20}{\sqrt{41}}\Big[\frac{1}{(s+\frac{11}{2}-\frac{\sqrt{41}}{2})} - \frac{1}{(s+\frac{11}{2}+\frac{\sqrt{41}}{2})}\Big]$$

giving the current $i_2(t)$ in the secondary loop as

$$i_2(t) = \mathcal{L}^{-1}\{I_2(s)\} = \frac{20}{\sqrt{41}}\big[e^{-(11+\sqrt{41})t/2} - e^{-(11-\sqrt{41})t/2}\big]$$

- **6(a)**

(i)

$$\mathcal{L}\{\cos(wt+\phi)\} = \mathcal{L}\{\cos\phi\cos wt - \sin\phi\sin wt\}$$
$$= \cos\phi\frac{s}{s^2+w^2} - \sin\phi\frac{w}{s^2+w^2}$$
$$= (s\cos\phi - w\sin\phi)/(s^2+w^2)$$

(ii)

$$\mathcal{L}\{e^{-wt}\sin(wt+\phi)\} = \mathcal{L}\{e^{-wt}\sin wt\cos\phi + e^{-wt}\cos wt\sin\phi\}$$
$$= \cos\phi\frac{w}{(s+w)^2+w^2} + \sin\phi\frac{s+w}{(s+w)^2+w^2}$$
$$= [\sin\phi + w(\cos\phi+\sin\phi)]/(s^2+2sw+2w^2)$$

6(b) Taking Laplace transforms

$$(s^2+4s+8)X(s) = (2s+1)+8+\frac{s}{s^2+4}$$
$$= \frac{2s^3+9s^2+9s+36}{(s^2+4)(s^2+4s+8)}$$
$$= \frac{1}{20}\cdot\frac{s+4}{s^2+4} + \frac{1}{20}\cdot\frac{39s+172}{s^2+4s+8}$$
$$= \frac{1}{20}\cdot\frac{s+4}{s^2+4} + \frac{1}{20}\cdot\frac{39(s+2)+47(2)}{(s+2)^2+(2)^2}$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{20}(\cos 2t + 2\sin 2t) + \frac{1}{20}e^{-2t}(39\cos 2t + 47\sin 2t)$.

- **7(a)**

$$\mathcal{L}^{-1}\Big[\frac{s-4}{s^2+4s+13}\Big] = \mathcal{L}^{-1}\Big[\frac{(s+2)-2(3)}{(s+2)^2+3^2}\Big]$$
$$= e^{-2t}[\cos 3t - 2\sin 3t]$$

7(b) Taking Laplace transforms

$$(s+2)Y(s) = -3 + \frac{4}{s} + \frac{2s}{s^2+1} + \frac{4}{s^2+1}$$

$$Y(s) = \frac{-3s^3 + 6s^2 + s + 4}{s(s+2)(s^2+1)}$$

$$= \frac{2}{s} - \frac{5}{s+2} + \frac{2}{s^2+1}$$

Therefore, $y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2 - 5e^{-2t} + 2\sin t$

■ **8** Taking Laplace transforms

$$(s+5)X(s) + 3Y(s) = 1 + \frac{5}{s^2+1} - \frac{2s}{s^2+1} = \frac{s^2 - 2s + 6}{s^2+1}$$

$$5X(s) + (s+3)Y(s) = \frac{6}{s^2+1} - \frac{3s}{s^2+1} = \frac{6 - 3s}{s^2+1}$$

Eliminating $Y(s)$

$$[(s+5)(s+3) - 15]X(s) = \frac{(s+3)(s^2 - 2s + 6)}{s^2+1} - \frac{3(6-3s)}{s^2+1}$$

$$(s^2 + 8s)X(s) = \frac{s^3 + s^2 + 9s}{s^2+1}$$

$$X(s) = \frac{s^2 + s + 9}{(s+8)(s^2+1)} = \frac{1}{s+8} + \frac{1}{s^2+1}$$

so $x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{-8t} + \sin t$

From the first differential equation

$$3y = 5\sin t - 2\cos t - 5x - \frac{dx}{dt} = 3e^{-8t} - 3\cos t$$

Thus, $x(t) = e^{-8t} + \sin t,\ y(t) = e^{-8t} - \cos t\,.$

■ **9** Taking Laplace transforms

$$(s^2+300s+2\times10^4)Q(s) = 200\cdot\frac{100}{s^2+10^4}$$
$$(s+100)(s+200)Q(s) = 10^4\cdot\frac{2}{s^2+10^4}$$
$$Q(s) = \frac{2.10^4}{(s+100)(s+200)(s^2+10^4)}$$
$$= \frac{1}{100}\cdot\frac{1}{s+100} - \frac{2}{500}\cdot\frac{1}{s+200} - \frac{1}{500}\cdot\frac{3s-100}{s^2+10^4}$$

giving $q(t) = \mathcal{L}^{-1}\{Q(s)\} = \frac{1}{100}e^{-100t} - \frac{2}{500}e^{-200t} - \frac{1}{500}(3\cos 100t - \sin 100t)$
that is,

$$q(t) = \frac{1}{500}[5e^{-100t} - 2e^{-200t}] - \frac{1}{500}[3\cos 100t - \sin 100t]$$
$$\uparrow \qquad\qquad \uparrow$$
transient steady state

Steady state current $= \frac{3}{5}\sin 100t + \frac{1}{5}\cos 100t \equiv A\sin(100t+\alpha)$
where $\alpha = \tan^{-1}\frac{1}{5} \simeq 18\frac{1}{2}^{\circ}$.
Hence, the current leads the applied emf by about $18\frac{1}{2}^{\circ}$.

■ **10**

$$4\frac{dx}{dt} + 6x + y = 2\sin 2t \qquad \text{(i)}$$
$$\frac{d^2x}{dt^2} + x - \frac{dy}{dt} = 3e^{-2t} \qquad \text{(ii)}$$

Given $x = 2$ and $\frac{dx}{dt} = -2$ when $t = 0$ so from (i) $y = -4$ when $t = 0$.
Taking Laplace transforms

$$(4s+6)X(s) + Y(s) = 8 + \frac{4}{s^2+4} = \frac{8s^2+36}{s^2+4}$$
$$(s^2+1)X(s) - sY(s) = 2s - 2 + 4 + \frac{3}{s+2} = \frac{2s^2+6s+7}{s+2}$$

Eliminating $Y(s)$

$$[s(4s+6) + (s^2+1)]X(s) = \frac{8s^2+36}{s^2+4} + \frac{2s^2+6s+7}{s+2}$$

$$\begin{aligned} X(s) &= \frac{8s^2+36}{s(s^2+4)(s+1)(s+\frac{1}{5})} + \frac{2s^2+6s+7}{5(s+2)(s+1)(s+\frac{1}{5})} \\ &= \frac{\frac{11}{5}}{s+1} - \frac{\frac{227}{505}}{s+\frac{1}{5}} - \frac{1}{505}\cdot\frac{76s-96}{s^2+4} + \frac{\frac{1}{3}}{s+2} - \frac{\frac{3}{4}}{s+1} + \frac{\frac{49}{60}}{s+\frac{1}{5}} \\ &= \frac{\frac{29}{20}}{s+1} + \frac{\frac{1}{3}}{s+2} + \frac{\frac{445}{1212}}{s+\frac{1}{5}} - \frac{1}{505}\Big[\frac{76s-96}{s^2+4}\Big] \end{aligned}$$

giving

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{29}{20}e^{-t} + \frac{1}{3}e^{-2t} + \frac{445}{1212}e^{-\frac{1}{5}t} - \frac{1}{505}(76\cos 2t - 48\sin 2t)$$

■ **11(a)** Taking Laplace transforms

$$\begin{aligned} (s^2+8s+16)\Phi(s) &= \frac{2}{s^2+4} \\ \Phi(s) &= \frac{2}{(s+4)^2(s^2+4)} \\ &= \frac{1}{25}\cdot\frac{1}{s+4} + \frac{1}{10}\cdot\frac{1}{(s+4)^2} - \frac{1}{50}\cdot\frac{2s-3}{s^2+4} \end{aligned}$$

so $\theta(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \frac{1}{25}e^{-4t} + \frac{1}{10}\cdot te^{-4t} - \frac{1}{100}(4\cos 2t - 3\sin 2t)$

that is, $\theta(t) = \frac{1}{100}(4e^{-4t} + 10te^{-4t} - 4\cos 2t + 3\sin 2t)$

11(b) Taking Laplace transforms

$$\begin{aligned} (s+2)I_1(s) + 6I_2(s) &= 1 \\ I_1(s) + (s-3)I_2(s) &= 0 \end{aligned}$$

Eliminating $I_2(s)$

$$\begin{aligned} [(s+2)(s-3)-6]I_1(s) &= s-3 \\ I_1(s) &= \frac{s-3}{(s-4)(s+3)} = \frac{\frac{1}{7}}{s-4} + \frac{\frac{6}{7}}{s+3} \end{aligned}$$

giving $i_1(t) = \mathcal{L}^{-1}\{I_1(s)\} = \frac{1}{7}(e^{4t} + 6e^{-3t})$

Then from the first differential equation

$$6i_2 = -2i_1 - \frac{di_1}{dt} = -\frac{6}{7}e^{4t} + \frac{6}{7}e^{-3t}$$
$$\text{giving } i_2(t) = \frac{1}{7}(e^{-3t} - e^{4t}),\ i_1(t) = \frac{1}{7}(e^{4t} + 6e^{-3t}).$$

■ **12** The differential equation

$$LCR\,\frac{d^2i}{dt^2} + L\frac{di}{dt} + Ri = V$$

follows using Kirchhoff's second law.
Substituting $V = E$ and $L = 2R^2C$ gives

$$2R^3C^2\frac{d^2i}{dt^2} + 2R^2C\frac{di}{dt} + Ri = E$$

which on substituting $CR = \dfrac{1}{2n}$ leads to

$$\frac{1}{2n^2}\frac{d^2i}{dt^2} + \frac{1}{n}\frac{di}{dt} + i = \frac{E}{R}$$

and it follows that

$$\frac{d^2i}{dt^2} + 2n\frac{di}{dt} + 2n^2i = 2n^2\frac{E}{R}$$

Taking Laplace transforms

$$(s^2 + 2ns + 2n^2)I(s) = \frac{2n^2E}{R}\cdot\frac{1}{s}$$

$$I(s) = \frac{E}{R}\Big[\frac{2n^2}{s(s^2 + 2ns + 2n^2)}\Big]$$
$$= \frac{E}{R}\Big[\frac{1}{s} - \frac{s + 2n}{(s + n)^2 + n^2}\Big]$$

so that

$$i(t) = \frac{E}{R}[1 - e^{-nt}(\cos nt + \sin nt)]$$

■ **13** The equations are readily deduced by applying Kirchhoff's second law to the left- and right-hand circuits.

Note that from the given initial conditions we deduce that $i_2(0) = 0$.

Taking Laplace transforms then gives

$$\begin{aligned}(sL + 2R)I_1(s) - RI_2(s) &= \frac{E}{s}\\ -RI_1(s) + (sL + 2R)I_2(s) &= 0\end{aligned}$$

Eliminating $I_2(s)$

$$\begin{aligned}[(sL + 2R)^2 - R^2]I_1(s) &= \frac{E}{s}(sL + 2R)\\ (sL + 3R)(sL + R)I_1(s) &= \frac{E}{s}(sL + 2R)\end{aligned}$$

$$\begin{aligned}I_1(s) &= \frac{E}{L}\Big[\frac{s + \frac{2R}{L}}{s(s + \frac{R}{L})(s + \frac{3R}{L})}\Big]\\ &= \frac{E}{R}\Big[\frac{\frac{2}{3}}{s} - \frac{\frac{1}{2}}{s + \frac{R}{L}} - \frac{\frac{1}{6}}{s + \frac{3R}{L}}\Big]\\ \text{giving } i_1(t) = \mathcal{L}^{-1}\{I_1(s)\} &= \frac{1}{6}\frac{E}{R}\big[4 - 3e^{-\frac{R}{L}t} - e^{-\frac{3R}{L}t}\big]\end{aligned}$$

For large t, the exponential terms are approximately zero and

$$i_1(t) \simeq \frac{2}{3}\frac{E}{R}$$

From the first differential equation

$$Ri_2 = 2Ri_1 + L\frac{di_1}{dt} - E$$

Ignoring the exponential terms we have that for large t

$$i_2 \simeq \frac{4}{3}\frac{E}{R} - \frac{E}{R} = \frac{1}{3}\frac{E}{R}$$

■ **14** Taking Laplace transforms

$$(s^2+2)X_1(s) - X_2(s) = \frac{2}{s^2+4}$$
$$-X_1(s) + (s^2+2)X_2(s) = 0$$

Eliminating $X_1(s)$

$$[(s^2+2)^2 - 1]X_2(s) = \frac{2}{s^2+4}$$

$$X_2(s) = \frac{2}{(s^2+4)(s^2+1)(s^2+3)}$$
$$= \frac{\frac{2}{3}}{s^2+4} + \frac{\frac{1}{3}}{s^2+1} - \frac{1}{s^2+3}$$
$$\text{so } x_2(t) = \mathcal{L}^{-1}\{X_2(s)\} = \frac{1}{3}\sin 2t + \frac{1}{3}\sin t - \frac{1}{\sqrt{3}}\sin\sqrt{3}t$$

Then from the second differential equation

$$x_1(t) = 2x_2 + \frac{d^2x_2}{dt^2} = \frac{2}{3}\sin 2t + \frac{2}{3}\sin t - \frac{2}{\sqrt{3}}\sin\sqrt{3}t - \frac{4}{3}\sin 2t - \frac{1}{3}\sin t + \sqrt{3}\sin\sqrt{3}t$$
$$\text{or } x_1(t) = -\frac{2}{3}\sin 2t + \frac{1}{3}\sin t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t$$

■ **15(a)**

(i)

$$\mathcal{L}^{-1}\Big\{\frac{s+4}{s^2+2s+10}\Big\} = \mathcal{L}^{-1}\Big\{\frac{(s+1)+3}{(s+1)^2+3^2}\Big\}$$
$$= e^{-t}(\cos 3t + \sin 3t)$$

(ii)

$$\mathcal{L}^{-1}\Big\{\frac{s-3}{(s-1)^2(s-2)}\Big\} = \mathcal{L}^{-1}\Big\{\frac{1}{(s-1)} + \frac{2}{(s-1)^2} - \frac{1}{s-2}\Big\}$$
$$= e^t + 2te^t - e^{2t}$$

15(b) Taking Laplace transforms

$$\begin{aligned}(s^2+2s+1)Y(s) &= 4s+2+8+\mathcal{L}\{3te^{-t}\}\\ (s+1)^2Y(s) &= 4s+10+\frac{3}{(s+1)^2}\\ Y(s) &= \frac{4s+10}{(s+1)^2}+\frac{3}{(s+1)^4}\\ &= \frac{4}{s+1}+\frac{6}{(s+1)^2}+\frac{3}{(s+1)^4}\\ \text{giving } y(t)=\mathcal{L}^{-1}\{Y(s)\} &= 4e^{-t}+6te^{-t}+\frac{1}{2}t^3e^{-t}\\ \text{that is, } y(t) &= \frac{1}{2}e^{-t}(8+12t+t^3)\end{aligned}$$

- **16(a)**

$$F(s)=\frac{5}{s^2-14s+53}=\frac{5}{2}\cdot\frac{2}{(s-7)^2+2^2}$$

$$\text{Therefore, } f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{5}{2}e^{7t}\sin 2t$$

16(b) $\dfrac{d^2\theta}{dt^2}+2K\dfrac{d\theta}{dt}+n^2\theta=\dfrac{n^2i}{K}$, $\theta(0)=\dot\theta(0)=0$, i const.

Taking Laplace transforms

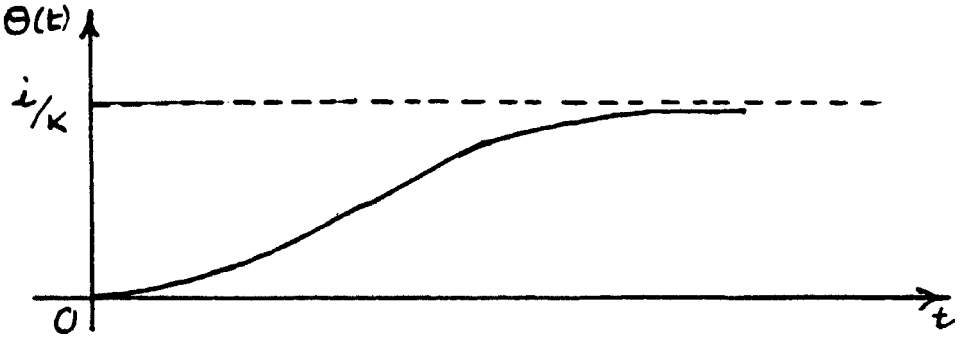
$$\begin{aligned}(s^2+2Ks+n^2)\Phi(s) &= \frac{n^2}{K}\frac{i}{s}\\ \text{Therefore, } \Phi(s) &= \frac{n^2i}{Ks(s^2+2Ks+n^2)}\end{aligned}$$

For the case of critical damping $n=K$ giving

$$\Phi(s)=\frac{Ki}{s(s+K)^2}=Ki\big[\tfrac{\frac{1}{K^2}}{s}-\tfrac{\frac{1}{K^2}}{s+K}-\tfrac{\frac{1}{K}}{(s+K)^2}\big]$$

Thus,

$$\theta(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \frac{i}{K}[1 - e^{-Kt} - Kte^{-Kt}]$$



- **17(a)**

(i)

$$\begin{aligned}\mathcal{L}\{\sin tH(t-\alpha)\} &= \mathcal{L}\{\sin[(t-\alpha)+\alpha]H(t-\alpha)\} \\ &= \mathcal{L}\{[\sin(t-\alpha)\cos\alpha + \cos(t-\alpha)\sin\alpha]H(t-\alpha)\} \\ &= \frac{\cos\alpha + s\sin\alpha}{s^2+1}.\, e^{-\alpha s}\end{aligned}$$

(ii)

$$\begin{aligned}\mathcal{L}^{-1}\frac{se^{-\alpha s}}{s^2+2s+5} &= \mathcal{L}^{-1}\Big\{e^{-\alpha s}\frac{(s+1)-1}{(s+1)^2+4}\Big\} \\ &= \mathcal{L}^{-1}\Big\{e^{\alpha s}\mathcal{L}[e^{-t}(\cos 2t - \frac{1}{2}\sin 2t)]\Big\} \\ &= e^{-(t-\alpha)}[\cos 2(t-\alpha) - \frac{1}{2}\sin 2(t-\alpha)]H(t-\alpha)\end{aligned}$$

17(b) Taking Laplace transforms

$$\begin{aligned}(s^2+2s+5)Y(s) &= \frac{1}{s^2+1} - \Big[\frac{-e^{-s\pi}}{s^2+1}\Big] \text{ by (i) above in part (a)} \\ &= \frac{1+e^{-\pi s}}{s^2+1}\end{aligned}$$

$$Y(s) = \frac{1+e^{-\pi s}}{(s^2+1)(s^2+2s+5)} = \Big[-\frac{1}{10}\frac{s-2}{s^2+1} + \frac{1}{10}\frac{s}{s^2+2s+5}\Big](1+e^{-\pi s})$$

giving

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \frac{1}{10}[2\sin t - \cos t + e^{-(t-\pi)}[2\sin(t-\pi) - \cos(t-\pi)]H(t-\pi)] \\ &\quad + e^{-t}(\cos 2t - \frac{1}{2}\sin 2t) + e^{-(t-\pi)}[\cos 2(t-z) \\ &\quad - \frac{1}{2}\sin 2(t-\pi)]H(t-\pi)] \\ &= \frac{1}{10}\big[e^{-t}(\cos 2t - \frac{1}{2}\sin 2t) + 2\sin t - \cos t \\ &\quad + [e^{-(t-\pi)}(\cos 2t - \frac{1}{2}\sin 2t) + \cos t - 2\sin t]H(t-\pi)\big] \end{aligned}$$

■ **18** By theorem 5.5

$$\begin{aligned} \mathcal{L}\{v(t)\} = V(s) &= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}v(t)dt \\ &= \frac{1}{1-e^{-sT}}\left[\int_0^{T/2} e^{-st}dt - \int_{T/2}^{T} e^{-st}dt\right] \\ &= \frac{1}{1-e^{-sT}}\left\{\big[-\frac{1}{s}e^{-st}\big]_0^{T/2} - \big[-\frac{1}{s}e^{-st}\big]_{T/2}^{T}\right\} \\ &= \frac{1}{s}\cdot\frac{1}{1-e^{-sT}}(e^{-sT} - e^{-sT/2} - e^{-sT/2} + 1) \\ &= \frac{1}{s}\frac{(1-e^{-sT/2})^2}{(1-e^{-sT/2})(1+e^{-sT/2})} = \frac{1}{s}\big[\frac{1-e^{-sT/2}}{1+e^{-sT/2}}\big] \end{aligned}$$

Equation for current flowing is

$$250i + \frac{1}{C}(q_0 + \int_0^t i(\tau)d\tau) = v(t), \qquad q_0 = 0$$

Taking Laplace transforms

$$\begin{aligned} 250I(s) + \frac{1}{10^{-4}}\cdot\frac{1}{s}\cdot I(s) = V(s) &= \frac{1}{s}\big[\frac{1-e^{-sT/2}}{1+e^{-sT/2}}\big] \\ (s+40)I(s) &= \frac{1}{250}\big[\frac{1-e^{-sT/2}}{1+e^{-sT/2}}\big] \\ \text{or } I(s) &= \frac{1}{250(s+40)}\cdot\frac{1-e^{-sT/2}}{1+e^{-sT/2}} \end{aligned}$$

$$I(s) = \frac{1}{250(s+40)}(1 - e^{-sT/2})(1 - e^{-sT/2} + e^{-sT} - e^{-\frac{3}{2}sT} + e^{-2sT} \ldots)$$
$$= \frac{1}{250(s+40)}[1 - 2e^{-sT/2} + 2e^{-sT} - 2e^{-\frac{3}{2}sT} + 2e^{-2sT} \ldots]$$

Since $\mathcal{L}^{-1}\{\frac{1}{250(s+40)}\} = \frac{1}{250}e^{-40t}$ using the second shift theorem gives

$$i(t) = \frac{1}{250}\Big[e^{-40t} - 2H\big(t - \frac{T}{2}\big)e^{-40(t-T/2)} + 2H(t-T)e^{-40(t-T)}$$
$$-2H\big(t - \frac{3T}{2}\big)e^{-40(t-3T/2)} + \ldots\Big]$$

If $T = 10^{-3}$ s then the first few terms give a good representation of the steady state since the time constant $\frac{1}{4}$ of the circuit is large compared to the period T.

■ **19** The impulse response $h(t)$ is the solution of

$$\frac{d^2h}{dt^2} + \frac{2dh}{dt} + 2h = \delta(t)$$

subject to the initial conditions $h(0) = \dot{h}(0) = 0$. Taking Laplace transforms

$$(s^2 + 2s + s)H(s) = \mathcal{L}\{\delta(t)\} = 1$$
$$H(s) = \frac{1}{(s+1)^2 + 1}$$
$$\text{that is, } h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-t}\sin t.$$

Using the convolution integral the step response $x_s(t)$ is given by

$$x_s(t) = \int_0^t h(\tau)u(t-\tau)d\tau$$

with $u(t) = 1H(t)$; that is,

$$x_s(t) = \int_0^t 1.e^{-\tau}\sin\tau d\tau$$
$$= -\frac{1}{2}[e^{-\tau}\cos\tau + e^{-\tau}\sin\tau]_0^t$$
$$\text{that is, } x_s(t) = \frac{1}{2}[1 - e^{-t}(\cos t + \sin t)].$$

Solving $\dfrac{d^2x_s}{dt^2} + \dfrac{2dx_s}{dt} + 2x_s = 1$ directly we have taking Laplace transforms

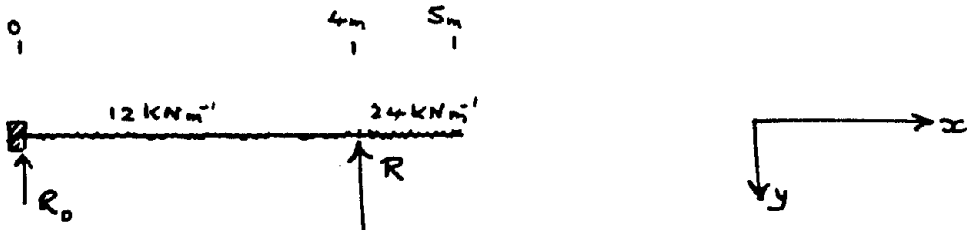
$$(s^2 + 2s + 2)X_s(s) = \frac{1}{s}$$

$$\begin{aligned} X_s(s) &= \frac{1}{s(s^2 + 2s + 2)} \\ &= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\Big[\frac{s+2}{(s+1)^2 + 1}\Big] \end{aligned}$$

giving as before

$$x_s(t) = \frac{1}{2} - \frac{1}{2}e^{-t}(\cos t + \sin t)$$

■ **20**



$$EI\frac{d^4y}{dx^4} = 12 + 12H(x-4) - R\delta(x-4)$$
$$y(0) = y'(0) = 0,\ y(4) = 0,\ y''(5) = y'''(5) = 0$$

With $y''(0) = A, y'''(0) = B$ taking Laplace transforms

$$EIs^4Y(s) = EI(sA + B) + \frac{12}{s} + \frac{12}{s}e^{-4s} - Re^{-4s}$$
$$Y(s) = \frac{A}{s^3} + \frac{B}{s^4} + \frac{12}{EI}\cdot\frac{1}{s^5} + \frac{12}{EI}\cdot\frac{1}{s^5}e^{-4s} - \frac{R}{EI}\cdot\frac{1}{s^4}e^{-4s}$$

giving

$$\begin{aligned} y(x) = \mathcal{L}^{-1}\{Y(s)\} =& \frac{A}{2}x^2 + \frac{B}{6}x^3 + \frac{1}{2EI}x^4 + \frac{1}{2EI}(x-4)^4H(x-4) \\ &- \frac{R}{6EI}(x-4)^3H(x-4) \end{aligned}$$

or

$$EIy(x) = \frac{1}{2}A_1x^2 + \frac{1}{6}B_1x^3 + \frac{1}{2}x^4 + \frac{1}{2}(x-4)^4H(x-4) - \frac{R}{6}(x-4)^3H(x-4)$$

$$\begin{aligned} y(4) = 0 &\Rightarrow 0 = 8A_1 + \frac{32}{3}B_1 + 128 \Rightarrow 3A_1 + 4B_1 = -48 \\ y''(5) = 0 &\Rightarrow 0 = A_1 + 5B_1 + 6(25) + 6 - R \Rightarrow A_1 + 5B_1 - R = -156 \\ y'''(5) = 0 &\Rightarrow 0 = B_1 + 12(5) + 12 - R \Rightarrow B_1 - R = -72 \end{aligned}$$

which solve to give $A_1 = 18,\ B_1 = -25.5,\ R = 46.5$

Thus,

$$y(x) = \begin{cases} \dfrac{1}{2}x^4 - 4.25x^3 + 9x^2,\ 0 \le x \le 4 \\ \dfrac{1}{2}x^4 - 4.25x^3 + 9x^2 + \dfrac{1}{2}(x-4)^4 - 7.75(x-4)^3,\ 4 \le x \le 5 \end{cases}$$

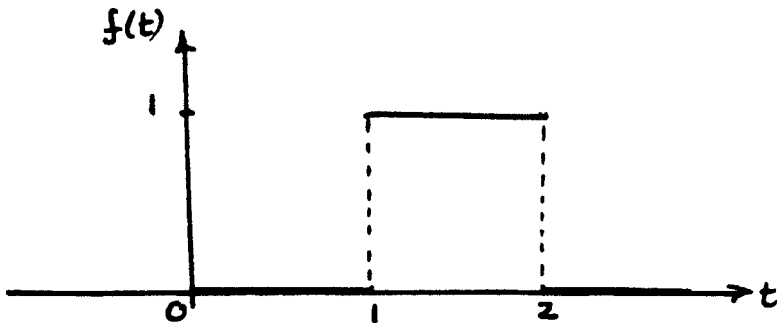
$$R_0 = -EIy'''(0) = 25.5\text{kN},\ M_0 = EIy''(0) = 18\text{kN.m}$$

Check: $R_0 + R = 72\text{kN}$, Total load $= 12 \times 4 + 24 = 72\text{kN}$ $\surd$

Moment about $x = 0$ is

$$12 \times 4 \times 2 + 24 \times 4.5 - 4R = 18 = M_0 \quad \surd$$

- **21(a)**



$f(t) = H(t-1) - H(t-2)$

and $\mathcal{L}\{f(t)\} = F(s) = \dfrac{e^{-s}}{s} - \dfrac{e^{-2s}}{s}$

Taking Laplace transforms throughout the differential equation

$$\begin{aligned}(s+1)X(s) &= \frac{1}{s}(e^{-s}-e^{-2s})\\ X(s) &= \frac{1}{s(s+1)}(e^{-s}-e^{-2s})\\ &= \big[\frac{1}{s}-\frac{1}{s+1}\big]e^{-s}-\big[\frac{1}{s}-\frac{1}{s+1}\big]e^{-2s}\end{aligned}$$

giving $x(t)=\mathcal{L}^{-1}\{X(s)\}=[1-e^{-(t-1)}]H(t-1)-[1-e^{-(t-2)}]H(t-2)$

21(b) $I(s)=\dfrac{E}{s[Ls+R/(1+Cs)]}$

(i) By the initial value theorem

$$\lim_{t\to 0} i(t)=\lim_{s\to\infty} sI(s)=\lim_{s\to\infty}\frac{E}{Ls+R/(1+Cs)}=0$$

(ii) Since $sI(s)$ has all its poles in the left half of the s-plane the conditions of the final value theorem hold so

$$\lim_{t\to\infty} i(t)=\lim_{s\to 0} sI(s)=\frac{E}{R}$$

■ **22** We have that for a periodic function $f(t)$ of period T

$$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-sT}}\int_0^T e^{-sT}f(t)dt$$

Thus, the Laplace transform of the half-rectified sine wave is

$$\begin{aligned}\mathcal{L}\{v(t)\} &= \frac{1}{1-e^{-2\pi s}}\int_0^\pi e^{-sT}\sin t dt\\ &= I_m\big\{\frac{1}{1-e^{-2\pi s}}\int_0^\pi e^{(j-s)t}dt\big\}\\ &= I_m\big\{\frac{1}{1-e^{-2\pi s}}\big[\frac{e^{(j-s)t}}{j-s}\big]_0^\pi\big\}\\ &= I_m\big\{\frac{1}{1-e^{-2\pi s}}\big[\frac{(-e^{-\pi s}-1)(-j-s)}{(j-s)(-j-s)}\big]\big\}=\frac{1+e^{-\pi s}}{(1-e^{-2\pi s})(1+s^2)}\end{aligned}$$

that is, $\mathcal{L}\{v(t)\}=\dfrac{1}{(1+s^2)(1-e^{-\pi s})}$

Applying Kirchoff's law to the circuit the current is determined by

$$\frac{di}{dt} + i = v(t)$$

which on taking Laplace transforms gives

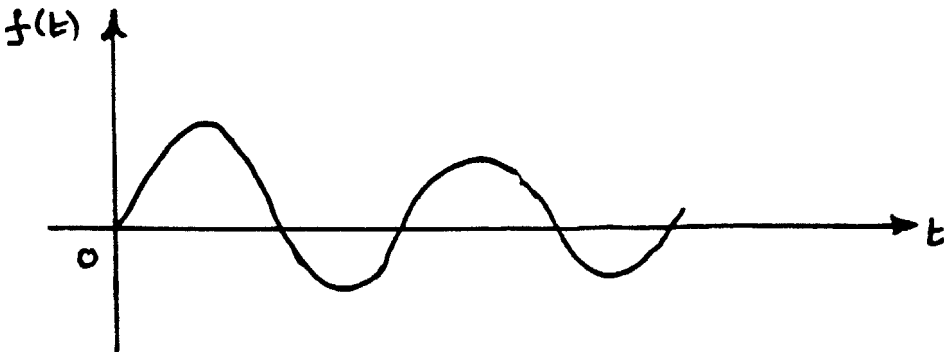
$$\begin{aligned}(s+1)I(s) &= \frac{1}{(1+s^2)(1-e^{-\pi s})} \\ I(s) &= \frac{1}{1-e^{-\pi s}}\Big[\frac{1}{s+1} - \frac{s+1}{s^2+1}\Big]\cdot\frac{1}{2} \\ &= \frac{1}{2}\Big[\frac{1}{s+1} - \frac{s+1}{s^2+1}\Big]\big[1 + e^{-\pi s} + e^{-2\pi s} + \ldots\big]\end{aligned}$$

Since $\mathcal{L}^{-1}\big\{\frac{1}{2}\big[\frac{1}{s+1} - \frac{s+1}{s^2+1}\big]\big\} = \frac{1}{2}(\sin t - \cos t + e^{-t})H(t) = f(t)$

we have by the second shift theorem that

$$i(t) = f(t) + f(t-\pi) + f(t-2\pi) + \ldots = \sum_{n=0}^{\infty} f(t-n\pi)$$

The graph may be plotted by computer and should take the form



■ **23(a)** Since $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2}$

taking $f(t) = t$ and $g(t) = te^{-t}$ in the convolution theorem

$$\mathcal{L}^{-1}[F(s)G(s)] = f * g(t)$$

gives

$$\begin{aligned}\mathcal{L}^{-1}\big[\frac{1}{s^2}\cdot\frac{1}{(s+1)^2}\big] &= \int_0^t f(t-\tau)g(\tau)d\tau \\ &= \int_0^t (t-\tau)\tau e^{-\tau} \\ &= \big[-(t-\tau)\tau e^{-\tau} - (t-2\tau)e^{-\tau} + 2e^{-\tau}\big]_0^t \\ \text{i.e. } \mathcal{L}^{-1}\big[\frac{1}{s^2}\cdot\frac{1}{(s+2)^2}\big] &= t-2+2e^{-t}+te^{-t}.\end{aligned}$$

23(b) $y(t) = t + 2\int_0^t y(u)\cos(t-u)du$

Taking $f(t) = y(t), g(t) = \cos t \Rightarrow F(s) = Y(s), G(s) = \dfrac{s}{s^2+1}$ giving on taking transforms

$$\begin{aligned}Y(s) &= \frac{1}{s^2} + 2Y(s)\frac{s}{s^2+1} \\ (s^2+1-2s)Y(s) &= \frac{s^2+1}{s^2} \\ \text{or } Y(s) = \frac{s^2+1}{s^2(s-1)^2} &= \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{2}{(s-1)^2} \\ \text{and } y(t) = \mathcal{L}^{-1}\{Y(s)\} &= 2 + t - 2e^t + 2te^t.\end{aligned}$$

Taking transforms

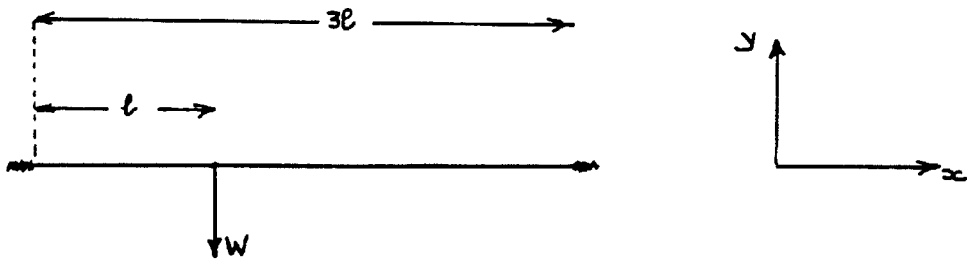
$$\begin{aligned}(s^2Y(s) - sy(0) - y'(0))(sY(s) - y(0)) &= Y(s) \\ \text{or } (s^2Y(s) - y_1)(sY(s)) &= Y(s) \\ \text{giving } Y(s) = 0 \text{ or } Y(s) &= \frac{y_1}{s^2} + \frac{1}{s^3}\end{aligned}$$

which on inversion gives

$$y(t) = 0 \text{ or } y(t) = \frac{1}{2}t^2 + ty_1$$

In the second of these solutions the condition on $y'(0)$ is arbitrary.

■ **24**



Equation for displacement is

$$EI\frac{d^4y}{dx^4} = -W\delta(x-\ell)$$

with $y(0)=0,\ y(3\ell)=0,\ y'(0)=y'(3\ell)=0$
with $y''(0)=A,\ y'''(0)=B$ then taking Laplace transforms gives

$$\begin{aligned} EIs^4Y(s) &= EI(sA+B) - We^{-\ell s} \\ Y(s) &= \frac{-W}{EIs^4}e^{-\ell s} + \frac{A}{s^3} + \frac{B}{s^4} \\ \text{giving } y(x) &= \frac{-W}{6EI}(x-\ell)^3.H(x-\ell) + \frac{A}{2}x^2 + \frac{B}{6}x^3 \end{aligned}$$

For $x>\ell, y'(x) = \dfrac{-3W}{6EI}(x-\ell)^2 + Ax + \dfrac{B}{2}x^2$
so $y'(3\ell)=0$ and $y(3\ell)=0$ gives

$$\begin{aligned} 0 &= -\frac{2W\ell^2}{EI} + 3A\ell + 9B\frac{\ell^2}{2} \\ 0 &= -\frac{4W\ell^3}{3EI} + \frac{9}{2}A\ell^2 + \frac{9}{2}B\ell^3 \\ \text{giving } A &= -\frac{4W\ell}{9EI} \text{ and } B = \frac{20}{27}\frac{W}{EI} \end{aligned}$$

Thus, deflection $y(x)$ is

$$y(x) = -\frac{W}{6EI}(x-\ell)^3H(x-\ell) - \frac{2}{9}\frac{W\ell}{EI}x^2 + \frac{10}{81}\frac{W}{EI}x^3$$

With the added uniform load the differential equation governing the deflection is

$$EI\frac{d^4y}{dx^4} = -W\delta(x-\ell) - w[H(x) - H(x-\ell)]$$

■ **25(a)** Taking Laplace transforms

$$(s^2 - 3s + 3)X(s) = \frac{1}{s}e^{-as}$$

$$X(s) = \frac{1}{s(s^2-3s+3)}\cdot e^{-as} = \big[\tfrac{\frac{1}{6}}{s} - \frac{\frac{1}{6}s - \frac{1}{2}}{s^2-3s+3}\big]\cdot e^{-as}$$

$$= \frac{1}{6}\big[\frac{1}{s} - \frac{(s-\frac{3}{2}) - \sqrt{3}(\frac{\sqrt{3}}{2})}{(s-\frac{3}{2})^2 + (\frac{\sqrt{3}}{2})^2}\big]e^{-as}$$

$$= \frac{e^{-as}}{6}\mathcal{L}\left\{1 - e^{-\frac{3}{2}t}\big(\cos\frac{\sqrt{3}}{2}t - \sqrt{3}\sin\frac{\sqrt{3}}{2}t\big)\right\}$$

giving

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{6}\left[1 - e^{-\frac{3}{2}(t-a)}\big(\cos\frac{\sqrt{3}}{2}(t-a) - \sqrt{3}\sin\frac{\sqrt{3}}{2}(t-a)\big)\right]H(t-a)$$

25(b)

$$X(s) = G(s)\mathcal{L}\{\sin wt\} = G(s)\frac{w}{s^2+w^2}$$

$$= \frac{w}{(s+jw)(s-jw)}G(s)$$

Since the system is stable all the poles of $G(s)$ have negative real part. Expanding in partial fractions and inverting gives

$$x(t) = 2R_e\big[\frac{F(jw)w}{2jw}\cdot e^{jwt}\big] + \text{ terms from } G(s) \text{ with negative exponentials}$$

Thus, as $t \to \infty$ the added terms tend to zero and $x(t) \to x_s(t)$ with

$$x_s(t) = R_e\big[\frac{e^{jwt}F(jw)}{j}\big]$$

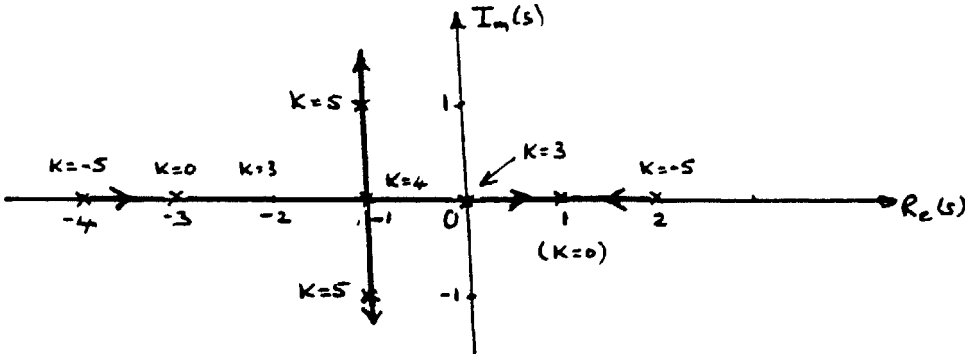
26(a) In the absence of feedback the system has poles at

$$s = -3 \text{ and } s = 1$$

and is therefore unstable.

26(b) $G_1(s) = \dfrac{G(s)}{1 + KG(s)} = \dfrac{1}{(s-1)(s+3) + K} = \dfrac{1}{s^2 + 2s + (K-3)}$

26(c) Poles $G_1(s)$ given by $s = -1 \pm \sqrt{4 - K}$.
These may be plotted in the s-plane for different values of K. Plot should be as in the figure



26(d) Clearly from the plot in (c) all the poles are in the left half plane when $K > 3$. Thus system stable for $K > 3$.

26(e)

$$\begin{matrix} a_2 & & a_1 & & a_0 \\ 1s^2 & + & 2s & + & (K-3) = 0 \end{matrix}$$

Routh–Hurwitz determinants are

$$\Delta_1 = 2 > 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_2 \\ 0 & a_0 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & K-3 \end{vmatrix} = 2(K-3) > 0 \text{ if } K > 3$$

thus, confirming the result in (d).

■ **27(a)** Closed loop transfer function is

$$G_1(s) = \frac{G(s)}{1+G(s)} = \frac{2}{s^2+\alpha s+5}$$

Thus $\mathcal{L}^{-1}\{\dfrac{2}{s^2+\alpha s+5}\} = h(t) = 2e^{-2t}\sin t$
i.e. $\mathcal{L}^{-1}\{\dfrac{2}{(s+\frac{\alpha}{2})^2+(5-\frac{\alpha^2}{4})}\} = 2e^{-\frac{\alpha}{2}t}\sin\sqrt{(5-\frac{\alpha^2}{4})}t = 2e^{-2t}\sin t$
giving $\alpha = 4$

27(b) Closed loop transfer function is

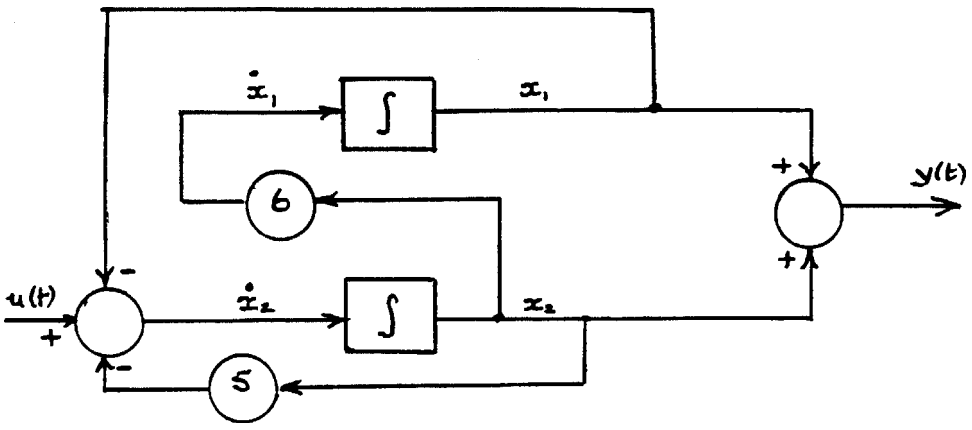
$$G(s) = \frac{\frac{10}{s(s-1)}}{1-\frac{(1+Ks)10}{s(s-1)}} = \frac{10}{s^2+(10K-1)s+10}$$

Poles of the system are given by

$$s^2+(10K-1)s+10 = 0$$

which are both in the negative half plane of the s-plane provided $(10K-1) > 0$; that is, $K > \frac{1}{10}$. Thus the critical value of K for stability of the closed loop system is $K = \frac{1}{10}$.

■ **28(a)**



28(b)

$$\mathcal{L}\{e^{\mathbf{A}t}\} = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{(s+2)(s+3)} \begin{bmatrix} s+5 & 6 \\ -1 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3}{s+2} - \frac{2}{s+3} & \frac{6}{s+2} - \frac{6}{s+3} \\ \frac{1}{s+3} - \frac{1}{s+2} & \frac{-2}{s+2} - \frac{1}{s+3} \end{bmatrix}$$

Taking inverse transforms gives

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 6e^{-2t} - 6e^{-3t} \\ e^{-3t} - e^{-2t} & 3e^{-3t} - 2e^{-2t} \end{bmatrix}$$

28(c) Taking Laplace transforms

$$[s\mathbf{I} - \mathbf{A}]\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{b}U(s) \; ; \; Y(s) = \mathbf{c}^T\mathbf{X}(s)$$

With $\mathbf{x}(0) = 0$ and $U(s) = 1$ the transform $\mathbf{X}_\delta(s)$ of the impulse response is

$$\mathbf{X}_\delta(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b} \; , \; Y_\delta(s) = \mathbf{c}^T[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}$$

Inverting then gives the impulse response as

$$y_\delta(t) = [1 \; 1] \begin{bmatrix} 6e^{-2t} - 6e^{-3t} \\ 3e^{-3t} - 2e^{-2t} \end{bmatrix} = 4e^{-2t} - 3e^{-3t}, \; t \geq 0$$

With $\mathbf{x}(0) = [1 \; 0]^T$ and $X(s) = \dfrac{1}{s}$

$$Y(s) = [1 \; 1] \left[[s\mathbf{I} - \mathbf{A}]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + [sI - A]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \right]$$
$$= [1 \; 1] \begin{bmatrix} \frac{3}{s+2} - \frac{2}{s+3} + 6\left(\frac{1/6}{s} - \frac{1/2}{s+2} + \frac{1/3}{s+3}\right) \\ \frac{1}{s+3} - \frac{1}{s+2} + \left(\frac{1}{s+2} - \frac{1}{s+3}\right) \end{bmatrix}$$
$$\text{so } y(t) = [1 \; 1] \begin{bmatrix} 3e^{-2t} - 2e^{-3t} + 1 - 3e^{-2t} + 2e^{-3t} \\ e^{-3t} - e^{-2t} + e^{-2t} - e^{-3t} \end{bmatrix}$$
$$\text{that is, } \; y(t) = 1, \; t \geq 0$$

■ **29** $\mathcal{L}\{e^{\mathbf{A}t}\} = [s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s}{(s+1)^2+1} & \frac{-1}{(s+1)^2+1} \\ \frac{2}{(s+1)^2+1} & \frac{s+2}{(s+1)^2+1} \end{bmatrix}$

Thus $e^{\mathbf{A}t} = \begin{bmatrix} e^{-t}(\cos t - \sin t) & -e^{-t}\sin t \\ 2e^{-t}\sin t & e^{-t}(\cos t + \sin t) \end{bmatrix}$
and $e^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{0}$ since $\mathbf{x}(0) = 0$
With $U(s) = \mathcal{L}\{u(t)\} = \dfrac{1}{s}$ we have

$$[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}U(s) = \begin{bmatrix} \frac{s}{s^2+2s+2} & \frac{-1}{s^2+2s+2} \\ \frac{2}{s^2+2s+2} & \frac{s+2}{s^2+2s+2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{s} = \begin{bmatrix} \frac{1}{s^2+2s+2} \\ \frac{2}{s(s^2+2s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(s+1)^2+1} \\ \frac{1}{s} - \frac{s+2}{(s+1)^2+1} \end{bmatrix}$$

$$\text{so } \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U(s)\} = \begin{bmatrix} e^{-t}\sin t \\ 1 - e^{-t}(\cos t + \sin t) \end{bmatrix}$$

$$\text{Thus} \quad x(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U(s)\}$$
$$= \begin{bmatrix} e^{-t}\sin t \\ 1 - e^{-t}(\cos t + \sin t) \end{bmatrix}$$

For the transfer function, we have,

$$\mathbf{Y}(s) = \mathbf{c}\ \mathbf{X}(s) \text{ and when } \mathbf{x}(0) = \mathbf{0}$$
$$\mathbf{Y}(s) = \mathbf{c}[s\ \mathbf{I}\ - \mathbf{A}]^{-1}\mathbf{b}U(s) = \mathbf{H}U(s)$$
$$\text{where } \mathbf{H} = \mathbf{c}[s\ \mathbf{I}\ -\ \mathbf{A}]^{-1}\mathbf{b}$$

For this system, we have $\mathbf{H}(s) = [1\ 1] \begin{bmatrix} \frac{s}{s^2+2s+1} \\ \frac{2}{s^2+2s+1} \end{bmatrix} = \dfrac{s+2}{(s+1)^2+1}$

When $u(t) = \delta(t), U(s) = 1$ and so the impulse response $y_\delta(t)$ is given by

$$\mathcal{L}\{y_\delta(t)\} = Y_\delta(s) = \frac{s+2}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$
$$\Rightarrow\ y_\delta(t) = e^{-t}(\cos t + \sin t)$$

■ **30** The controllability question can be answered by either reducing to canonical form as in section 6.7.8 of the text or by using the Kalman matrix criterion given in Exercise 61 of the text. Adopting the Kalman matrix approach

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -3 & -3 & -2 \end{bmatrix} \text{ and}$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{A}\ \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}, \ \mathbf{A}^2\ \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$$

so the controllability Kalman matrix is

$$[\mathbf{b} \ \ \mathbf{A}\ \mathbf{b} \ \ \mathbf{A}^2\ \mathbf{b}] = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & -3 & 9 \end{bmatrix} = \mathbf{C}$$

Since $\det \mathbf{C} \neq 0$, rank $\mathbf{C} = 3$ so the system is controllable.
The eigenvalues of $\mathbf{A}$ are given by

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & -1-\lambda & 0 \\ -3 & -3 & -2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -(1+\lambda) & 0 \\ -3 & -(2+\lambda) \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda)(2+\lambda) = 0$$

so that the eigenvalues are $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1$. The system is therefore unstable with $\lambda_3 = 1$ corresponding to the unstable mode. The corresponding eigenvectors of $\mathbf{A}$ are given by

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{e}_i = 0$$

and are readily determined as

$$\mathbf{e}_1 = [0\ 0\ 1]^T$$
$$\mathbf{e}_2 = [1\ -1\ 0]^T$$
$$\mathbf{e}_3 = [1\ 0\ -1]^T$$

To determine the control law to relocate $\lambda_3 = 1$ at -5 we need to determine the eigenvector $\mathbf{v}_3$ of $\mathbf{A}^T$ corresponding to $\lambda_3 = 1$. This is readily obtained as

$$\mathbf{v}_3 = [1\ 1\ 0]^T$$

Thus, the required control law is

$$u(t) = K\mathbf{v}_3^T\mathbf{x}(t) = K[1\ 1\ 0]^T\mathbf{x}(t)$$
$$\text{where } K = \frac{p_3 - \lambda_3}{v_3^T b} = \frac{(-5) - 1}{[1\ 1\ 0]\begin{bmatrix}0\\1\\0\end{bmatrix}} = \frac{6}{1} = -6$$

So $u(t) = -6(x_1(t) + x_2(t))$

■ **31**(a) Let $\mathbf{x}\,[\,x_1 \quad x_2\,]^T$ then state − space model is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \Rightarrow \begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix} = \begin{bmatrix}-2 & 4\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}u$$
$$y = \mathbf{c}^T\mathbf{x} \Rightarrow y = [\,1 \quad 0\,]\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

(b) $G(s) = \frac{Y(s)}{U(s)} = \mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$

$$det(s\mathbf{I} - \mathbf{A}) = \Delta = \begin{vmatrix}s+2 & -4\\0 & s-1\end{vmatrix} = (s+2)(s-1)$$
$$adj(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix}s-1 & 4\\0 & s+2\end{bmatrix}$$
$$G(s) = \frac{1}{\Delta}[\,1 \quad 0\,]\begin{bmatrix}s-1 & 4\\0 & s+2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{s+3}{(s+2)(s-1)}$$

System has positive pole $s = 1$ and is therefore is unstable.

(c) $u(t) = r(t) - ky(t) \Rightarrow \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}(r - k\mathbf{c}^T\mathbf{x}) = (\mathbf{A} - k\mathbf{b}\mathbf{c}^T)\mathbf{x} + \mathbf{b}r$

$$(\mathbf{A} - k\mathbf{b}\mathbf{c}^T) = \begin{bmatrix}-2 & 4\\0 & 1\end{bmatrix} - k\begin{bmatrix}1\\1\end{bmatrix}[\,1 \quad 0\,] = \begin{bmatrix}-2-k & 4\\-k & 1\end{bmatrix}$$
$$\Rightarrow \text{Eigenvalues given by } \begin{vmatrix}\lambda+2+k & 4\\-k & \lambda-1\end{vmatrix} = 0 \Rightarrow$$
$$\lambda^2 + (k+1)\lambda + 3k - 2 = 0$$
$$\text{so system is stable if and only if } 3k - 2 > 0 \text{ or } k > \tfrac{2}{3}$$

(d) $r(t) = H(t) \Rightarrow R(s) = \frac{1}{s} \Rightarrow \mathrm{Y}(s) = \mathbf{c}^T(s\mathbf{I} - \mathbf{A} + k\mathbf{b}\mathbf{c}^T)^{-1}\mathbf{b}\frac{1}{s}$
Since $k > \frac{2}{3}$, system stable, the final value theorem gives

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) = \lim_{s\to 0}[\mathbf{c}^T(s\mathbf{I} - \mathbf{A} + k\mathbf{b}\mathbf{c}^T)^{-1}\mathbf{b}] = -\mathbf{c}^T(\mathbf{A} - k\mathbf{b}\mathbf{c}^T)^{-1}\mathbf{b} \\ &= -\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}-2-k & 4\\ -k & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix} = -\begin{bmatrix}1 & 0\end{bmatrix}\frac{1}{3k-2}\begin{bmatrix}1 & -4\\ k & -2-k\end{bmatrix} \\ &= \frac{3}{3k-2}\end{aligned}$$

Thus, $\lim_{t\to\infty} y(t) = 1$ if and only if $\frac{3}{3k-2} = 1 \Rightarrow k = \frac{5}{3}$

■ **32(a)** Overall closed loop transfer function is

$$G(s) = \frac{\frac{K}{s(s+1)}}{1 + \frac{K}{s(s+1)}(1 + K_1 s)} = \frac{K}{s^2 + s(1 + KK_1) + K}$$

32(b) Assuming zero initial conditions step response $x(t)$ is given by

$$\begin{aligned}X(s) = G(s)\mathcal{L}\{1.H(t)\} &= \frac{K}{s[s^2 + s(1 + KK_1) + K]} \\ &= \frac{w_n}{s[s^2 + 2\xi w_n s + w_n^2]} \\ &= \frac{1}{s} - \frac{s + 2\xi w_n}{s^2 + 2\xi w_n s + w_n^2} \\ &= \frac{1}{s} - \left[\frac{(s + \xi w_n) + \xi w_n}{(s + \xi w_n)^2 + [w_n^2(1 - \xi^2)]}\right] \\ &= \frac{1}{s} - \left[\frac{(s + \xi w_n) + \xi w_n}{(s + \xi w_n)^2 + w_d^2}\right]\end{aligned}$$

giving $x(t) = \mathcal{L}^{-1}\{X(s)\} = 1 - e^{-\xi w_n t}\big[\cos w_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin w_d t\big], t \geq 0.$

■ **32(c)** The peak time t_p is given by the solution of $\frac{dx}{dt}\big|_{t=t_p} = 0$

$$\begin{aligned}\frac{dx}{dt} &= e^{-\xi w_n t}\left[\big(\xi w_n - \frac{\xi w_d}{\sqrt{1-\xi^2}}\big)\cos w_d t\big(\frac{\xi^2 w_n}{\sqrt{1-\xi^2}} + w_d\big)\sin w_d t\right] \\ &= e^{-\xi w_n t}\frac{w_n}{\sqrt{1-\xi^2}}\sin w_d t\end{aligned}$$

Thus, t_p given by the solution of

$$e^{-\xi w_n t_p}\frac{w_n}{\sqrt{1-\xi^2}}\sin w_d t_p = 0$$

$$\text{i.e. } \sin w_d t_p = 0$$

Since the peak time corresponds to the first peak overshoot

$$w_d t_p = \pi \text{ or } t_p = \frac{\pi}{w_d}$$

The maximum overshoot M_p occurs at the peak time t_p. Thus

$$M_p = x(t_p) - 1 = e^{-\frac{\xi w_n \pi}{w_d}}\left[\cos\pi + \frac{\xi}{\sqrt{1-\xi^2}}\sin\pi\right]$$

$$= e^{-\frac{\xi w_n \pi}{w_d}} = e^{-\xi\pi/\sqrt{1-\xi^2}}\pi$$

We wish M_p to be 0.2 and t_p to be 1s, thus

$$e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.2 \text{ giving } \xi = 0.456$$

and

$$t_p = \frac{\pi}{w_d} = 1 \text{ giving } w_d = 3.14$$

Then it follows that $w_n = \dfrac{w_d}{\sqrt{1-\xi^2}} = 3.53$ from which we deduce that

$$K = w_n^2 = 12.5$$

$$\text{and } K_1 = \frac{2w_n\xi - 1}{K} = 0.178.$$

32(d) The rise time t_r is given by the solution of

$$x(t_r) = 1 = 1 - e^{-\xi w_n t_r}\left[\cos w_d t_r + \frac{\xi}{\sqrt{1-\xi^2}}\sin w_d t_r\right]$$

Since $e^{-\xi w_n t_r} \neq 0$

$$\cos w_d t_r + \frac{\xi}{\sqrt{1-\xi^2}}\sin w_d t_r = 0$$

giving $\tan w_d t_r = -\dfrac{\sqrt{1-\xi^2}}{\xi}$

$$\text{or} \qquad t_r = \frac{1}{w_d}\tan^{-1}\left(-\frac{\sqrt{1-\xi^2}}{\xi}\right) = \frac{\pi - 1.10}{w_d} = 0.65\text{s}.$$

The response $x(t)$ in (b) may be written as

$$x(t) = 1 - \frac{e^{-\xi w_n t}}{\sqrt{1-\xi^2}}\sin\left[w_\alpha t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

so the curves $1 \pm \dfrac{e^{-\xi w_n t}}{\sqrt{1-\xi^2}}$ are the envelope curves of the transient response to a unit step input and have a time constant $T = \dfrac{1}{\xi w_n}$. The settling time t_s may be measured in terms of T. Using the 2% criterion t_s is approximately 4 times the time constant and for the 5% criterion it is approximately 3 times the time constant. Thus,

$$2\% \text{ criterion } : \; t_s = 4T = \frac{4}{\xi w_n} = 2.48\text{s}$$
$$5\% \text{ criterion } : \; t_s = 3T = \frac{3}{\xi w_n} = 1.86\text{s}$$

Footnote: This is intended to be an extended exercise with students being encouraged to carry out simulation studies in order to develop a better understanding of how the transient response characteristics can be used in system design.

■ **33** As for Exercise 32 this is intended to be an extended problem supported by simulation studies. The following is simply an outline of a possible solution.
Figure 5.67(a) is simply a mass-spring damper system represented by the differential equation

$$M_1\frac{d^2x}{dt^2} + B\frac{dx}{dt} + K_1 x = \sin wt$$

Assuming that it is initially in a quiescent state taking Laplace transforms

$$X(s) = \frac{1}{M_1 s^2 + Bs + K_1}\cdot\frac{w}{s^2 + w^2}$$

The steady state response will be due to the forcing term and determined by the $\dfrac{\alpha s+\beta}{s^2+w^2}$ term in the partial fractions expansion of $X(s)$. Thus, the steady state response will be of the form $A\sin(wt+\delta)$; that is, a sinusoid having the same frequency as the forcing term but with a phase shift δ and amplitude scaling A.
In the situation of Figure 5.67(b) the equations of motion are

$$M_1\frac{d^2x}{dt^2} = -K_1x - B\frac{dx}{dt} + K_2(y-x) + \sin wt$$
$$M_2\frac{d^2y}{dt^2} = -K_2(y-x)$$

Assuming an initial quiescent state taking Laplace transforms gives

$$[M_1s^2 + Bs + (K_1+K_2)]X(s) - K_2Y(s) = w/(s^2+w^2)$$
$$-K_2X(s) + (s^2M_2 + K_2)Y(s) = 0$$

Eliminating $Y(s)$ gives

$$X(s) = \frac{w(s^2M_2+K_2)}{(s^2+w^2)p(s)}$$

where $p(s) = (M_1s^2 + Bs + K_1 + K_2)(s^2M_2 + K_2)$.
Because of the term (s^2+w^2) in the denominator $x(t)$ will contain terms in $\sin wt$ and $\cos wt$. However, if $(s^2M_2+K_2)$ exactly cancels (s^2+w^2) this will be avoided. Thus choose $K_2 = M_2w^2$. This does make practical sense for if the natural frequency of the secondary system is equal to the frequency of the applied force then it may resonate and therefore damp out the steady state vibration of M_1.
It is also required to show that the polynomial $p(s)$ does not give rise to any undamped oscillations. That is, it is necessary to show that $p(s)$ does not possess purely imaginary roots of the form $j\theta, \theta$ real, and that it has no roots with a positive real part. This can be checked using the Routh–Hurwitz criterion.
To examine the motion of the secondary mass M_2 solve for $Y(s)$ giving

$$Y(s) = \frac{K_2w}{(s^2+w^2)p(s)}$$

Clearly due to the term (s^2+w^2) in the denominator the mass M_2 possesses an undamped oscillation. Thus, in some sense the secondary system has absorbed the energy produced by the applied sinusoidal force $\sin wt$.

■ **34** Again this is intended to be an extended problem requiring wider exploration by the students. The following is an outline of the solution.

34(a) Students should be encouraged to plot the Bode plots using the steps used in Example 5.65 of the text and using a software package. Sketches of the magnitude and phase Bode plots are given in the figures below.

34(b) With unity feedback the amplifier is unstable. Since the $-180°$ crossover gain is greater than 0dB (from the plot it is +92dB).

34(c) Due to the assumption that the amplifier is ideal it follows that for marginal stability the value of $\dfrac{1}{\beta}$ must be 92dB (that is, the plot is effectively lowered by 92dB). Thus

$$20\log\frac{1}{\beta} = 92$$

$$\frac{1}{\beta} = \text{ antilog } \left(\frac{92}{20}\right) \Rightarrow \beta \simeq 2.5 \times 10^{-5}$$

34(d) From the amplitude plot the effective 0dB axis is now drawn through the 100dB point. Comparing this to the line drawn through the 92dB point, corresponding to marginal stability, it follows that

$$\text{Gain margin } = -8\text{dB}$$

$$\text{and Phase margin } = 24°.$$

34(e)

$$G(s) = \frac{K}{(1+s\tau_1)(1-s\tau_2)(1+s\tau_3)}$$

Given low frequency gain $K = 120$dB so

$$20\log K = 120 \;\Rightarrow\; K = 10^6$$

$T_i = \dfrac{1}{f_i}$ where f_i is the oscillating frequency in cycles per second of the pole.

Since 1MHz = 10 cycles per second

$$\tau_1 = \frac{1}{f_1} = \frac{1}{10^6} \text{ since } f_1 = 1\text{MHz}$$
$$\tau_2 = \frac{1}{f_2} = \frac{1}{10.10^6} \text{ since } f_2 = 10\text{MHz}$$
$$\tau_3 = \frac{1}{f_3} = \frac{1}{25.10^6} \text{ since } f_3 = 25\text{MHz}$$

Thus,

$$G(s) = \frac{10^6}{(1+\frac{s}{10^6})(1+\frac{s}{10.10^6})(1+\frac{s}{25.10^6})}$$
$$= \frac{250.10^{24}}{(s+10^6)(s+10^7)(s+\frac{5}{2}.10^7)}$$

The closed loop transfer function $G_1(s)$ is

$$G(s) = \frac{G(s)}{1+\beta G(s)}$$

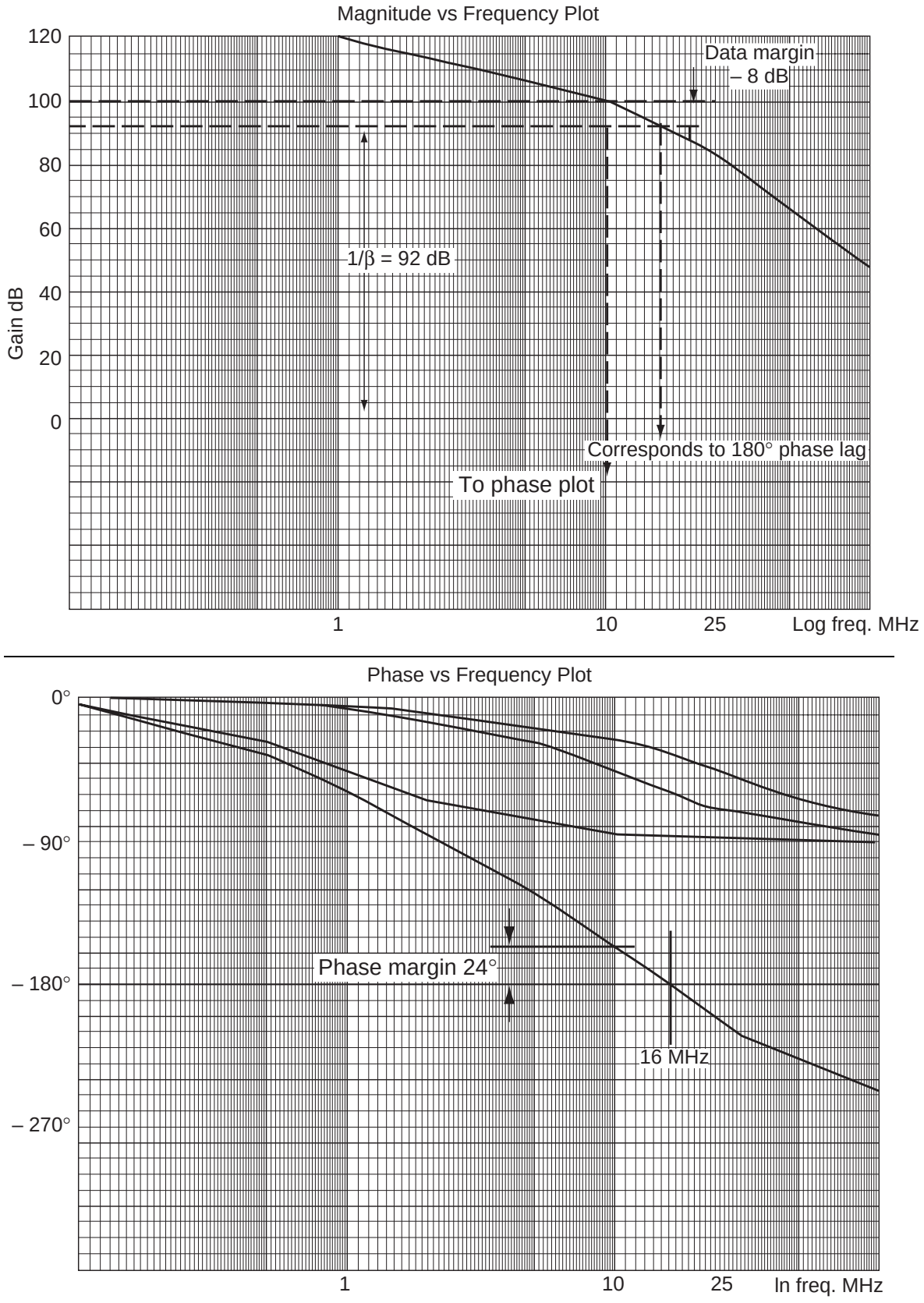
34(f) The characteristic equation for the closed loop system is

$$(s+10^6)(s+10^7)(s+\tfrac{5}{2}.10^7) + \beta 25.10^{25} = 0$$

or

$$s^3 + 36(10^6)s^2 + (285)10^{12}s + 10^{19}(25 + 25\beta 10^6) = 0$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$A_1 \qquad A_2 \qquad A_3$$

By Routh–Hurwitz criterion system stable provided $A_1 > 0$ and $A_1 A_2 > A_3$. If $\beta = 1$ then $A_1 A_2 < A_3$ and the system is unstable as determined in (b). For marginal stability $A_1 A_2 = A_3$ giving $\beta = 1.40^{-5}$ (compared with $\beta = 2.5.10^{-5}$ using the Bode plot).



6

The z Transform

Exercises 6.2.3

- **1(a)**

$$F(z) = \sum_{k=0}^{\infty} \frac{(1/4)^k}{z^k} = \frac{1}{1 - 1/4z} = \frac{4z}{4z - 1} \quad \text{if } \mid z \mid > 1/4$$

1(b)

$$F(z) = \sum_{k=0}^{\infty} \frac{3^k}{z^k} = \frac{1}{1 - 3/z} = \frac{z}{z - 3} \quad \text{if } \mid z \mid > 3$$

1(c)

$$F(z) = \sum_{k=0}^{\infty} \frac{(-2)^k}{z^k} = \frac{1}{1 - (-2)/z} = \frac{z}{z + 2} \quad \text{if } \mid z \mid > 2$$

1(d)

$$F(z) = \sum_{k=0}^{\infty} \frac{-(2)^k}{z^k} = -\frac{1}{1 - 2/z} = -\frac{z}{z - 2} \quad \text{if } \mid z \mid > 2$$

1(e)

$$\mathcal{Z}\{k\} = \frac{z}{(z-1)^2} \quad \text{if } \mid z \mid > 1$$

from (6.6) whence

$$\mathcal{Z}\{3k\} = 3\frac{z}{(z-1)^2} \quad \text{if } \mid z \mid > 1$$

- **2**

$$u_k = e^{-2\omega kT} = \left(e^{-2\omega T}\right)^k$$

whence

$$U(Z) = \frac{z}{z - e^{-2\omega T}}$$

Exercises 6.3.6

- **3**

$$\mathcal{Z}\{\sin k\omega T\} = \frac{1}{2\jmath}\frac{z}{z-e^{\jmath\omega T}} - \frac{1}{2\jmath}\frac{z}{z-e^{-\jmath\omega T}}$$
$$= \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}$$

- **4**

$$\mathcal{Z}\{\left(\frac{1}{2}\right)^k\} = \frac{2z}{2z-1}$$

so

$$\mathcal{Z}\{y_k\} = \frac{1}{z^3}\times\frac{2z}{2z-1} = \frac{2}{z^2(2z-1)}$$

Proceeding directly

$$\mathcal{Z}\{y_k\} = \sum_{k=3}^{\infty}\frac{x_{k-3}}{z^k} = \sum_{r=0}^{\infty}\frac{x_r}{z^{r+3}} = \frac{1}{z^3}\times\mathcal{Z}\{x_k\} = \frac{2}{z^2(2z-1)}$$

- **5(a)**

$$\mathcal{Z}\left\{-\frac{1}{5}\right\} = \sum_{r=0}^{\infty}\left(\frac{-1}{5z}\right)^r = \frac{5z}{5z+1} \quad |\,z\,| > \frac{1}{5}$$

5(b)

$$\{\cos k\pi\} = \left\{(-1)^k\right\}$$

so

$$\mathcal{Z}\{\cos k\pi\} = \frac{z}{z+1} \quad |\,z\,| > 1$$

- **6**

$$\mathcal{Z}\left\{\left(\frac{1}{2}\right)^k\right\} = \frac{2z}{2z-1}$$

By (3.5)

$$\mathcal{Z}\left\{(a^k)\right\} = \frac{z}{z-a}$$

so

$$\mathcal{Z}\left\{(ka^{k-1})\right\} = \frac{z}{(z-a)^2}$$

thus

$$\mathcal{Z}\left\{(ka^k)\right\} = \frac{az}{(z-a)^2}$$

whence

$$\mathcal{Z}\left\{k\left(\frac{1}{2}\right)^k\right\} = \frac{2z}{(2z-1)^2}$$

- **7(a)**

$$\sinh k\alpha = \frac{1}{2}(e^{\alpha})^k - \frac{1}{2}(e^{-\alpha})^k$$

so

$$\mathcal{Z}\left\{\sinh k\alpha\right\} = \frac{1}{2}\left(\frac{z}{z-e^{\alpha}} - \frac{z}{z-e^{-\alpha}}\right) = \frac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$$

7(b)

$$\cosh k\alpha = \frac{1}{2}(e^{\alpha})^k + \frac{1}{2}(e^{-\alpha})^k$$

then proceed as above.

- **8(a)**

$$u_k = \left(e^{-4kT}\right) = \left(e^{-4T}\right)^k; \quad \mathcal{Z}\left\{u_k\right\} = \frac{z}{z-e^{-4T}}$$

8(b)

$$u_k = \frac{1}{2\jmath}\left(e^{\jmath kT} - e^{-\jmath kT}\right)$$

$$\mathcal{Z}\left\{u_k\right\} = \frac{1}{2\jmath}\left(\frac{z}{z-e^{\jmath T}} - \frac{z}{z-e^{-\jmath T}}\right) = \frac{z\sin T}{z^2 - 2z\cos T + 1}$$

8(c)

$$u_k = \frac{1}{2}\left(e^{\jmath 2kT} + e^{-\jmath 2kT}\right)$$

then proceed as in 8(b) to give

$$Z\{u_k\} = \frac{z(z-\cos 2T)}{z^2 - 2z\cos 2T + 1}$$

- **9** Initial value theorem: obvious from definition.

■ **9** Final value theorem

$$(1-z^{-1})X(z) = \sum_{r=0}^{\infty} \frac{x_r - x_{r-1}}{z^r}$$

$$= x_0 + \frac{x_1 - x_0}{z} + \frac{x_2 - x_1}{z^2} + \ldots + \frac{x_r - x_{r-1}}{z^r} + \ldots$$

As $z \to 1$ and if $\lim_{r\to\infty} x_r$ exists, then

$$\lim_{z\to 1}(1-z^{-1})X(z) = \lim_{r\to\infty} x_r$$

■ **10** Multiplication property (6.19): Let $\mathcal{Z}\{x_k\} = \sum_{k=0}^{\infty} \frac{x_k}{z^k} = X(z)$ then

$$\mathcal{Z}\left\{a^k x_k\right\} = \sum_{k=0}^{\infty} \frac{a^k x_k}{z^k} = X(z/a)$$

---

10 Multiplication property (6.20)

$$-z\frac{d}{dz}X(z) = -z\frac{d}{dz}\sum_{k=0}^{\infty}\frac{x_k}{z^k} = \sum_{k=0}^{\infty}\frac{kx_k}{z^k} = \mathcal{Z}\{kx_k\}$$

The general result follows by induction.

Exercises 6.4.2

■ **11(a)**

$$\frac{z}{z-1}; \quad \text{from tables } u_k = 1$$

11(b)

$$\frac{z}{z+1} = \frac{z}{z-(-1)}; \quad \text{from tables } u_k = (-1)^k$$

11(c)

$$\frac{z}{z-1/2}; \quad \text{from tables } u_k = (1/2)^k$$

11(d)

$$\frac{z}{3z+1} = \frac{1}{3}\frac{z}{z+1/3} \longleftrightarrow \frac{1}{3}(-1/3)^k$$

11(e)

$$\frac{z}{z-\jmath}; \quad \text{from tables } u_k = (\jmath)^k$$

11(f)

$$\frac{z}{z+\jmath\sqrt{2}} = \frac{z}{z-(-\jmath\sqrt{2})} \longleftrightarrow (-\jmath\sqrt{2})^k$$

11(g)

$$\frac{1}{z-1} = \frac{1}{z}\frac{z}{z-1} \longleftrightarrow \begin{cases} 0; & k=0 \\ 1; & k>0 \end{cases}$$

using first shift property.

11(h)

$$\frac{z+2}{z+1} = 1 + \frac{1}{z}\frac{z}{z+1} \longleftrightarrow \begin{cases} 1; & k=0 \\ (-1)^{k-1}; & k>0 \end{cases}$$

$$= \begin{cases} 1; & k=0 \\ (-1)^{k+1}; & k>0 \end{cases}$$

- **12(a)**

$$Y(z)/z = \frac{1}{3}\frac{1}{z-1} - \frac{1}{3}\frac{1}{z+2}$$

so

$$Y(z) = \frac{1}{3}\frac{z}{z-1} - \frac{1}{3}\frac{z}{z+2} \longleftrightarrow \frac{1}{3}\left(1-(-2)^k\right)$$

12(b)

$$Y(z) = \frac{1}{7}\left(\frac{z}{z-3} - \frac{z}{z+1/2}\right) \longleftrightarrow \frac{1}{7}\left((3)^k - (-1/2)^k\right)$$

12(c)

$$Y(z) = \frac{1}{3}\frac{z}{z-1} + \frac{1}{6}\frac{z}{z+1/2} \longleftrightarrow \frac{1}{3} + \frac{1}{6}(-1/2)^k$$

12(d)

$$Y(z) = \frac{2}{3}\frac{z}{z-1/2} - \frac{2}{3}\frac{z}{z+1} \longleftrightarrow \frac{2}{3}(1/2)^k - \frac{2}{3}(-1)^k$$
$$= \frac{2}{3}(1/2)^k + \frac{2}{3}(-1)^{k+1}$$

12(e)

$$Y(z) = \frac{1}{2\jmath}\left(\frac{z}{z-\jmath} - \frac{z}{z-(-\jmath)}\right)$$
$$= \frac{1}{2\jmath}\left(\frac{z}{z-e^{\jmath\pi/2}} - \frac{z}{z-e^{-\jmath\pi/2}}\right)$$
$$\longleftrightarrow \frac{1}{2\jmath}\left((e^{\jmath\pi/2})^k - (e^{-\jmath\pi/2})^k\right) = \sin k\pi/2$$

12(f)

$$Y(z) = \frac{z}{\left(z-(\sqrt{3}+\jmath)\right)\left(z-(\sqrt{3}-\jmath)\right)}$$
$$= \frac{1}{2\jmath}\left(\frac{z}{z-(\sqrt{3}+\jmath)} - \frac{z}{z-(\sqrt{3}-\jmath)}\right)$$
$$= \frac{1}{2\jmath}\left(\frac{z}{z-2e^{\jmath\pi/6}} - \frac{z}{z-2e^{-\jmath\pi/6}}\right)$$
$$\longleftrightarrow \frac{1}{2\jmath}\left(2^k e^{\jmath k\pi/6} - 2^k e^{-\jmath k\pi/6}\right) = 2^k \sin k\pi/6$$

12(g)

$$Y(z) = \frac{5}{2}\frac{z}{(z-1)^2} + \frac{1}{4}\frac{z}{z-1} - \frac{1}{4}\frac{z}{z-3}$$
$$\longleftrightarrow \frac{5}{2}k + \frac{1}{4}\left(1-3^k\right)$$

12(h)

$$Y(z)/z = \frac{z}{(z-1)^2(z^2-z+1)} = \frac{1}{(z-1)^2} - \frac{1}{z^2-z+1}$$

so

$$Y(z) = \frac{z}{(z-1)^2} - \frac{1}{\sqrt{3}\jmath}\left(\frac{z}{z-\dfrac{1+\sqrt{3}\jmath}{2}} - \frac{z}{z-\dfrac{1-\sqrt{3}\jmath}{2}}\right)$$

$$= \frac{z}{(z-1)^2} - \frac{1}{\sqrt{3}\jmath}\left(\frac{z}{z-e^{\jmath\pi/3}} - \frac{z}{z-e^{-\jmath\pi/3}}\right)$$

$$\longleftrightarrow k - \frac{2}{\sqrt{3}}\sin k\pi/3 = k + \frac{2}{\sqrt{3}}\cos(k\pi/3 - 3\pi/2)$$

13(a)

$$Y(z) = \sum_{k=0}^{\infty}\frac{x_k}{z^k} = \frac{1}{z} + \frac{2}{z^7}$$

whence $x_0 = 0,\ x_1 = 1,\ x_2 = x_3 = \ldots = x_6 = 0,\ x_7 = 2$ and $x_k = 0,\ k > 7$; giving $Y(z) \leftrightarrow \{0, 1, 0, 0, 0, 0, 0, 2, \cdots\}$

13(b) Proceed as in Exercise 13(a) to give

$$Y(z) \leftrightarrow \{1, 0, 3, 0, 0, 0, 0, 0, 0, -2, \cdots\}$$

13(c) Observe that

$$\frac{3z + z^2 + 5z^5}{z^5} = 5 + \frac{1}{z^3} + \frac{3}{z^4}$$

and proceed as in Exercise 13(a) to give $Y(z) \leftrightarrow \{5, 0, 0, 1, 3, \cdots\}$

13(d)

$$Y(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{z}{z+1/3}$$

$$\longleftrightarrow \{0, 0, 1, 1\} + \{(-1/3)^k\}$$

13(e)

$$Y(z) = 1 + \frac{3}{z} + \frac{1}{z^2} - \frac{1/2}{z+1/2}$$

$$\longleftrightarrow \{1, 3, 1\} - \frac{1}{2}\begin{cases} 0, & k = 0 \\ (-1/2)^k, & k \geq 1 \end{cases}$$

$$= \begin{cases} 1, & k = 0 \\ 5/2, & k = 1 \\ 5/4, & k = 2 \\ -\dfrac{1}{2}(-1/2)^{k-1}, & k \geq 3 \end{cases} \qquad = \begin{cases} 1, & k = 0 \\ 5/2, & k = 1 \\ 5/4, & k = 2 \\ -\dfrac{1}{8}(-1/2)^{k-3}, & k \geq 3 \end{cases}$$

13(f)

$$Y(z) = \frac{1}{z-1} - \frac{2}{(z-1)^2} + \frac{1}{z-2}$$

$$\longleftrightarrow \begin{cases} 0, \ k = 0 \\ 1 - 2(k-1) + 2^{k-1}, \ k \geq 1 \end{cases}$$

$$= \begin{cases} 0, \ k = 0 \\ 3 - 2k + 2^{k-1}, \ k \geq 1 \end{cases}$$

13(g)

$$Y(z) = \frac{2}{z-1} - \frac{1}{z-2}$$

$$\longleftrightarrow \begin{cases} 0, \ k = 0 \\ 2 - 2^{k-1}, \ k \geq 1 \end{cases}$$

Exercises 6.5.3

14(a) If the signal going into the left D-block is w_k and that going into the right D-block is v_k, we have

$$y_{k+1} = v_k, \quad v_{k+1} = w_k = x_k - \frac{1}{2}v_k$$

so

$$y_{k+2} = v_{k+1} = x_k - \frac{1}{2}v_k$$

$$= x_k - \frac{1}{2}v_k = x_k - \frac{1}{2}y_{k+1}$$

that is,

$$y_{k+2} + \frac{1}{2}y_{k+1} = x_k$$

14(b) Using the same notation

$$y_{k+1} = v_k, \quad v_{k+1} = w_k = x_k - \frac{1}{4}v_k - \frac{1}{5}y_k$$

Then

$$y_{k+2} = x_k - \frac{1}{4}y_{k+1} - \frac{1}{5}y_k$$

or

$$y_{k+2} + \frac{1}{4}y_{k+1} + \frac{1}{5}y_k = x_k$$

■ **15(a)**

$$z^2Y(z) - z^2y_0 - zy_1 - 2(zY(z) - zy_0) + Y(z) = 0$$

with $y_0 = 0$, $y_1 = 1$

$$Y(z) = \frac{z}{(z-1)^2}$$

so $y_k = k, k \geq 0$.

15(b) Transforming and substituting for y_0 and y_1,

$$Y(z)/z = \frac{2z - 15}{(z-9)(z+1)}$$

so

$$Y(z) = \frac{3}{10}\frac{z}{z-9} - \frac{17}{10}\frac{z}{z+1}$$

thus

$$y_k = \frac{3}{10}9^k - \frac{17}{10}(-1)^k,\ k \geq 0$$

15(c) Transforming and substituting for y_0 and y_1,

$$Y(z) = \frac{z}{(z - 2\jmath)(z + 2\jmath)}$$

$$= \frac{1}{4\jmath}\left(\frac{z}{z - 2e^{\jmath\pi/2}} - \frac{z}{z - 2e^{-\jmath\pi/2}}\right)$$

thus

$$y_k = \frac{1}{4\jmath}2^k\left(e^{\jmath k\pi/2} - e^{-\jmath k\pi/2}\right) = 2^{k-1}\sin k\pi/2,\ k \geq 0$$

15(d) Transforming, substituting for y_0 and y_1, and rearranging

$$Y(z)/z = \frac{6z-11}{(2z+1)(z-3)}$$

so

$$Y(z) = 2\frac{z}{z+1/2} + \frac{z}{z-3}$$

thus

$$y_k = 2(-1/2)^k + 3^k,\ k \geq 0$$

- **16(a)**

$$6y_{k+2} + y_{k+1} - y_k = 3, \quad y_0 = y_1 = 0$$

Transforming with $y_0 = y_1 = 0$,

$$(6z^2 + z - 1)Y(z) = \frac{3z}{z-1}$$

so

$$Y(z)/z = \frac{3}{(z-1)(3z-1)(2z+1)}$$

and

$$Y(z) = \frac{1}{2}\frac{z}{z-1} - \frac{9}{10}\frac{z}{z-1/3} + \frac{2}{5}\frac{z}{z+1/2}$$

Inverting

$$y_k = \frac{1}{2} - \frac{9}{10}(1/3)^k + \frac{2}{5}(-1/2)^k$$

16(b) Transforming with $y_0 = 0$, $y_1 = 1$,

$$(z^2 - 5z + 6)Y(z) = z + 5\frac{z}{z-1}$$

whence

$$Y(z) = \frac{5}{2}\frac{z}{z-1} + \frac{7}{2}\frac{z}{z-3} - 6\frac{z}{z-2}$$

so

$$y_k = \frac{5}{2} + \frac{7}{2}(3)^k - 6\,(2)^k$$

16(c) Transforming with $y_0 = y_1 = 0$,

$$(z^2 - 5z + 6)Y(z) = \frac{z}{z-1/2}$$

so

$$Y(z) = \frac{4}{15}\frac{z}{z-1/2} - \frac{2}{3}\frac{z}{z-2} + \frac{2}{5}\frac{z}{z-3}$$

whence

$$y_n = \frac{4}{15}(1/2)^k - \frac{2}{3}(2)^k + \frac{2}{5}(3)^k$$

16(d) Transforming with $y_0 = 1$, $y_1 = 0$,

$$(z^2 - 3z + 3)Y(z) = z^2 - 3z + \frac{z}{z-1}$$

so

$$Y(z) = \frac{z}{z-1} - \frac{z}{z^2 - 3z + 3}$$

$$= \frac{z}{z-1} - \frac{1}{\sqrt{3}\mathrm{j}}\left\{\frac{z}{z - \dfrac{3+\sqrt{3}\mathrm{j}}{2}} - \frac{z}{z - \dfrac{3-\sqrt{3}\mathrm{j}}{2}}\right\}$$

$$= \frac{z}{z-1} - \frac{1}{\sqrt{3}\mathrm{j}}\left\{\frac{z}{z - \sqrt{3}e^{\mathrm{j}\pi/6}} - \frac{z}{z - \sqrt{3}e^{-\mathrm{j}\pi/6}}\right\}$$

so

$$y_n = 1 - \frac{2}{\sqrt{3}}(\sqrt{3})^k \frac{e^{\mathrm{j}n\pi/6} - e^{-\mathrm{j}n\pi/6}}{2\mathrm{j}} = 1 - 2(\sqrt{3})^{n-1}\sin n\pi/6$$

16(e) Transforming with $y_0 = 1$, $y_1 = 2$,

$$(2z^2 - 3z - 2)Y(z) = 2z^2 + z + 6\frac{z}{(z-1)^2} + \frac{z}{z-1}$$

so

$$Y(z) = \frac{z}{z-2} + z\left\{\frac{z+5}{(z-1)^2(2z+1)(z-2)}\right\}$$

$$= \frac{12}{5}\frac{z}{z-2} - \frac{2}{5}\frac{z}{z+1/2} - \frac{z}{z-1} - 2\frac{z}{(z-1)^2}$$

so

$$y_n = \frac{12}{5}(2)^n - \frac{2}{5}(-1/2)^n - 1 - 2n$$

16(f) Transforming with $y_0 = y_1 = 0$,

$$(z^2 - 4)Y(z) = 3\frac{z}{(z-1)^2} - 5\frac{z}{z-1}$$

so

$$Y(z) = \frac{z}{z-1} - \frac{z}{(z-1)^2} - \frac{1}{2}\frac{z}{z-2} - \frac{1}{2}\frac{z}{z+2}$$

and

$$y_n = 1 - n - \frac{1}{2}(2)^n - \frac{1}{2}(-2)^n$$

■ **17(a)** Write the transformed equations in the form

$$\begin{pmatrix} z-3/2 & 1 \\ -0.21 & z-1/2 \end{pmatrix} \begin{pmatrix} c(z) \\ e(z) \end{pmatrix} = \begin{pmatrix} zC_0 \\ zE_0 \end{pmatrix}$$

Then

$$\begin{pmatrix} c(z) \\ e(z) \end{pmatrix} = \frac{1}{z^2 - 2z + 0.96} \begin{pmatrix} z-1/2 & -1 \\ 0.21 & z-3/2 \end{pmatrix} \begin{pmatrix} zC_0 \\ zE_0 \end{pmatrix}$$

Solve for $c(z)$ as

$$c(z) = 1200\frac{z}{z-1.2} + 4800\frac{z}{z-0.8}$$

and

$$C_k = 1200(1.2)^k + 4800(0.8)^k$$

This shows the 20% growth in C_k in the long term as required.

(b) Then

$$E_k = 1.5C_k - C_{k+1}$$

$$= 1800(1.2)^k + 7200(0.8)^k - 1200(1.2)^{k+1} - 4800(0.8)^{k+1}$$

Differentiate wrt k and set to zero giving

$$0.6\log(1.2) + 5.6x\log(0.8) = 0 \text{ where } x = (0.8/1.2)^k$$

Solving, $x = 0.0875$ and so

$$k = \frac{\log 0.0875}{\log(0.8/1.2)} = 6.007$$

The nearest integer is $k = 6$, corresponding to the seventh year in view of the labelling, and $C_6 = 4841$ approximately.

■ **18** Transforming and rearranging

$$Y(z)/z = \frac{z-4}{(z-2)(z-3)} + \frac{1}{(z-1)(z-2)(z-3)}$$

so

$$Y(z) = \frac{1}{2}\frac{z}{z-1} + \frac{z}{z-2} - \frac{1}{2}\frac{z}{z-3}$$

thus

$$y_k = \frac{1}{2} + 2^k - \frac{1}{2}3^k$$

■ **19**

$$I_k = C_k + P_k + G_k$$

$$= aI_{k-1} + b(C_k - C_{k-1}) + G_k$$

$$= aI_{k-1} + ba(I_{k-1} - I_{k-2}) + G_k$$

so

$$I_{k+2} - a(1+b)I_{k+1} + abI_k = G_{k+2}$$

Thus substituting

$$I_{k+2} - I_{k+1} + \frac{1}{2}I_k = G$$

Using lower case for the z transform, we obtain

$$(z^2 - z + \frac{1}{2})i(z) = (2z^2 + z)G + G\frac{z}{z-1}$$

whence

$$i(z)/z = G\left[\frac{1}{z^2 - z + \frac{1}{2}} + \frac{2}{z-1}\right]$$

$$= G\left[\frac{2}{z-1} + \frac{1}{(z - \frac{1+\jmath}{2})(z - \frac{1-\jmath}{2})}\right]$$

so

$$i(z) = G\left[2\frac{z}{z-1} + \frac{2}{2\jmath}\left\{\frac{z}{z - \frac{1}{\sqrt{2}}e^{\jmath\pi/4}} - \frac{z}{z - \frac{1}{\sqrt{2}}e^{-\jmath\pi/4}}\right\}\right]$$

Thus

$$I_k = G\left[2 + \frac{2}{2\jmath}(\frac{1}{\sqrt{2}})^k\left\{e^{\jmath k\pi/4} - e^{-\jmath k\pi/4}\right\}\right]$$

$$= 2G\left[1 + \left(\frac{1}{\sqrt{2}}\right)^k \sin k\pi/4\right]$$

■ **20** Elementary rearrangement leads to

$$i_{n+2} - 2\cosh\alpha\, i_{n+1} + i_n = 0$$

with $\cosh\alpha = 1 + R_1/2R_2$. Transforming and solving for $I(z)/z$ gives

$$I(z)/z = \frac{zi_0 + (i_1 - 2i_0\cosh\alpha)}{(z - e^{\alpha})(z - e^{-\alpha})}$$

$$= \frac{1}{2\sinh\alpha}\left[\frac{i_0e^{\alpha} + (i_1 - 2i_0\cosh\alpha)}{z - e^{\alpha}} - \frac{i_0e^{-\alpha} + (i_1 - 2i_0\cosh\alpha)}{z - e^{-\alpha}}\right]$$

Thus

$$i_k = \frac{(i_0e^{\alpha} + (i_1 - 2i_0\cosh\alpha))e^{n\alpha} - (i_0e^{-\alpha} + (i_1 - 2i_0\cosh\alpha))e^{-n\alpha}}{2\sinh\alpha}$$

$$= \frac{1}{\sinh\alpha}\{i_1\sinh n\alpha - i_0\sinh(n-1)\alpha\}$$

Exercises 6.6.5

■ **21** Transforming in the quiescent state and writing as $Y(z) = H(z)U(z)$, then

21(a)

$$H(z) = \frac{1}{z^2 - 3z + 2}$$

21(b)

$$H(z) = \frac{z - 1}{z^2 - 3z + 2}$$

21(c)

$$H(z) = \frac{1 + 1/z}{z^3 - z^2 + 2z + 1}$$

■ **22** For the first system, transforming from a quiescent state, we have

$$(z^2 + 0.5z + 0.25)Y(z) = U(z)$$

The diagram for this is the standard one for a second-order system and is shown in Figure 6.1 and where $Y(z) = P(z)$, that is $y_k = p_k$.

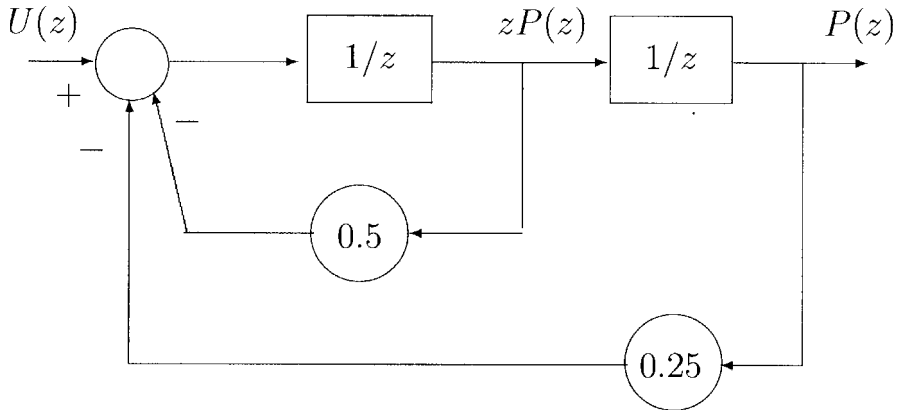


Figure 6.1: The block diagram for the basic system of Exercise 22.

Transforming the second system in the quiescent state, we obtain

$$(z^2 + 0.5z + 0.25)Y(z) = (1 - 0.6)U(z)$$

Clearly

$$(z^2 + 0.5z + 0.25)(1 - 0.6z)P(z) = (1 - 0.6z)U(z)$$

indicating that we should now set $Y(z) = P(z) - 0.6zP(z)$ and this is shown in Figure 6.2.

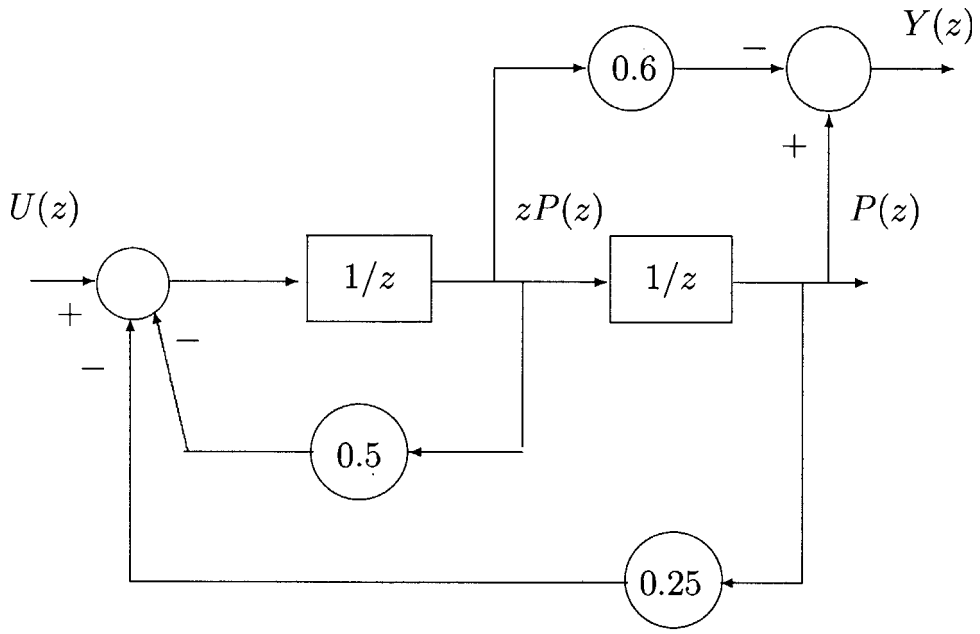


Figure 6.2: The block diagram for the second system of Exercise 22.

■ **23(a)**

$$Y_\delta(z)/z = \frac{1}{(4z+1)(2z+1)}$$

so

$$Y_\delta(z) = \frac{1}{2}\frac{z}{z+1/4} - \frac{1}{2}\frac{z}{z+1/2}$$

$$y_k = \frac{1}{2}(-1/4)^k - \frac{1}{2}(1/2)^k$$

23(b)

$$Y_\delta(z)/z = \frac{z}{z^2-3z+3}$$

whence

$$Y_\delta(z) = \frac{3+\sqrt{3}\jmath}{2\sqrt{3}\jmath}\frac{z}{z-\frac{(3+\sqrt{3}\jmath)}{2}} - \frac{3-\sqrt{3}\jmath}{2\sqrt{3}\jmath}\frac{z}{z-\frac{(3-\sqrt{3}\jmath)}{2}}$$

so

$$y_k = \frac{3+\sqrt{3}\jmath}{2\sqrt{3}\jmath}(\sqrt{3})^k e^{\jmath\, k\pi/6} - \frac{3-\sqrt{3}\jmath}{2\sqrt{3}\jmath}(\sqrt{3})^k e^{-\jmath\, k\pi/6}$$

$$= 2(\sqrt{3})^k\left[\frac{\sqrt{3}}{2}\sin k\pi/6 + \frac{1}{2}\cos k\pi/6\right]$$

$$= 2(\sqrt{3})^k \sin(k+1)\pi/6$$

■ **23(c)**

$$Y_\delta(z)/z = \frac{z}{(z-0.4)(z+0.2)}$$

so

$$Y_\delta(z) = \frac{2}{3}\frac{z}{z-0.4} + \frac{1}{3}\frac{z}{z+0.2}$$

then

$$y_k = \frac{2}{3}(0.4)^k + \frac{1}{3}(-0.2)^k$$

23(d)

$$Y_\delta(z)/z = \frac{5z-12}{(z-2)(z-4)}$$

so

$$Y_\delta(z) = \frac{z}{z-2} + 4\frac{z}{z-4}$$

and

$$y_k = (2)^k + (4)^{k+1}$$

- **24(a)**

$$Y_\delta(z) = \frac{1}{z^2 - 3z + 2}$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

$$y_k = \begin{cases} 0, & k = 0 \\ 2^{k-1} - 1, & k \geq 1 \end{cases}$$

24(b)

$$Y_\delta(z) = \frac{1}{z-2}$$

so

$$y_k = \begin{cases} 0, & k = 0 \\ 2^{k-1}, & k \geq 1 \end{cases}$$

- **25** Examining the poles of the systems, we find

25(a) Poles at $z = -1/3$ and $z = -2/3$, both inside $\mid z \mid = 1$ so the system is stable.

25(b) Poles at $z = -1/3$ and $z = 2/3$, both inside $\mid z \mid = 1$ so the system is stable.

25(c) Poles at $z = 1/2 \pm 1/2\jmath$, $\mid z \mid = 1/\sqrt{2}$, so both inside $\mid z \mid = 1$ and the system is stable.

25(d) Poles at $z = -3/4 \pm \sqrt{17}/4$, one of which is outside $\mid z \mid = 1$ and so the system is unstable.

25(e) Poles at $z = -1/4$ and $z = 1$ thus one pole is on $\mid z \mid = 1$ and the other is inside and the system is marginally stable.

■ **26** To use the convolution result, calculate the impulse response as $y_{\delta,k} - (1/2)^k$. Then the step response is

$$y_k = \sum_{j=0}^{k} 1 \times (1/2)^{k-j} = (1/2)^k \sum_{j=0}^{k} 1 \times (2)^j = (1/2)^k \frac{1-(2)^{k+1}}{1-2}$$

$$= (1/2)^k (2^{k+1} - 1) = 2 - (1/2)^k$$

Directly,

$$Y(z)/z = \frac{z}{(z-1/2)(z-1)} = \frac{2}{z-1} - \frac{1}{z-1/2}$$

so

$$y_k = 2 - (1/2)^k$$

■ **27**

$$G(s) = \frac{k}{s(s\tau+1)} = \frac{k}{s} - \frac{k}{s+\frac{1}{\tau}} \Rightarrow f(t) = k - ke^{-\frac{t}{\tau}}$$

$$\Rightarrow G(z) = k\left[\frac{k}{z-1} - \frac{z}{z-e^{-\frac{T}{\tau}}}\right] = \frac{kz(1-e^{-\frac{T}{\tau}})}{(z-1)(z-e^{-\frac{T}{\tau}})}$$

Characteristic equation is $1 + G(z) = 0$

$$\Rightarrow (z-1)(z-e^{-\frac{T}{\tau}}) + kz(1-e^{-\frac{T}{\tau}}) = 0$$

$$\Rightarrow z^2 + [k(1-e^{-a}) - (1+e^{-a})]z + e^{-a} = 0, \text{ where } a = \frac{T}{\tau}$$

$$\Rightarrow z^2 + Kz + e^{-a} = 0, \text{ where } K = k(1-e^{-a}) - (1+e^{-a})$$

Using Jury's procedure:

$$F(z) = z^2 + Kz + e^{-a}$$

$$F(1) = 1 + K + e^{-a} = k(1-e^{-a}) > 0 \text{ since } k > 0, (1-e^{-a}) > 0$$

$$(-1)^2 F(-1) = 1 - Kz + e^{-a} > 0 \Rightarrow 2(1+e^{-a}) - k(1-e^{-a}) > 0$$

$$\Rightarrow k < \frac{2(1+e^{-\alpha})}{(1-e^{-a})} = 2\coth\left(\frac{a}{2}\right) = 2\coth\left(\frac{T}{2\tau}\right)$$

$$\Delta_1 = \begin{vmatrix} 1 & e^{-a} \\ e^{-a} & 1 \end{vmatrix} = 1 - e^{-2a} > 0$$

Thus system is stable if and only if $0 < k < 2\coth\left(\frac{T}{2\tau}\right)$

■ **28** Substituting

$$y_{n+1} - y_n + Ky_{n-1} = K/2^n$$

or

$$y_{n+2} - y_{n+1} + Ky_n = K/2^{n+1}$$

Taking z transforms from the quiescent state, the characteristic equation is

$$z^2 - z + K = 0$$

with roots

$$z_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1-4K} \text{ and } z_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1-4K}$$

For stability, both roots must be inside $\mid z \mid = 1$ so if $K < 1/4$ then

$$z_1 < 1 \Rightarrow \frac{1}{2} + \frac{1}{2}\sqrt{1-4K} < 1 \Rightarrow K > 0$$

and

$$z_2 > -1 \Rightarrow \frac{1}{2} - \frac{1}{2}\sqrt{1-4K} > -1 \Rightarrow k > -2$$

If $K > 1/4$ then

$$\mid \frac{1}{2} + \jmath \frac{1}{2}\sqrt{4K-1} \mid^2 < 1 \Rightarrow K < 1$$

The system is then stable for $0 < K < 1$.

When $k = 2/9$, we have

$$y_{n+2} - y_{n+1} + \frac{2}{9}y_n = \frac{1}{9}$$

Transforming with a quiescent initial state,

$$(z^2 - z + \frac{2}{9})Y(z) = \frac{1}{9}\frac{z}{z-1/2}$$

so

$$Y(z) = z\frac{1}{9}\left[\frac{1}{(z-1/2)(z-1/3)(z-2/3)}\right]$$

$$= 2\frac{z}{z-1/3} + 2\frac{z}{z-2/3} - 4\frac{z}{z-1/2}$$

which inverts to

$$y_n = 2(1/3)^n + 2(2/3)^n - 4(1/2)^n$$

■ **29**

$$z^2 + 2z + 2 = (z - (-1 + \jmath))(z - (-1 + \jmath))$$

establishing the pole locations. Then

$$Y_\delta(z) = \frac{1}{2\jmath}\frac{z}{z - (-1 + \jmath)} - \frac{1}{2\jmath}\frac{z}{z - (-1 - \jmath)}$$

So since $(-1 \pm \jmath) = \sqrt{2}e^{\pm 3\jmath\,\pi/4}$ etc.,

$$y_k = (\sqrt{2})^k \sin 3k\pi/4$$

Exercises 6.8.3

■ **30(a)**

$$z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z & -1 \\ -4 & z \end{bmatrix}$$

$$(z\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(z-2)(z+2)}\begin{bmatrix} z & 1 \\ 4 & z \end{bmatrix} = \begin{bmatrix} \frac{1/2}{z-2} + \frac{1/2}{z+2} & \frac{1/4}{z-2} - \frac{1/4}{z+2} \\ \frac{1}{z-2} - \frac{1}{z+2} & \frac{1/2}{z-2} + \frac{1/2}{z+2} \end{bmatrix}$$

$$\mathbf{A}^k = Z^{-1}\{z(z\mathbf{I} - \mathbf{A})^{-1}\} = Z^{-1}\begin{bmatrix} \frac{1}{2}\frac{1/2}{z-2} + \frac{1}{2}\frac{1/2}{z+2} & \frac{1}{4}\frac{z}{z-2} - \frac{1}{4}\frac{z}{z+2} \\ \frac{z}{z-2} - \frac{z}{z+2} & \frac{1}{2}\frac{z}{z-2} + \frac{z}{z+2} \end{bmatrix}$$

$$= \frac{1}{4}\left\{2^k\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} + (-2)^k\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\right\}$$

30(b)

$$z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z+1 & -3 \\ -3 & z+1 \end{bmatrix}$$

$$(z\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(z+4)(z-2)}\begin{bmatrix} z+1 & 3 \\ 3 & z+1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1/2}{z+4} + \frac{1/2}{z-2} & \frac{-1/2}{z+4} + \frac{1/2}{z-2} \\ -\frac{1/2}{z+4} + \frac{1/2}{z-2} & \frac{1/2}{z-2} + \frac{1/2}{z+2} \end{bmatrix}$$

$$\mathbf{A}^k = Z^{-1}\{z(z\mathbf{I} - \mathbf{A})^{-1}\} = Z^{-1}\begin{bmatrix} \frac{1}{2}\frac{z}{z+4} + \frac{1}{2}\frac{z}{z-2} & -\frac{1}{2}\frac{z}{z+4} + \frac{1}{2}\frac{z}{z-2} \\ -\frac{1}{2}\frac{z}{z+4} + \frac{1}{2}\frac{z}{z-2} & \frac{1}{2}\frac{z}{z-2} + \frac{1}{2}\frac{z}{z-2} \end{bmatrix}$$

$$= \frac{1}{2}\left\{(-4)^k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + (2)^k\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$$

30(c)

$$z\mathbf{I}-\mathbf{A}=\begin{bmatrix} z+1 & -1 \\ 0 & z+1 \end{bmatrix}$$

$$(z\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{(z+1)^2}\begin{bmatrix} z+1 & 1 \\ 0 & z+1 \end{bmatrix}=\begin{bmatrix} \frac{1}{z+1} & \frac{1}{(z+1)^2} \\ 0 & \frac{1}{z+1} \end{bmatrix}$$

$$\mathbf{A}^k=Z^{-1}\{z(z\mathbf{I}-\mathbf{A})^{-1}\}=Z^{-1}\begin{bmatrix} \frac{z}{z+1} & \frac{z}{(z+1)^2} \\ 0 & \frac{z}{z+1} \end{bmatrix}=(-1)^k\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$$

■ **31** Taking $x_1=x$ and $x_2=y$ we can express equations in the form

$$\mathbf{x}(k+1)=\begin{bmatrix} -7 & 4 \\ -8 & 1 \end{bmatrix}\mathbf{x}(k) \text{ with } \mathbf{x}(0)=[1\ 2]$$

The solution is given by

$$\mathbf{x}(k+1)=\mathbf{A}^k\mathbf{x}(0),\ \mathbf{A}=\begin{bmatrix} -7 & 4 \\ -8 & 1 \end{bmatrix}$$

where $\mathbf{A}^k=\alpha_1\mathbf{I}+\alpha_1\mathbf{A}$. That is, the solution is

$$\mathbf{x}(k)=\begin{bmatrix} \alpha_0-7\alpha_1 & 4\alpha_1 \\ -8\alpha_1 & \alpha_0+\alpha_1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} \alpha_0+\alpha_1 \\ 2\alpha_0-6\alpha_1 \end{bmatrix}$$

The eigenvalues of $\mathbf{A}$ are given by

$$\lambda^2-6\lambda+25=0 \text{ so } \lambda=3\pm j4$$

or, in polar form, $\lambda_1=5e^{j\theta},\ \lambda_2=5e^{-j\theta}$ where $\theta=\cos^{-1}(-\frac{3}{5})$.
Thus, α_0 and α_1 are given by

$$5^ke^{jk\theta}=\alpha_0+\alpha_1 5e^{j\theta},\ 5^ke^{-jk\theta}=\alpha_0+\alpha_1 5e^{-j\theta}$$

which are readily solved to give

$$\alpha_0=-\frac{(5)^k\sin(k-1)\theta}{\sin\theta}\ ,\ \alpha_1=\frac{1}{5}(5)^k\frac{\sin k\theta}{\sin\theta}$$

$$\text{Then,} \quad \alpha_0 + \alpha_1 = \frac{(5)^k}{\sin\theta}\left[\frac{1}{5}\sin k\theta - \sin(k-1)\theta\right]$$
$$= \frac{5}{4}(5)^k\left[\frac{1}{5}\sin k\theta + \frac{3}{5}\sin k\theta + \frac{4}{5}\cos k\theta\right]$$
$$= (5)^k[\sin k\theta + \cos k\theta]$$
$$2\alpha_0 - 6\alpha_1 = \frac{(5)^k}{\sin\theta}\left[-2(\sin k\theta\cos\theta - \cos k\theta\sin\theta) - \frac{6}{5}\sin k\theta\right]$$
$$= (5)^k[2\cos(k\theta)]$$

Thus, solution is

$$x(k) = (5)^k[\sin k\theta + \cos k\theta]$$
$$y(k) = (5)^k[2\cos(k\theta)]$$

We have

$$x(1) = 1\ ,\ y(1) = -6\ ,\ x(2) = -31\ ,\ y(2) = -14$$

■ **32** $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}$ $z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z & -1 \\ 0.16 & z+1 \end{bmatrix}$

$$(z\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{z+1}{(z+0.2)(z+0.8)} & \frac{1}{(z+0.2)(z+0.8)} \\ \frac{-0.16}{(z+0.2)(z+0.8)} & \frac{z}{(z+0.2)(z+0.8)} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{4}{3}\cdot\frac{1}{z+0.2} - \frac{1}{3}\cdot\frac{1}{z+0.8} & \frac{5}{3}\cdot\frac{1}{z+0.2} - \frac{5}{3}\cdot\frac{1}{z+0.8} \\ \frac{-0.8}{3}\frac{1}{z+0.2} + \frac{0.8}{3}\frac{1}{z+0.8} & -\frac{1}{3}\frac{1}{z+0.2} + \frac{4}{3}\cdot\frac{1}{z+0.8} \end{bmatrix}$$

$$\mathbf{A}^k = Z^{-1}\{z(z\mathbf{I} - \mathbf{A})^{-1}\} = \begin{bmatrix} \frac{4}{3}(-0.2)^k - \frac{1}{3}(-0.8)^k & \frac{5}{3}(-0.2)^k - \frac{5}{3}(-0.8)^k \\ \frac{-0.8}{3}(-0.2)^k + \frac{0.8}{3}(-0.8)^k & -\frac{1}{3}(-0.2)^k + \frac{4}{3}(-0.8)^k \end{bmatrix}$$

$$\mathbf{A}^k\mathbf{x}(0) = \mathbf{A}^k[1\ \ -1]^T = \begin{bmatrix} -\frac{1}{3}(-0.2)^k + \frac{4}{3}(-0.8)^k \\ \frac{0.2}{3}(-0.2)^k - \frac{3.2}{3}(-0.8)^k \end{bmatrix}$$

$$U(z) = Z\{u(k)\} = z/(z-1)$$
$$(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U(z) = \frac{1}{(z+0.2)(z+0.8)}\begin{bmatrix} z+1 & 1 \\ -0.16 & z \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{z}{z-1}$$

$$= \frac{z}{(z+0.2)(z+0.8)(z-1)} \begin{bmatrix} z+2 \\ z-0.16 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{2}\frac{z}{z+0.2} + \frac{10}{9}\frac{z}{z+0.8} + \frac{25}{18}\frac{z}{z-1} \\ \frac{1}{2}\frac{z}{z+0.2} - \frac{8}{9}\frac{z}{z+0.8} + \frac{7}{18}\frac{z}{z-1} \end{bmatrix}$$

$$Z^{-1}\{(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}U(z)\} = \begin{bmatrix} -\frac{5}{2}(-0.2)^k + \frac{10}{9}(-0.8)^k + \frac{25}{18} \\ \frac{1}{2}(-0.2)^k - \frac{8}{9}(-0.8)^k + \frac{7}{18} \end{bmatrix}$$

Thus, solution is

$$\mathbf{x}(k) = \mathbf{A}^k\mathbf{x}(0) + Z^{-1}\{(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}U(t)\}$$

$$= \begin{bmatrix} -\frac{17}{6}(-0.2)^k + \frac{22}{9}(-0.8)^k + \frac{25}{18} \\ \frac{3.4}{6}(-0.2)^k - \frac{17.6}{9}(-0.8)^k + \frac{7}{18} \end{bmatrix}$$

■ **33** Let $x_1(k) = y(k),\ x_2(k) = x_1(k+1) = y(k+1)$ then the difference equation may be written

$$\mathbf{x}(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},\ \mathbf{x}(0) = [0\ 1]^T$$

Taking $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ its eigenvalues are $\lambda_1 = \dfrac{1+\sqrt{5}}{2},\ \lambda_2 = \dfrac{1\sqrt{5}}{2}$

$\mathbf{A}^k = \alpha_1\mathbf{I} + \alpha_1\mathbf{A}$ where α_0 and α_1 satisfy

$$\left(\frac{1+\sqrt{5}}{2}\right)^k = \alpha_0 + \left(\frac{1+\sqrt{5}}{2}\right)\alpha_1,\ \left(\frac{1-\sqrt{5}}{2}\right)^k = \alpha_0 + \left(\frac{1-\sqrt{5}}{2}\right)\alpha_1$$

giving

$$\alpha_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

$$\alpha_0 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{\sqrt{5}-1}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{1+\sqrt{5}}{2}\right)\right]$$

Solution to the difference equation is

$$\mathbf{x}(k) = \begin{bmatrix} y(k) \\ y(k+1) \end{bmatrix} = \begin{bmatrix} \alpha_0 & \alpha_1 \\ \alpha_1 & \alpha_0+\alpha_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_0+\alpha_1 \end{bmatrix}$$

so $y(k) = \alpha_1 = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^k - \left(\dfrac{1-\sqrt{5}}{2}\right)^k\right]$

[Note that, $y(k+1) = \alpha_0 + \alpha_1 = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^k - \left(\dfrac{1-\sqrt{5}}{2}\right)^{k+1}\right]$ using above values.]

As $k \to \infty, \left(\dfrac{1-\sqrt{5}}{2}\right)^k \to 0$ and $y(k+1)/y(k) \to \dfrac{(\frac{1+\sqrt{5}}{2})^{k+1}}{(\frac{1+\sqrt{5}}{2})^k} = \dfrac{1}{2}(\sqrt{5}+1)$

Exercises 6.9.3

■ **34** $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix} \Rightarrow (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\Rightarrow \mathbf{G} = e^{-\mathbf{A}T} = L^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\} = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2T}) \\ 0 & e^{-2T} \end{bmatrix}$$

$$\mathbf{H} = \int_0^T e^{\mathbf{A}t}\mathbf{B}dt = \begin{bmatrix} t & \frac{1}{2}t + \frac{1}{4}e^{-2t} \\ 0 & -\frac{1}{2}e^{-2t} \end{bmatrix}_0^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}T + \frac{1}{4}e^{-2T} - \frac{1}{4} \\ \frac{1}{2} - \frac{1}{2}e^{-2T} \end{bmatrix}$$

Discretized form is:

$$\begin{bmatrix} x_1[(k+1)T] \\ x_2[(k+1)T] \end{bmatrix} = \begin{bmatrix} 1 & \frac{!}{2}(1 - e^{-2T}) \\ 0 & e^{-2T} \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix} + \begin{bmatrix} \frac{1}{2}T + \frac{1}{4}(e^{-2T} - 1) \\ \frac{1}{2}(1 - e^{-2T}) \end{bmatrix} u(kT)$$

In the particular case, when sampling period is $T = 1$, this reduces to

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.432 \\ 0 & 0.135 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.284 \\ 0.432 \end{bmatrix} u(k)$$

In MATLAB the commands:

$$A = [0, 1; 0, -2]; B = [0; 1];$$
$$[G, H] = c2d(A, B, 1)$$

return

$$G = \begin{matrix} 1.0000 & 0.4323 \\ 0 & 0.1353 \end{matrix}$$
$$H = \begin{matrix} 0.2838 \\ 0.4324 \end{matrix}$$

which check with the answer.

■ **35**

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(a) Using (6.89),

$$\mathbf{G}_1 = (T\mathbf{A} + \mathbf{I}) = \begin{bmatrix} 1 & T \\ -T & 1-T \end{bmatrix}$$

$$\mathbf{H}_1 = T\mathbf{B} = \begin{bmatrix} 0 \\ T \end{bmatrix}$$

giving the Euler discretized form of state-space model as

$$\mathbf{x}[(k+1)T] = \begin{bmatrix} x_1[(k+1)T] \\ x_2[(k+1)T] \end{bmatrix} = \begin{bmatrix} 1 & T \\ -T & 1-T \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix} + \begin{bmatrix} 0 \\ T \end{bmatrix} u(kT)$$

$$y(kT) = [\,1 \quad 0\,]\,\mathbf{x}(kT)$$

(b)

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix} \Rightarrow (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + s + 1} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}$$

$$\Rightarrow (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \dfrac{(s+\frac{1}{2}) + \frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} & \dfrac{1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \\ \dfrac{-1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} & \dfrac{(s+\frac{1}{2}) - \frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \end{bmatrix}$$

$$\Rightarrow \mathbf{G} = L^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}$$

$$= e^{-\frac{T}{2}} \begin{bmatrix} \cos(\frac{\sqrt{3}}{2}T) + \frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}T) & \frac{2}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}T) \\ -\frac{2}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}T) & \cos(\frac{\sqrt{3}}{2}T) - \frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}T) \end{bmatrix}$$

Since $\det(\mathbf{A}) \neq 0$ the matrix $\mathbf{H}$ is best determined using (6.95),

$$\mathbf{H} = (\mathbf{G} - \mathbf{I})\mathbf{A}^{-1}\mathbf{B} = (\mathbf{G} - \mathbf{I}) \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{H} = \begin{bmatrix} 1 - e^{-\frac{T}{2}}\cos(\frac{\sqrt{3}}{2}T) - \frac{1}{\sqrt{3}}e^{-\frac{T}{2}}\sin(\frac{\sqrt{3}}{2}T) \\ \frac{2}{\sqrt{3}}e^{-\frac{T}{2}}\sin(\frac{\sqrt{3}}{2}T) \end{bmatrix}$$

giving the step-invariant discretized form of the state-space model as

$$\mathbf{x}[(k+1)] = \mathbf{G}\mathbf{x}(kT) + \mathbf{H}u(kT)$$

$$y(kT) = [\,1 \quad 0\,]\mathbf{x}(kT)$$

■ **36(a)** The eigenvalues of the matrix $\mathbf{A}$ are given by

$$\det(\lambda\mathbf{I}-\mathbf{A})=0\Rightarrow\begin{vmatrix}\lambda+1 & -1\\ 1 & \lambda+2\end{vmatrix}=0$$
$$\Rightarrow\lambda^2+3\lambda+1=0\Rightarrow\lambda_1=-\frac{3}{2}+j\sqrt{3},\lambda_2=-\frac{3}{2}-j\sqrt{3}$$

Both eigenvalues have negative real parts so the matrix $\mathbf{A}$ represents a stable system.

(b) From (6.89), the corresponding Euler discretized state matrix $\mathbf{A}_{\mathrm{d}}$ is given by

$$\mathbf{A}_d=\mathbf{G}_1=\mathbf{I}+T\mathbf{A}=\begin{bmatrix}1-T & T\\ -T & 1-2T\end{bmatrix}$$

(c) The eigenvalues of the matrix $\mathbf{A}_{\mathrm{d}}$ are given by

$$\begin{vmatrix}\lambda-1+T & -T\\ T & \lambda-1+2T\end{vmatrix}=0$$
$$\Rightarrow\lambda^2+(3T-2)\lambda+(3T^2-3T+1)=0$$

Let $\mathrm{F}(\lambda)=\lambda^2+(3T-2)\lambda+(3T^2-3T+1)$ then
$F(1)=1+(3T-2)+(1-3T+3T^2)=3T^2>0$ if $T>0$ (since T is non-negative by definition).
$(-1)^2F(-1)=1-((3T-2)+1-3T+3T^2=3T^2-6T+4>0$ all T.
Taking $a_1=(3T-2)$ and $a_0=(1-3T+3T^2)$ gives

$$\mathrm{F}(\lambda)=\lambda^2+a_1\lambda+a_0$$

leading to Jury table:

$$\begin{matrix}1 & a_1 & a_0\\ a_0 & a_1 & 1\end{matrix}$$

$$\begin{matrix}\Delta_1=1-a_0^2 & a_1(1-a_0)\\ a_1(1-a_0) & 1-a_0^2\end{matrix}$$
$$\Delta_2=(1-a_0^2)^2-a_1^2(1-a_0)^2=(1-a_0)^2[(1+a_0)^2-a_1^2]$$

with $\Delta_1>0$ if $1-a_0^2>0\Rightarrow|a_0|<1$
and $\Delta_2>0$ if $[(1+a_0)^2-a_1^2]>0\Rightarrow 1+a_0>|a_1|$ or $a_0>-1+a_1$ and $a_0>-1-a_1$.
In terms of T these conditions become:

$\Delta_1 > 0 \Rightarrow 3T^2 - 3T + 1 < 1 \Rightarrow T(T-1) < 0 \Rightarrow T < 1$

$\Delta_2 > 0 \Rightarrow 3T^2 - 3T + 1 > 3T - 3 \Rightarrow 3T^2 - 6T + 4 > 0$ all T (as above);

and $3T^2 - 3T + 1 > -3T - 1 \Rightarrow T^2 > 0$

Thus discrete system stable provided $0 < T < 1$.

- **37(a)**

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{B} = \begin{bmatrix} k_1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s+1 & 0 \\ -1 & s \end{bmatrix} \Rightarrow (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s(s+1)} \begin{bmatrix} s & 0 \\ 1 & s+1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \dfrac{1}{s} - \dfrac{1}{s+1} & \dfrac{1}{s} \end{bmatrix}$$

$$\Rightarrow \mathbf{G} = e^{\mathbf{A}T} = L^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\} = \begin{bmatrix} e^{-T} & 0 \\ (1 - e^{-T}) & 1 \end{bmatrix}$$

$$\mathbf{H} = \int_0^T e^{\mathbf{A}t}\mathbf{B}\, dt = \begin{bmatrix} -e^{-t} & c \\ t + e^{-t} & t \end{bmatrix}_0^T \begin{bmatrix} k_1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} k_1(1 - e^{-T}) & 0 \\ k_1(e^{-T} + T - 1) & -T \end{bmatrix}$$

Thus discrete form of model is

$$\mathbf{x}[(k+1)T] = \begin{bmatrix} e^{-T} & 0 \\ (1 - e^{-T}) & 1 \end{bmatrix} \mathbf{x}(kT) + \begin{bmatrix} k_1(1 - e^{-T}) & 0 \\ k_1(e^{-T} + T - 1) & -T \end{bmatrix} \mathbf{u}(kT)$$

In the particular case $T = 1$, the model becomes

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.368 & 0 \\ 0.632 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.632\, k_1 & 0 \\ 0.368 k_1 & -1 \end{bmatrix} \mathbf{u}(k)$$

(b) Taking sampling period $T = 1$ and feedback control policy,

$$u_1(k) = k_c - x_2(k)$$

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.368 & 0 \\ 0.632 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.632 k_1 & 0 \\ 0.368 k_1 & -1 \end{bmatrix} \begin{bmatrix} k_c - x_2(k) \\ u_2(k) \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(k+1) = \begin{bmatrix} 0.368 & -0.632 k_1 \\ 0.632 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.632 k_1 & 0 \\ 0.368 k_1 & -1 \end{bmatrix} \begin{bmatrix} k_c \\ u_2(k) \end{bmatrix}$$

Given $u_2 = 1.1 x_1(0)$ and $k_1 = \frac{3}{16}$, the discrete-time state-equation becomes

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.368 & -0.1185 \\ 0.632 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.1185 & 0 \\ 0.069 & -1 \end{bmatrix} \begin{bmatrix} k_c \\ 1.1x_1(0) \end{bmatrix}$$

(c) Adopting the feedback control policy

$$u_1(t) = k_c - x_2(t)$$

the given continuous-time state model becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & k_1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} k_1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} k_c \\ u_2 \end{bmatrix}$$

Taking $k_1 = \frac{3}{16}$ and $u_2 = 1.1x_1(0)$ this reduces to

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -\frac{3}{16} \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{3}{16} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} k_c \\ 1.1x_1(0) \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A}_c) = \begin{bmatrix} s+1 & \frac{3}{16} \\ -1 & s \end{bmatrix} \Rightarrow (s\mathbf{I} - \mathbf{A}_c)^{-1} = \frac{1}{s^2 + s + \frac{3}{16}} \begin{bmatrix} s & -\frac{3}{16} \\ 1 & s+1 \end{bmatrix}$$

$$\Rightarrow (s\mathbf{I} - \mathbf{A}_c)^{-1} = \begin{bmatrix} \dfrac{-\frac{1}{2}}{s+\frac{1}{4}} + \dfrac{\frac{3}{2}}{s+\frac{3}{4}} & \dfrac{-\frac{3}{8}}{s+\frac{1}{4}} + \dfrac{\frac{3}{8}}{s+\frac{3}{4}} \\ \dfrac{2}{s+\frac{1}{4}} - \dfrac{2}{s+\frac{3}{4}} & \dfrac{\frac{3}{2}}{s+\frac{1}{4}} - \dfrac{\frac{1}{2}}{s+\frac{3}{4}} \end{bmatrix}$$

giving

$$e^{\mathbf{A}_c t} = L^{-1}\{(s\mathbf{I} - \mathbf{A}_c)^{-1}\} = \begin{bmatrix} -\frac{1}{2}e^{-\frac{1}{4}t} + \frac{3}{2}e^{-\frac{3}{4}t} & -\frac{3}{8}e^{-\frac{1}{4}t} + \frac{3}{8}e^{-\frac{3}{4}t} \\ 2e^{-\frac{1}{4}t} - 2e^{-\frac{3}{4}t} & \frac{3}{2}e^{-\frac{1}{4}t} - \frac{1}{2}e^{-\frac{3}{4}t} \end{bmatrix}$$

The response of the continuous feedback system is

$$\mathbf{x}(t) = e^{\mathbf{A}_c t} \begin{bmatrix} x_1(0) \\ k_c \end{bmatrix} + \int_0^t e^{\mathbf{A}(t-\tau)} d\tau \mathbf{B} \begin{bmatrix} k_c \\ 1.1x_1(0) \end{bmatrix}$$

Carrying out the integration and simplifying gives the response

$$x_1(t) = x_1(0)[1.1 - 2.15e^{-\frac{1}{4}t} + 2.05e^{-\frac{3}{4}t}]$$
$$x_2(t) = k_c + x_1(0)[-5.867 + 8.6e^{-\frac{1}{4}t} - 2.714e^{-\frac{3}{4}t}]$$

Exercises 6.11.6

■ **38**

$$H(s) = \frac{1}{s^2 + 3s + 2}$$

Replace s with $\dfrac{2}{\Delta}\,\dfrac{z-1}{z+1}$ to give

$$\tilde{H}(z) = \frac{\Delta^2(z+1)^2}{4(z-1)^2 + 6\Delta(z^2-1) + 2\Delta^2(z+1)^2}$$

$$= \frac{\Delta^2(z+1)^2}{(4 + 6\Delta + 2\Delta^2)z^2 + (4\Delta^2 - 8)z + (4 - 6\Delta + 2\Delta^2)}$$

This corresponds to the difference equation

$$(Aq^2 + Bq + C)y_k = \Delta^2(q^2 + 2q + 1)u_k$$

where

$$A = 4 + 6\Delta + 2\Delta^2 \quad B = 4\Delta^2 - 8 \quad C = 4 - 6\Delta + 2\Delta^2$$

Now put $q = 1 + \Delta\delta$ to get

$$(A\Delta^2\delta^2 + (2A + B)\Delta\delta + A + B + C)y_k$$

$$= \Delta^2(\Delta^2\delta^2 + 4\Delta\delta + 4)u_k$$

With $t = 0.01$ in the q form the system poles are at $z = 0.9048$ and $z = 0.8182$, inside $\mid z \mid = 1$. When $t = 0.01$ these move to $z = 0.9900$ and $z = 0.9802$, closer to the stability boundary. Using the δ form with $t = 0.1$, the poles are at $\nu = -1.8182$ and $\nu = -0.9522$, inside the circle centre $(-10, 0)$ in the ν-plane with radius 10. When $t = 0.01$ these move to $\nu = -1.9802$ and $\nu = -0.9950$, within the circle centre $(-100, 0)$ with radius 100, and the closest pole to the boundary has moved slightly further from it.

■ **39** The transfer function is

$$H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

To discretize using the bi-linear form use $s \to \dfrac{2}{T}\dfrac{z-1}{z+1}$ to give

$$\tilde{H}(z) = \frac{T^3(z+1)^3}{Az^3 + Bz^2 + Cz + D}$$

and thus the discrete-time form

$$(Aq^3 + Bq^2 + Cq + D)y_k = T^3(q^3 + 3q^2 + 3q + 1)u_k$$

where

$$A = T^3 + 4T^2 + 8T + 8, \quad B = 3T^3 + 4T^2 - 8T - 3,$$

$$C = 3T^3 - 4T^2 - 8T + 3, \quad D = T^3 - 4T^2 + 8T - 1$$

To obtain the δ form use $s \to \dfrac{2\delta}{2 + \Delta\delta}$ giving the δ transfer function as

$$\frac{(2 + \Delta\delta)^3}{A\delta^3 + B\delta^2 + C\delta + D}$$

This corresponds to the discrete-time system

$$(A\delta^3 + B\delta^2 + C\delta + D)y_k = (\Delta^3\delta^3 + 2\Delta^2\delta^2 + 4\Delta\delta + 8)u_k$$

where

$$A = \Delta^3 + 4\Delta^2 + 8\Delta + 8, \quad B = 6\Delta^2 + 16\Delta + 16,$$

$$C = 12\Delta + 16, \quad D = 8$$

■ **40** Making the given substitution and writing the result in vector–matrix form, we obtain

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

and

$$y(t) = [1, 0]\mathbf{x}(t)$$

This is in the general form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$$

$$y = \mathbf{c}^T\mathbf{x}(t) + d\,u(t)$$

The Euler discretization scheme gives at once

$$\mathbf{x}((k+1)\Delta) = \mathbf{x}(k\,\Delta) + \Delta\,[\mathbf{A}\mathbf{x}(k\,\Delta) + \mathbf{b}u(k\,\Delta)]$$

Using the notation of Exercise 38 write the simplified δ form equation as

$$\left[\delta^2 + \frac{12+8\Delta}{A}\delta + \frac{8}{A}\right] y_k = \frac{1}{A}\left[\Delta^2\delta^2 + 4\Delta\delta + 4\right] u_k$$

Now, as usual, consider the related system

$$\left[\delta^2 + \frac{12+8\Delta}{A}\delta + \frac{8}{A}\right] p_k = u_k$$

and introduce the state variables $x_1(k) = p_k$, $x_2(k) = \delta p_k$ together with the redundant variable $x_3(k) = \delta^2 p_k$. This leads to the representation

$$\delta\mathbf{x}(k) = \begin{bmatrix} 0 & 1 \\ -\dfrac{8}{A} & -\dfrac{12+8\Delta}{A} \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y_k = \left[\left(\frac{4}{A} - \frac{8\Delta^2}{A^2}\right), \left(\frac{4\Delta}{A} - \frac{(12+8\Delta)\Delta^2}{A^2}\right)\right] \mathbf{x}(k) + \frac{\Delta^2}{A}u(k)$$

or

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \Delta\,[\mathbf{A}(\Delta)\mathbf{x}(k) + \mathbf{b}u(k)]$$

$$y_k = \mathbf{c}^T(\Delta)\mathbf{x}(k) + d(\Delta)u_k$$

Since $A(0) = 4$ it follows that using $\mathbf{A}(0)$, $\mathbf{c}(0)$ and $d(0)$ generates the Euler Scheme when $\mathbf{x}(k) = \mathbf{x}(k\Delta)$ etc.

■ **41(a)** In the z form substitution leads directly to

$$H(z) = \frac{12(z^2 - z)}{(12+5\Delta)z^2 + (8\Delta - 12)z - \Delta}$$

When $\Delta = 0.1$, this gives

$$H(z) = \frac{12(z^2 - z)}{12.5z^2 + -11.2z - 0.1}$$

(b) The γ form is given by replacing z by $1 + \Delta\gamma$. Substitution and rearrangement gives

$$\tilde{H}(\gamma) = \frac{12\gamma(1 + \Delta\gamma)}{\gamma^2\Delta(12 + 5\Delta) + \gamma(8\Delta - 12) + 12}$$

when $\Delta = 0.1$, this gives

$$\tilde{H}(\gamma) = \frac{12\gamma(1 + 0.1\gamma)}{1.25\gamma^2 - 11.2\gamma + 12}$$

Review exercises 6.12

- **1**

$$\mathcal{Z}\{f(kT)\} = \mathcal{Z}\{kT\} = T\mathcal{Z}\{k\} = T\frac{z}{(z-1)^2}$$

- **2**

$$\begin{aligned}\mathcal{Z}\left\{a^k \sin k\omega\right\} &= \mathcal{Z}\left\{\frac{a^k(e^{\jmath k\omega} - e^{-\jmath k\omega})}{2\jmath}\right\} \\ &= \frac{1}{2\jmath}\mathcal{Z}\left\{(ae^{\jmath\omega})^k - (ae^{-\jmath\omega})^k\right\} \\ &= \frac{1}{2\jmath}\left(\frac{z}{z - ae^{\jmath\omega}} - \frac{z}{z - ae^{-\jmath\omega}}\right) \\ &= \frac{az\sin\omega}{z^2 - 2az\cos\omega + a^2}\end{aligned}$$

- **3** Recall that

$$\mathcal{Z}\left\{a^k\right\} = \frac{z}{(z-a)^2}$$

Differentiate twice wrt a, then put $a = 1$ to get the pairs

$$k \longleftrightarrow \frac{z}{(z-1)^2} \quad k(k-1) \longleftrightarrow \frac{2z}{(z-1)^3}$$

then

$$\mathcal{Z}\left\{k^2\right\} = \frac{2z}{(z-1)^3} + \frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3}$$

■ **4**

$$H(z) = \frac{3z}{z-1} + \frac{2z}{(z-1)^2}$$

so inverting, the impulse response is

$$\{3 + 2k\}$$

■ **5**

$$Y_{\text{STEP}}(z) = \frac{z}{(z+1)(z+2)(z-1)}$$
$$= -\frac{1}{2}\frac{z}{z+1} + \frac{1}{3}\frac{z}{z+2} + \frac{1}{6}\frac{z}{z-1}$$

Thus

$$y_{\text{STEP},k} = -\frac{1}{2}(-1)^k + \frac{1}{3}(-2)^k + \frac{1}{6}$$

■ **6**

$$F(s) = \frac{1}{s+1} = \frac{1}{s} - \frac{1}{s+1}$$

which inverts to

$$f(t) = (1 - e^{-t})\zeta(t)$$

where $\zeta(t)$ is the Heaviside step function, and so

$$\tilde{F}(z) = \mathcal{Z}\{f(kT)\} = \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$

Then

$$e^{-sT}F(s) \longleftrightarrow f((t-T))$$

which when sampled becomes $f((k-1)T)$ and

$$\mathcal{Z}\{f((k-1)T)\} = \sum_{k=0}^{\infty} \frac{f((k-1)T)}{z^k} = \frac{1}{z}\tilde{F}(z)$$

That is

$$e^{-sT}F(s) \to \frac{1}{z}\tilde{F}(z)$$

So the overall transfer function is

$$\frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right) = \frac{1-e^{-T}}{z-e^{-T}}$$

■ **7**

$$H(s) = \frac{s+1}{(s+2)(s+3)} = \frac{2}{s+3} - \frac{1}{s+2}$$

$$y_\delta(t) = 2e^{-3t} - e^{2t} \longrightarrow \{2e^{-3kT} - e^{2kT}\}$$

so

$$\tilde{H}(z) = 2\frac{z}{z - e^{-3T}} - \frac{z}{z - e^{-2T}}$$

■ **8(a)** Simple poles at $z = a$ and $z = b$. The residue at $z = a$ is

$$\lim_{z\to a}(z-a)z^{n-1}X(z) = \lim_{z\to a}(z-a)\frac{z^n}{(z-a)(z-b)} = \frac{a^n}{a-b}$$

The residue at $z = b$ is similarly $\dfrac{b^n}{b-a}$ and the inverse transform is the sum of these, that is

$$\left\{\frac{a^n - b^n}{a-b}\right\}$$

■ **8(b)**

(i) There is a only double pole at $z = 3$ and the residue is

$$\lim_{z\to 3}\frac{d}{dz}(z-3)^2\frac{z^n}{(z-3)^2} = \left\{n3^{n-1}\right\}$$

(ii) There are now simple poles at $z = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}\jmath$. The individual residues are thus given by

$$\lim_{z\to(1/2\pm\sqrt{3}/2\jmath)} \pm\frac{\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\jmath\right)^n}{\sqrt{3}\jmath}$$

Adding these and simplifying in the usual way gives the inverse transform as

$$\left\{\frac{2}{\sqrt{3}}\sin n\pi/3\right\}$$

■ **9**

$$H(z) = \frac{z}{z+1} - \frac{z}{z-2}$$

so

$$Y_{\text{STEP}}(z) = \left(\frac{z}{z+1} - \frac{z}{z-2}\right)\frac{z}{z-1}$$

$$= -\frac{3z}{(z-1)(z+1)(z-2)}.\ z$$
$$= \frac{3}{2}\frac{z}{z-1} + \frac{1}{2}\frac{z}{z+1} - 2\frac{z}{z-2}$$

so

$$y_{\text{STEP,k}} = \frac{3}{2} + \frac{1}{2}(-1)^k - 2^{k+1}$$

■ **10**

$$Y(z) = \frac{z^2}{(z+1)(z-1)} \times \left(1 - \frac{1}{z}\right) = \frac{z}{z+1}$$

so

$$y_k = (-1)^k$$

■ **11**

$$Y(z) = \frac{z^2}{(z-\alpha)(z-\beta)} \times \left(1 - \frac{\alpha+\beta}{z} + \frac{\alpha\beta}{z^2}\right) = 1$$

so

$$y_k = \{\delta_k\} = \{1, 0, 0, \ldots\}$$

The response of the system with $H(z) = \dfrac{z}{(z-\alpha)(z-\beta)}$ is clearly given by $Y(z) = 1/z$, which transforms to

$$y_k = \{\delta_{k-1}\} = \{0, 1, 0, 0, \ldots\}$$

■ **12** From $H(s) = \dfrac{s}{(s+1)(s+2)}$ the impulse response is calculated as

$$y_\delta(t) = (2e^{-2t} - e^{-t})\ t \geq 0$$

Sampling gives

$$\{y_\delta(nT)\} = \left\{2e^{-2nT} - e^{nTt}\right\}$$

with z transform

$$\mathcal{Z}\{y_\delta(nT)\} = 2\frac{z}{z-e^{-2T}} - \frac{z}{z-e^{-T}} = D(z)$$

Setting $Y(z) = TD(z)X(z)$ gives

$$Y(z) = T\left[2\frac{z}{z-e^{-2T}} - \frac{z}{z-e^{-T}}\right] X(z)$$

Substituting for T and simplifying gives

$$Y(z) = \frac{1}{2}z\left[\frac{z - 0.8452}{z^2 - 0.9744z + 0.2231}\right]X(z)$$

so

$$(z^2 - 0.9744z + 0.2231)Y(z) = (0.5z^2 - 0.4226z)X(x)$$

leading to the difference equation

$$y_{n+2} - 0.9744y_{n+1} + 0.2231y_n = 0.5x_{n+2} - 0.4226x_{n+1}$$

As usual (see Exercise 22), draw the block diagram for

$$p_{n+2} - 0.9744p_{n+1} + 0.2231p_n = x_n$$

then taking $y_n = 0.5p_{n+2} - 0.4226p_{n+1}$

$$y_{n+2} - 0.9744y_{n+1} + 0.2231y_n = 0.5p_{n+4} - 0.4226p_{n+3}$$

$$-0.9774(0.5p_{n+3} - 0.4226p_{n+2}) + 0.2231(0.5p_{n+2} - 0.4226p_{n+1})$$

$$= 0.5x_{n+2} - 0.4226x_{n+1}$$

■ **13**

$$\begin{aligned} y_{n+1} &= y_n + av_n \\ v_{n+1} &= v_n + bu_n \\ &= v_n + b(k_1(x_n - y_n) - k_2v_n) \\ &= bk_1(x_n - y_n) + (1 - bk_2)v_n \end{aligned}$$

so

$$y_{n+2} = y_{n+1} + a[bk_1(x_n - y_n) + (1 - bk_2)v_n]$$

(a) Substituting the values for k_1 and k_2, we get

$$y_{n+2} = y_{n+1} + \frac{1}{4}(x_n - y_n)$$

or

$$y_{n+2} - y_{n+1} + \frac{1}{4}y_n = \frac{1}{4}x_n$$

Transforming with relaxed initial conditions gives

$$Y(z) = \frac{1}{(2z-1)^2}X(z)$$

(b) When $X(z) = \dfrac{A}{z-1}$,

$$Y(z) = \frac{A}{4}\left[4\frac{z}{z-1} - 4\frac{z}{z-1/2} - 2\frac{z}{(z-1/2)^2}\right]$$

then

$$y_n = \frac{A}{4}\left[4 - 4(1/2)^n - 2n(1/2)^{n-1}\right]$$

■ **14** Substitution leads directly to

$$\frac{y_k - 2y_{k-1} + y_{k-2}}{T^2} + 3\frac{y_k - y_{k-1}}{T} + 2y_k = 1$$

Take the z transform under the assumption of a relaxed system to get

$$[(1 + 3Tz + 2T^2)z^2 - (2 + 3T)z + 1]Y(z) = T^2\frac{z^3}{z-1}$$

The characteristic equation is thus

$$(1 + 3Tz + 2T^2)z^2 - (2 + 3T)z + 1 = 0$$

with roots (the poles)

$$z = \frac{1}{1+T},\ z = \frac{1}{1+2T}$$

The general solution of the difference equation is a linear combination of these together with a particular solution. That is

$$y_k = \alpha\left(\frac{1}{1+T}\right)^k + \beta\left(\frac{1}{1+2T}\right)^k + \gamma$$

This can be checked by substitution which also shows that $\gamma = 1/2$. The condition $y(0) = 0$ gives $y_0 = 0$ and since $y'(t) \to \dfrac{y_k - y_{k-1}}{T}$, $y'(0) = 0$ implies $y_{k-1} = 0$. Using these we have

$$\alpha + \beta + \frac{1}{2} = 0$$

$$\alpha(1+T) + \beta(1+2T) + \frac{1}{2} = 0$$

with solution $\alpha = -1,\ \beta = 1/2$ so

$$y_k = -\left(\frac{1}{1+T}\right)^k + \frac{1}{2}\left(\frac{1}{1+2T}\right)^k + \frac{1}{2}$$

The differential equation is simply solved by inverting the Laplace transform to give

$$y(t) = \frac{1}{2}(e^{-2t} - 2e^{-t} + 1),\ t \geq 0$$

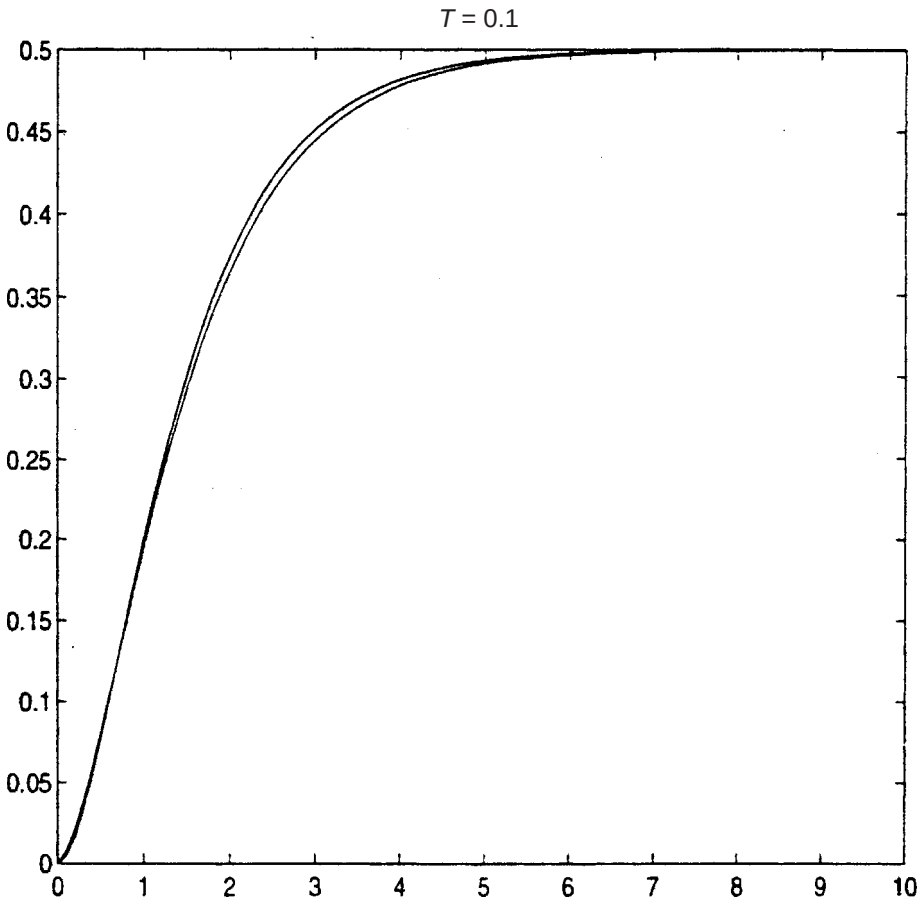


Figure 6.3: Response of continuous and discrete systems in Review Exercise 14 over 10 seconds when $T = 0.1$.

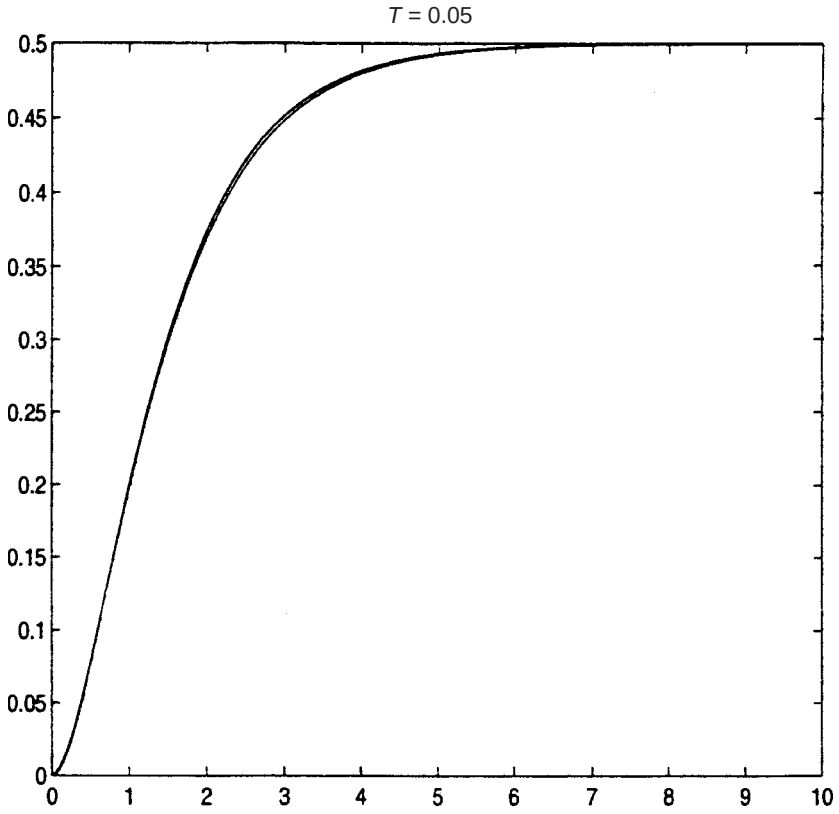


Figure 6.4: Response of continuous and discrete systems in Review Exercise 14 over 10 seconds when $T = 0.05$.

■ **15** Substitution for s and simplifying gives

$$[(4+6T+2T^2)z^2+(4T^2-8)z+(4-6T+2T^2)]Y(z)$$
$$=T^2(z+1)^2X(x)$$

The characteristic equation is

$$(4+6T+2T^2)z^2+(4T^2-8)z+(4-6T+2T^2)=0$$

with roots

$$z=\frac{8-4T^2\pm 4T}{2(4+6T+2T^2)}$$

That is,

$$z=\frac{1-T}{1+T} \text{ and } z=\frac{2-T}{2+T}$$

The general solution of the difference equation is then

$$y_k = \alpha\left(\frac{1-T}{1+T}\right)^k + \beta\left(\frac{2-T}{2+T}\right)^k + \gamma$$

This can be checked by substitution which also shows that $\gamma = 1/2$. The condition $y(0) = 0$ gives $y_0 = 0$ and since $y'(t) \to \dfrac{y_k - y_{k-1}}{T}$, $y'(0) = 0$ implies $y_{k-1} = 0$. Using these we have

$$\alpha + \beta + \frac{1}{2} = 0$$

$$\alpha\frac{1+T}{1-T} + \beta\frac{2+T}{2-T} + \frac{1}{2} = 0$$

with solution

$$\alpha = \frac{1-T}{2} \quad \beta = -\frac{2-T}{2}$$

Thus,

$$y_k = \frac{1-T}{2}\left(\frac{1-T}{1+T}\right)^k + -\frac{2-T}{2}\left(\frac{2-T}{2+T}\right)^k + \frac{1}{2}$$

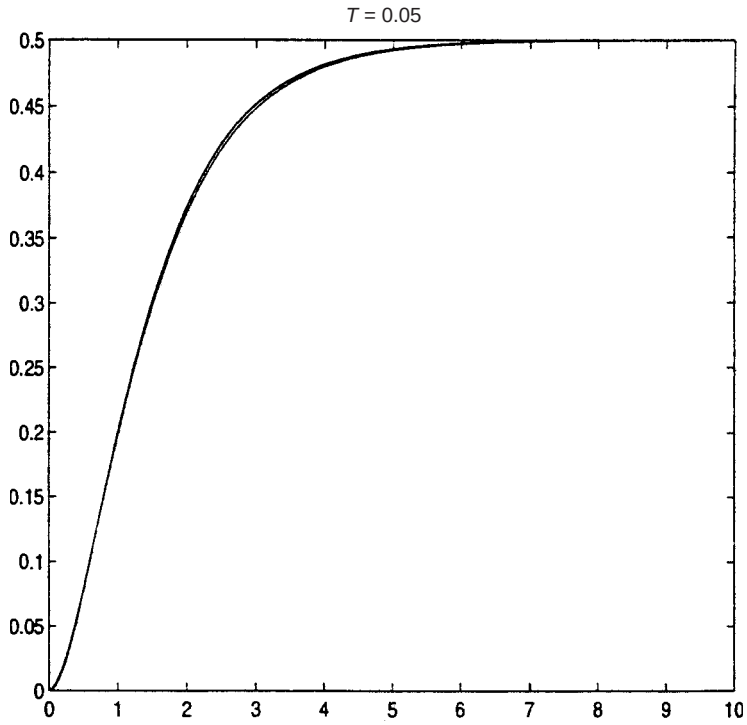


Figure 6.5: Response of continuous and discrete systems in Review Exercise 15 over 10 seconds when $T = 0.1$.

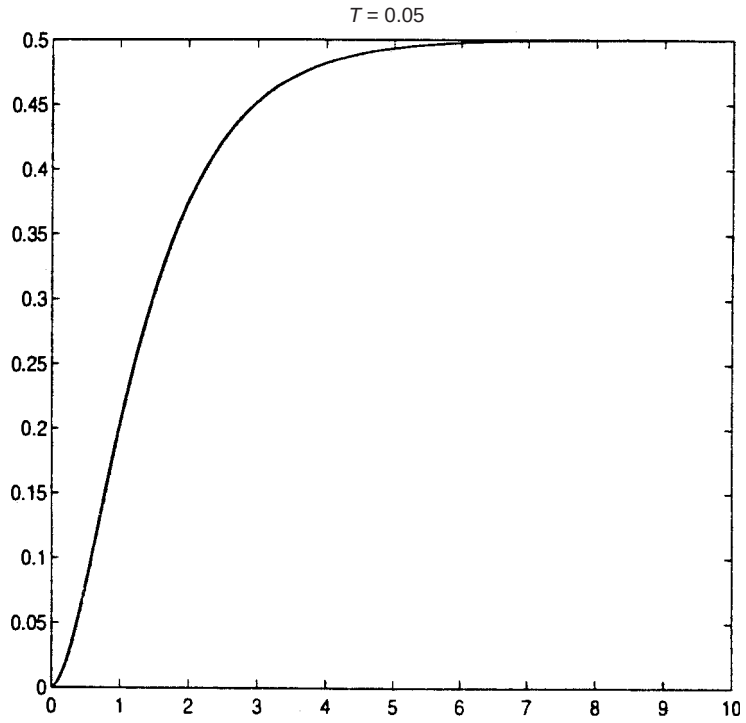


Figure 6.6: Response of continuous and discrete systems in Review Exercise 14 over 10 seconds when $T = 0.05$.

■ **16**

$$f(t) = t^2, \quad \{f(k\Delta)\} = \left\{k^2\Delta^2\right\},\ k \geq 0$$

Now

$$\mathcal{Z}\{k^2\} = -z\frac{\mathrm{d}}{\mathrm{d}z}\frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3}$$

So

$$\mathcal{Z}\{k^2\Delta^2\} = \frac{z(z+1)\Delta^2}{(z-1)^3}$$

To get $\mathcal{D}$-transform, put $z = 1 + \Delta\gamma$ to give

$$F'_\Delta(\gamma) = \frac{(1+\Delta\gamma)(2+\Delta\gamma)\Delta^2}{\Delta^3\gamma^3}$$

Then the $\mathcal{D}$-transform is

$$F_\Delta(\gamma) = \Delta F'_\Delta(\gamma) = \frac{(1+\Delta\gamma)(2+\Delta\gamma)}{\gamma^3}$$

■ **17** Eigenvalues are given by

$$0 = \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-\lambda-3)+(1-\lambda)$$
$$= (1-\lambda)(\lambda-2)(\lambda+1)$$

so eigenvalues are $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$

Corresponding eigenvectors are the solutions of

$$(1-\lambda_i)e_{i1} + e_{i2} - 2e_{i3} = 0$$
$$-e_{i1} + (2-\lambda_i)e_{i2} + e_{i3} = 0$$
$$0e_{i1} + e_{i2} - (1+\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ the eigenvectors are

$$\mathbf{e}_1 = [1\ 3\ 1]^T,\ \mathbf{e}_2 = [3\ 2\ 1]^T,\ \mathbf{e}_3 = [1\ 0\ 1]^T$$

The modal matrix $\mathbf{M}$ and spectral matrix $\mathbf{\Lambda}$ are

$$\mathbf{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ \mathbf{\Lambda} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$\mathbf{M}\,\mathbf{\Lambda} = \begin{bmatrix} 2 & 3 & -1 \\ 6 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix},\ \mathbf{A}\,\mathbf{M} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 3 & -1 \\ 6 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix} = \mathbf{M}\,\mathbf{\Lambda}$$

Substituting $\mathbf{x} = \mathbf{M}\,\mathbf{y}$ in $\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k)$ gives

$$\mathbf{M}\,\mathbf{y}(k+1) = \mathbf{A}\,\mathbf{M}\,\mathbf{y}(k)$$
$$\text{or } \mathbf{y}(k+1) = \mathbf{M}^{-1}\mathbf{A}\,\mathbf{M}\,\mathbf{y}^{(k)} = \mathbf{\Lambda}\,\mathbf{y}(k)$$
$$\text{so } \mathbf{y}(1) = \mathbf{\Lambda}\,\mathbf{y}(0),\ \mathbf{y}(2) = \mathbf{\Lambda}\,\mathbf{y}(1) = \mathbf{\Lambda}^2\mathbf{y}(0)$$
$$\Rightarrow\ \mathbf{y}(k) = \mathbf{\Lambda}^k\mathbf{y}(0)$$

Thus,

$$\mathbf{x}(k) = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^k \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \text{ say}$$

$$= \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha 2^k \\ \beta \\ \gamma(-1)^k \end{bmatrix}$$

$$= \begin{bmatrix} \alpha 2^k + 3\beta + \gamma(-1)^k \\ 3\alpha 2^k + 2\beta \\ \alpha 2^k + \beta + \gamma(-1)^k \end{bmatrix}$$

When $k = 0$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha + 3\beta + \gamma \\ 3\alpha + 2\beta \\ \alpha + \beta + \gamma \end{bmatrix}$$

which gives $\alpha = -\frac{1}{3},\ \beta = \frac{1}{2},\ \gamma = \frac{1}{6}$

so that

$$\mathbf{x}(k) = \begin{bmatrix} -\frac{1}{3}(2)^k + \frac{3}{2} + \frac{1}{6}(-1)^k \\ -(2)^k + 1 \\ -\frac{1}{3}(2)^k + \frac{1}{2} + \frac{1}{6}(-1)^k \end{bmatrix}$$

■ **18**

$$x_1(k+1) = u(k) - 3x_1(k) - 4x_2(k)$$
$$x_2(k+1) = -2x_1(k) - x_2(k)$$
$$y(k) = x_1(k) + x_2(k)$$

or in vector-matrix form

$$\mathbf{x}(k+1) = \begin{bmatrix} -3 & -4 \\ -2 & -1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k),\ y(k) = [1\ \ -1]\mathbf{x}(k) \qquad (1)$$

$$D(x) = \mathbf{c}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b} = \frac{1}{z^2 + 4z - 5}[1\ \ -1] \begin{bmatrix} z+1 & -4 \\ -2 & z+3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \frac{z+3}{z^2 + 4z - 5}$$

(i) $\mathbf{M}_c = \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}$

(ii) $\det \mathbf{M}_c = -2 \neq 0$ so $\mathbf{M}_c$ is of rank 2

(iii) $\mathbf{M}_c^{-1} = -\frac{1}{2}\begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}$

so $\mathbf{v}^T = [0 \ -\frac{1}{2}]$

(iv) $\mathbf{T} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$

(v) $\det \mathbf{T} \neq 0$ and $\mathbf{T}^{-1} = 2\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$

Substituting $\mathbf{z}(k) = \mathbf{T}\,\mathbf{x}(k)$ in (1) gives

$$\mathbf{T}^{-1}\mathbf{z}(k+1) = \mathbf{A}\,\mathbf{T}^{-1}\,\mathbf{z}(k) + \mathbf{b}u(k)$$

$$\text{or } \mathbf{z}(k+1) = \mathbf{T}\,\mathbf{A}\,\mathbf{T}^{-1}\,\mathbf{z}(k) + \mathbf{T}\,\mathbf{b}u(k)$$

$$= \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 & -4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}\mathbf{z}(k) + \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)$$

$$\text{thatis, } \mathbf{z}(k+1) = \begin{bmatrix} 0 & 1 \\ 5 & -4 \end{bmatrix}\mathbf{z}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k)$$

Thus $\mathbf{C}$ and $\mathbf{b}_c$ are of the required form with $\alpha = -5, \beta = 4$ which are coefficients in the characteristic polynomial of $D(z)$.

7
Fourier series

Exercises 7.2.6

- **1(a)**

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dt + \int_{0}^{\pi} t dt\right] \\
&= \frac{1}{\pi}\left[(-\pi t)_{-\pi}^{0} + \left(\frac{t^2}{2}\right)_{0}^{\pi}\right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\frac{\pi}{2} \\
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nt dt + \int_{0}^{\pi} t\cos nt dt\right] \\
&= \frac{1}{\pi}\left\{\left[-\frac{\pi}{n}\sin nt\right]_{-\pi}^{0} + \left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_{0}^{\pi}\right\} \\
&= \frac{1}{\pi n^2}(\cos n\pi - 1) = \begin{cases} -\dfrac{2}{\pi n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \\
b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nt dt + \int_{0}^{\pi} t\sin nt dt\right] \\
&= \frac{1}{\pi}\left\{\left[\frac{\pi}{n}\cos nt\right]_{-\pi}^{0} + \left[-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_{0}^{\pi}\right\} \\
&= \frac{1}{n}(1 - 2\cos n\pi) = \begin{cases} \dfrac{3}{n}, & n \text{ odd} \\ -\dfrac{1}{n}, & n \text{ even} \end{cases}
\end{aligned}
$$

Thus, the Fourier expansion of $f(t)$ is

$$
f(t) = -\frac{\pi}{4} + \sum_{n \text{ odd}} \left(-\frac{2}{\pi n^2}\right)\cos nt + \sum_{n \text{ odd}} \frac{3}{n}\sin nt - \sum_{n \text{ even}} \frac{1}{n}\sin nt
$$

$$
\text{i.e.} \quad f(t) = -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)t}{(2\pi - 1)^2} + 3\sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)} - \sum_{n=1}^{\infty} \frac{\sin 2nt}{2n}
$$

1(b)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(t+\pi)dt = \frac{1}{\pi}\left[\frac{t^2}{2}+\pi t\right]_{-\pi}^{0} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(t+\pi)\cos ntdt = \frac{1}{\pi}\left[(t+\pi)\frac{\sin nt}{n}+\frac{\cos nt}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi n^2}(1-\cos n\pi) = \begin{cases} 0, & n \text{ even} \\ \dfrac{2}{\pi n^2}, & n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(t+\pi)\sin ntdt = \frac{1}{\pi}\left[-(t+\pi)\frac{\cos nt}{n}+\frac{\sin nt}{n^2}\right]_{-\pi}^{0} = -\frac{1}{n}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{\pi}{4} + \sum_{n \text{ odd}} \frac{2}{\pi n^2}\cos nt - \sum_{n=1}^{\infty}\frac{1}{n}\sin nt$$

$$\text{i.e.} \quad f(t) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)t}{(2n-1)^2} - \sum_{n=1}^{\infty}\frac{\sin nt}{n}$$

1(c) From its graph we see that $f(t)$ is an odd function; so it has Fourier expansion

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{t}{\pi}\right)\sin ntdt$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\left(1-\frac{t}{\pi}\right)\cos nt - \frac{1}{\pi n^2}\sin nt\right]_0^{\pi} = \frac{2}{n\pi}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nt}{n}$$

1(d) From its graph $f(t)$ is seen to be an even function; so its Fourier expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

with

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} f(t)dt = \frac{2}{\pi}\int_0^{\pi/2} 2\cos t dt = \frac{2}{\pi}[2\sin t]_0^{\pi/2} = \frac{4}{\pi} \\
a_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt dt = \frac{2}{\pi}\int_0^{\pi/2} 2\cos t\cos nt dt \\
&= \frac{2}{\pi}\int_0^{\pi/2} [\cos(n+1)t + \cos(n-1)t]dt \\
&= \frac{2}{\pi}\left[\frac{\sin(n+1)t}{(n+1)} + \frac{\sin(n-1)t}{(n-1)}\right]_0^{\pi/2} \\
&= \frac{2}{\pi}\left[\frac{1}{(n+1)}\sin(n+1)\frac{\pi}{2} + \frac{1}{(n-1)}\sin(n-1)\frac{\pi}{2}\right] \\
&= \begin{cases} 0, & n \text{ odd} \\ -\frac{4}{\pi}\frac{1}{(n^2-1)}, & n = 4, 8, 12, \ldots \\ \frac{4}{\pi}\frac{1}{(n^2-1)}, & n = 2, 6, 10, \ldots \end{cases}
\end{aligned}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos 2nt}{4n^2 - 1}$$

1(e)

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\frac{t}{2} dt = \frac{1}{\pi}\left[2\sin\frac{t}{2}\right]_{-\pi}^{\pi} = \frac{4}{\pi} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\frac{t}{2}\cos nt dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos(n+\frac{1}{2})t + \cos(n-\frac{1}{2})t\right]dt \\
&= \frac{2}{2\pi}\left[\frac{2}{(2n+1)}\sin(n+\frac{1}{2})\pi + \frac{2}{(2n-1)}\sin(n-\frac{1}{2})\pi\right] \\
&= \begin{cases} \frac{4}{\pi(4n^2-1)}, & n = 1, 3, 5, \ldots \\ -\frac{4}{\pi(4n^2-1)}, & n = 2, 4, 6, \ldots \end{cases} \\
b_n &= 0
\end{aligned}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nt}{(4n^2-1)}$$

1(f) Since $f(t)$ is an even function, it has Fourier expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

with

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \mid t \mid dt = \frac{2}{\pi}\int_0^{\pi} tdt = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t\cos ntdt = \frac{2}{\pi}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1) = \begin{cases} 0, & n \text{ even} \\ -\dfrac{4}{\pi n^2}, & n \text{ odd} \end{cases}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}}\frac{1}{n^2}\cos nt$$

$$\text{that is,} \quad f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)t}{(2n-1)^2}$$

1(g)

$$a_0 = \frac{1}{\pi}\int_0^{\pi}(2t-\pi)dt = \frac{1}{\pi}\left[t^2 - \pi t\right]_0^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}(2t-\pi)\cos ntdt = \frac{1}{\pi}\left[\frac{(2t-\pi)}{n}\sin nt + \frac{2}{n^2}\cos nt\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1) = \begin{cases} -\dfrac{4}{\pi n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}(2t-\pi)\sin ntdt = \frac{1}{\pi}\left[-\frac{(2t-\pi)}{n}\cos nt + \frac{2}{n^2}\sin nt\right]_0^{\pi}$$

$$= -\frac{1}{n}(\cos n\pi + 1) = \begin{cases} 0, & n \text{ odd} \\ -\dfrac{2}{n}, & n \text{ even} \end{cases}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \sum_{n \text{ odd}} -\frac{4}{\pi n^2}\cos nt + \sum_{n \text{ even}} -\frac{2}{n}\sin nt$$

$$\text{that is,} \quad f(t) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)t}{(2n-1)^2} - \sum_{n=1}^{\infty}\frac{\sin 2nt}{n}$$

1(h)

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-t+e^t)dt + \int_0^{\pi}(t+e^t)dt\right] \\
&= \frac{1}{\pi}\left\{\left[-\frac{t^2}{2}+e^t\right]_{-\pi}^{0} + \left[\frac{t^2}{2}+e^t\right]_0^{\pi}\right\} \\
&= \frac{1}{\pi}\left[\pi^2 + (e^{\pi}-e^{-\pi})\right] = \pi + \frac{2}{\pi}\sinh\pi \\
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-t+e^t)\cos nt dt + \int_0^{\pi}(t+e^t)\cos nt dt\right] \\
&= \frac{1}{\pi}\left\{-\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_{-\pi}^{0} + \frac{1}{(n^2+1)}\left[ne^t\sin nt + e^t\cos nt\right]_{-\pi}^{0}\right. \\
&\qquad \left. + \left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_0^{\pi} + \frac{1}{(n^2+1)}\left[ne^t\sin nt + e^t\cos nt\right]_0^{\pi}\right\} \\
&= \frac{2}{\pi n^2}(-1+\cos n\pi) + \frac{2\cos n\pi}{\pi(n^2+1)}\left[\frac{e^{\pi}-e^{-\pi}}{2}\right] \\
&= \frac{2}{\pi}\left[\frac{(\cos\pi - 1)}{n^2} + \frac{\cos n\pi}{(n^2+1)}\sinh\pi\right], \ \cos n\pi = (-1)^n
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-t+e^t)\sin nt dt + \int_0^{\pi}(t+e^t)\sin nt dt\right] \\
&= \frac{1}{\pi}\left\{\left[\frac{t}{n}\cos nt - \frac{1}{n^2}\sin nt\right]_{-\pi}^{0} + \left[-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_0^{\pi}\right. \\
&\qquad \left. + \frac{n^2}{\pi^2+1}\left[-\frac{e^t\cos nt}{n} + \frac{e^t\sin nt}{n^2}\right]_{-\pi}^{\pi}\right\} \\
&= -\frac{n}{\pi(n^2+1)}\cos n\pi(e^{\pi}-e^{-\pi}) = -\frac{2n}{\pi(n^2+1)}\cos n\pi\sinh\pi, \cos n\pi = (-1)^n
\end{aligned}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \left(\frac{\pi}{2} + \frac{1}{\pi}\sinh\pi\right) + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2} + \frac{(-1)^n \sinh\pi}{n^2+1}\right]\cos nt - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{n^2+1}\sinh\pi\sin nt$$

■ **2** Since the periodic function $f(t)$ is an even function, its Fourier expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

with

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - t)^2 dt = \frac{2}{\pi}\left[-\frac{1}{3}(\pi - t)^3\right]_0^{\pi} = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - t)^2\cos nt dt = \frac{2}{\pi}\left[\frac{(\pi - t)^2}{n}\sin nt - \frac{2(\pi - t)}{n^2}\cos nt - \frac{2}{n^3}\sin nt\right]_0^{\pi}$$

$$= \frac{4}{n^2}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nt$$

Taking $t = \pi$ gives

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n$$

so that

$$\frac{1}{12}\pi^2 = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

■ **3** Since $q(t)$ is an even function, its Fourier expansion is

$$q(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

with

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} \frac{Qt}{\pi} dt = Q \\ a_n &= \frac{2}{\pi}\int_0^{\pi} \frac{Qt}{\pi}\cos nt dt = \frac{2Q}{\pi^2}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_0^{\pi} \\ &= \frac{2Q}{\pi^2 n^2}(\cos n\pi - 1) = \begin{cases} 0, & n \text{ even} \\ -\dfrac{4Q}{\pi^2 n^2}, & n \text{ odd} \end{cases} \end{aligned}$$

Thus, the Fourier expansion of $q(t)$ is

$$q(t) = Q\left[\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)t}{(2n-1)^2}\right]$$

■ **4**

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi} 5\sin t dt = \frac{1}{\pi}[-5\cos t]_0^{\pi} = \frac{10}{\pi} \\ a_n &= \frac{5}{\pi}\int_0^{\pi}\sin t\cos nt dt = \frac{5}{2\pi}\int_0^{\pi}[\sin(n+1)t - \sin(n-1)t]dt \\ &= \frac{5}{2\pi}\left[-\frac{\cos(n+1)t}{(n+1)} + \frac{\cos(n-1)t}{(n-1)}\right]_0^{\pi}, \ n \neq 1 \\ &= \frac{5}{2\pi}\left\{\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{(n-1)}\right] - \left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\right\} \\ &= -\frac{5}{\pi(n^2-1)}(\cos n\pi + 1) = \begin{cases} 0, & n \text{ odd}, n \neq 1 \\ -\dfrac{10}{\pi(n^2-1)}, & n \text{ even} \end{cases} \end{aligned}$$

Note that in this case we need to evaluate a_1 separately as

$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_0^{\pi} 5\sin t\cos t dt = \frac{5}{2\pi}\int_0^{\pi}\sin 2t dt = 0 \\ b_n &= \frac{5}{\pi}\int_0^{\pi}\sin t\sin nt dt = -\frac{5}{2\pi}\int_0^{\pi}[\cos(n+1)t - \cos(n-1)t]dt \\ &= -\frac{5}{2\pi}\left[\frac{\sin(n+1)t}{(n+1)} - \frac{\sin(n-1)t}{(n-1)}\right]_0^{\pi}, \ n \neq 1 \\ &= 0\ , \ n \neq 1 \end{aligned}$$

Evaluating b_1 separately,

$$b_1 = \frac{5}{\pi}\int_0^{\pi} \sin t \sin t dt = \frac{5}{2\pi}\int_0^{\pi}(1-\cos 2t)dt$$
$$= \frac{5}{2\pi}\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi} = \frac{5}{2}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{5}{\pi} + \frac{5}{2}\sin t - \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nt}{4n^2-1}$$

■ **5**

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}\pi^2 dt + \int_0^{\pi}(t-\pi)^2 dt\right]$$
$$= \frac{1}{\pi}\left\{\left[\pi^2 t\right]_{-\pi}^{0} + \left[\frac{1}{3}(t-\pi)^3\right]_0^{\pi}\right\} = \frac{4}{3}\pi^2$$
$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\pi^2\cos nt dt + \int_0^{\pi}(t-\pi)^2\cos nt dt\right]$$
$$= \frac{1}{\pi}\left\{\left[\frac{\pi^2}{n}\sin nt\right]_{-\pi}^{0} + \left[\frac{(t-\pi)^2}{n}\sin nt + \frac{2(t-\pi)}{n^2}\cos nt - \frac{2}{n^3}\sin nt\right]_0^{\pi}\right\}$$
$$= \frac{2}{n^2}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}\pi^2\sin nt dt + \int_0^{\pi}(t-\pi)^2\sin nt dt\right]$$
$$= \frac{1}{\pi}\left\{\left[-\frac{\pi^2}{n}\cos nt\right]_{-\pi}^{0} + \left[-\frac{(t-\pi)^2}{n}\cos nt + 2\frac{(t-\pi)}{n^2}\sin nt + \frac{2}{n^3}\cos nt\right]_0^{\pi}\right\}$$
$$= \frac{1}{\pi}\left\{\left[-\frac{\pi^2}{n} + \frac{\pi^2}{n}(-1)^n\right]\right\}$$
$$= \frac{\pi}{n}(-1)^n - \frac{2}{\pi n^3}[1-(-1)^n]$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}\cos nt + \frac{(-1)^n}{n}\pi \sin nt\right] - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)^3}$$

5(a) Taking $t = 0$ gives

$$\frac{\pi^2 + \pi^2}{2} = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{2}{n^2}$$

and hence the required result

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{6}\pi^2$$

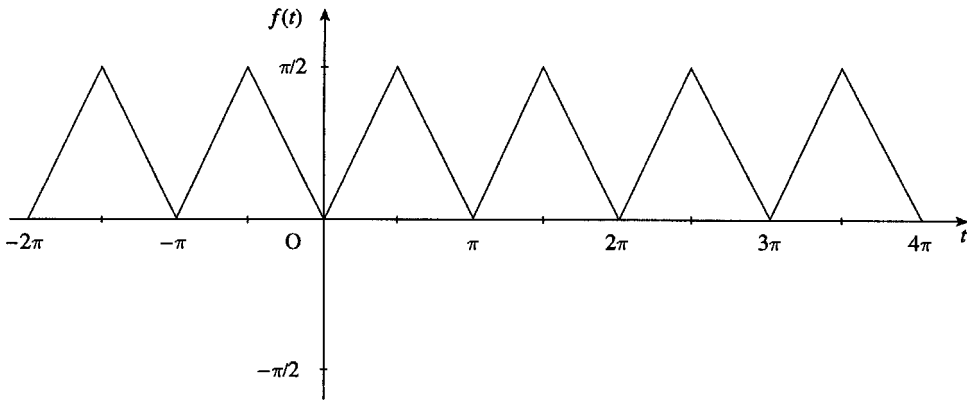
5(b) Taking $t = \pi$ gives

$$\frac{\pi^2 + 0}{2} = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{2}{n^2}(-1)^n$$

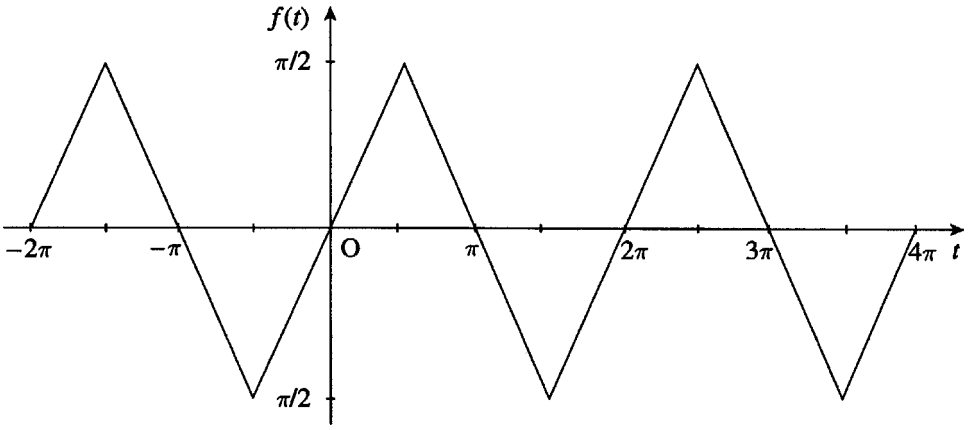
and hence the required result

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{1}{12}\pi^2$$

- **6(a)**



6(b)



The Fourier expansion of the even function (a) is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

with

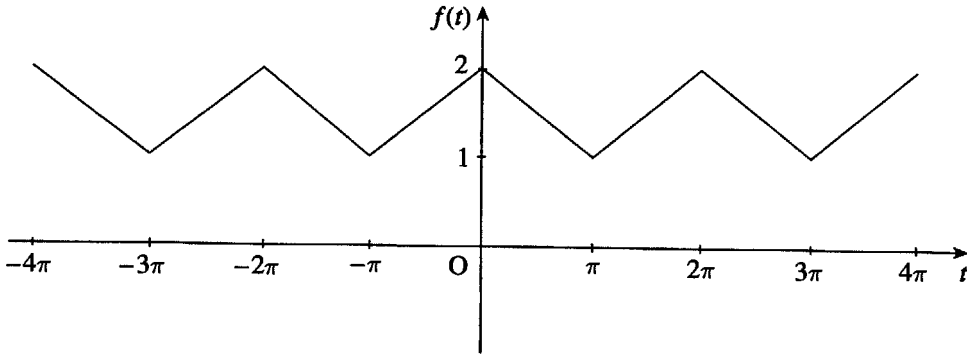
$$\begin{aligned}
a_0 &= \frac{2}{\pi}\left[\int_0^{\pi/2} t dt + \int_{\pi/2}^{\pi} (\pi - t) dt\right] \\
&= \frac{2}{\pi}\left\{\left[\frac{1}{2}t^2\right]_0^{\pi/2} + \left[-\frac{1}{2}(\pi - t)^2\right]_{\pi/2}^{\pi} = \frac{\pi}{2}\right. \\
a_n &= \frac{2}{\pi}\left[\int_0^{\pi/2} t\cos nt dt + \int_{\pi/2}^{\pi} (\pi - t)\cos nt dt\right] \\
&= \frac{2}{\pi}\left\{\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_0^{\pi/2} + \left[\frac{\pi - t}{n}\sin nt - \frac{1}{n^2}\cos nt\right]_{\pi/2}^{\pi}\right\} \\
&= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}(1 + (-1)^n)\right] \\
&= \begin{cases} 0, & n \text{ odd} \\ -\dfrac{8}{\pi n^2}, & n = 2, 6, 10, \ldots \\ 0, & n = 4, 8, 12, \ldots \end{cases}
\end{aligned}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(4n-2)t}{(2n-1)^2}$$

Taking $t = 0$ where $f(t) = 0$ gives the required result.

■ **7**



$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\left[\int_0^{\pi}(2-\frac{t}{\pi})dt + \int_{\pi}^{2\pi} t/\pi dt\right] \\
&= \frac{1}{\pi}\left\{\left[2t-\frac{t^2}{2\pi}\right]_0^{\pi} + \left[\frac{t^2}{2\pi}\right]_{\pi}^{2\pi}\right\} = 3 \\
a_n &= \frac{1}{\pi}\left[\int_0^{\pi}(2-\frac{t}{\pi})\cos ntdt + \int_{\pi}^{2\pi}\frac{t}{\pi}\cos ntdt\right] \\
&= \frac{1}{\pi}\left\{\left[\frac{2}{n}\sin nt - \frac{t}{\pi n}\sin nt - \frac{1}{\pi n^2}\cos nt\right]_0^{\pi} + \left[\frac{t}{\pi n}\sin nt + \frac{1}{\pi n^2}\cos nt\right]_{\pi}^{2\pi}\right\} \\
&= \frac{2}{\pi^2 n^2}[1-(-1)^n] \\
&= \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{\pi^2 n^2}, & n \text{ odd} \end{cases} \\
b_n &= \frac{1}{\pi}\left[\int_0^{\pi}(2-\frac{t}{\pi})\sin ntdt + \int_{\pi}^{2\pi}\frac{t}{\pi}\sin ntdt\right] \\
&= \frac{1}{\pi}\left\{\left[-\frac{2}{n}\cos nt + \frac{t}{\pi n}\cos nt - \frac{1}{\pi n^2}\sin nt\right]_0^{\pi} + \left[-\frac{t}{\pi n}\cos nt + \frac{1}{\pi n^2}\sin nt\right]_{\pi}^{2\pi}\right\} \\
&= 0
\end{aligned}
$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{3}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)t}{(2n-1)^2}$$

Replacing t by $t - \frac{1}{2}\pi$ gives

$$f(t-\frac{1}{2}\pi) = \frac{3}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)(t-\pi)}{(2n-1)^2}$$

Since

$$\cos(2n-1)(t-\frac{1}{2}\pi) = \cos(2n-1)t\cos(2n-1)\frac{\pi}{2} + \sin(2n-1)t\sin(2n-1)\frac{\pi}{2}$$
$$= (-1)^{n+1}\sin(2n-1)t$$

$$f(t-\frac{1}{2}\pi) - \frac{3}{2} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(2n-1)t}{(2n-1)^2}$$

The corresponding odd function is readily recognized from the graph of $f(t)$.

Exercises 7.2.8

■ **8** Since $f(t)$ is an odd function, the Fourier expansion is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{\ell}$$

with

$$b_n = \frac{2}{\ell}\int_0^{\ell} t\sin\frac{n\pi t}{\ell}dt = \frac{2}{\ell}\left[-\frac{t\ell}{n\pi}\cos\frac{n\pi t}{\ell} + \left(\frac{\ell}{n\pi}\right)^2\sin\frac{n\pi t}{\ell}\right]_0^{\ell}$$
$$= -\frac{2\ell}{n\pi}\cos n\pi$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{2\ell}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{\ell}$$

■ **9** Since $f(t)$ is an odd function (readily seen from a sketch of its graph) its Fourier expansion is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{\ell}$$

with

$$b_n = \frac{2}{\ell}\int_0^{\ell}\frac{K}{\ell}(\ell - t)\sin\frac{n\pi t}{\ell}tdt$$
$$= \frac{2}{\ell}\left[-\frac{K\ell}{n\pi}\cos\frac{n\pi t}{\ell} + \frac{Kt}{n\pi}\cos\frac{n\pi t}{\ell} - \frac{K\ell}{(n\pi)^2}\sin\frac{n\pi t}{\ell}\right]_0^{\ell}$$
$$= \frac{2K}{n\pi}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{2K}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi t}{\ell}$$

- **10**

$$a_0 = \frac{1}{5}\int_0^5 3dt = 3$$

$$a_n = \frac{1}{5}\int_0^5 3\cos\frac{n\pi t}{5}dt = \frac{1}{5}\left[\frac{15}{n\pi}\sin\frac{n\pi t}{5}\right]_0^5 = 0$$

$$b_n = \frac{1}{5}\int_0^5 3\sin\frac{n\pi t}{5}dt = \frac{1}{5}\left[-\frac{15}{n\pi}\cos\frac{n\pi t}{5}\right]_0^5$$

$$= \frac{3}{n\pi}[1-(-1)^n] = \begin{cases}\dfrac{6}{n\pi}, & n \text{ odd}\\ 0, & n \text{ even}\end{cases}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{3}{2} + \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)}{5}\pi t$$

- **11**

$$a_0 = \frac{2\omega}{2\pi}\int_0^{\pi/\omega} A\sin\omega t dt = \frac{\omega}{\pi}\left[-\frac{A}{\omega}\cos\omega t\right]_0^{\pi/\omega} = \frac{2A}{\pi}$$

$$a_n = \frac{A\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t\cos n\omega t dt = \frac{A\omega}{2\pi}\int_0^{\pi/\omega}[\sin(n+1)\omega t - \sin(n-1)\omega t]dt$$

$$= \frac{A\omega}{2\pi}\left[-\frac{\cos(n+1)\omega t}{(n+1)\omega} + \frac{\cos(n-1)\omega t}{(n-1)\omega}\right]_0^{\pi/\omega},\ n \neq 1$$

$$= \frac{A}{2\pi}\left[\frac{2(-1)^{n+1}}{n^2-1} - \frac{2}{n^2-1}\right] = \frac{A}{\pi(n^2-1)}[(-1)^{n+1}-1]$$

$$= \begin{cases}0, & n \text{ odd},\ n \neq 1\\ -\dfrac{2A}{\pi(n^2-1)}, & n \text{ even}\end{cases}$$

Evaluating a_1 separately,

$$a_1 = \frac{A\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t\cos\omega t dt = \frac{A}{2\pi}\int_0^{\pi/\omega}\sin 2\omega t dt = 0$$

$$b_n = \frac{A\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t \sin n\omega t dt = -\frac{A\omega}{2\pi}\int_0^{\pi/\omega}[\cos(n+1)\omega t - \cos(n-1)\omega t]dt$$
$$= -\frac{A\omega}{2\pi}\left[\frac{\sin(n+1)\omega t}{(n+1)\omega} - \frac{\sin(n-1)\omega t}{(n-1)\omega}\right]_0^{\pi/\omega}, \ n \neq 1$$
$$= 0, \ n \neq 1$$
$$b_1 = \frac{A\omega}{\pi}\int_0^{\pi/\omega} \sin^2\omega t dt = \frac{A\omega}{2\pi}\int_0^{\pi/\omega}(1-\cos 2\omega t)dt = \frac{A}{2}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{A}{\pi}\left[1 + \frac{\pi}{2}\sin\omega t - 2\sum_{n=1}^{\infty}\frac{\cos 2n\omega t}{4n^2-1}\right]$$

■ **12** Since $f(t)$ is an even function, its Fourier expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{T}$$

with

$$a_0 = \frac{2}{T}\int_0^T t^2 dt = \frac{2}{T}\left[\frac{1}{3}t^3\right]_0^T = \frac{2}{3}T^2$$
$$a_n = \frac{2}{T}\int_0^T t^2 \cos\frac{n\pi t}{T}dt = \frac{2}{T}\left[\frac{Tt^2}{n\pi}\sin\frac{n\pi t}{T} + \frac{2tT^2}{(n\pi)^2}\cos\frac{n\pi t}{T} - \frac{2T^3}{(n\pi)^3}\sin\frac{n\pi t}{T}\right]_0^T$$
$$= \frac{4T^2}{(n\pi)^2}(-1)^n$$

Thus, the Fourier series expansion of $f(t)$ is

$$f(t) = \frac{T^2}{3} + \frac{4T^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi t}{T}$$

■ **13**

$$a_0 = \frac{2}{T}\int_0^T \frac{E}{T}t dt = \frac{2E}{T^2}\left[\frac{1}{2}t^2\right]_0^T = E$$

$$a_n = \frac{2}{T}\int_0^T \frac{E}{T}t\cos\frac{2\pi nt}{T}dt$$

$$= \frac{2E}{T^2}\left[\frac{tT}{2\pi n}\sin\frac{2\pi nt}{T} + \left(\frac{T}{2\pi n}\right)^2\cos\frac{2\pi nt}{T}\right]_0^T = 0$$

$$b_n = \frac{2E}{T^2}\int_0^T t\sin\frac{2\pi nt}{T}dt$$

$$= \frac{2E}{T^2}\left[-\frac{tT}{2\pi n}\cos\frac{2\pi nt}{T} + \left(\frac{T}{2\pi n}\right)^2\sin\frac{2\pi nt}{T}\right]_0^T = -\frac{E}{\pi n}$$

Thus, the Fourier expansion of $e(t)$ is

$$e(t) = \frac{E}{2} - \frac{E}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2\pi nt}{T}$$

Exercises 7.3.3

■ **14** Half range Fourier sine series expansion is given by

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1\sin nt dt = \frac{2}{\pi}\left[-\frac{1}{n}\cos nt\right]_0^{\pi}$$

$$= -\frac{2}{n\pi}[(-1)^n - 1]$$

$$= \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{n\pi}, & n \text{ odd} \end{cases}$$

Thus, the half range Fourier sine series expansion of $f(t)$ is

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)}$$

Plotting the graphs should cause no problems.

■ **15** Half range Fourier cosine series expansion is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t$$

with

$$\begin{aligned} a_0 &= \frac{2}{1}\int_0^1 (2t-1)dt = 0 \\ a_n &= 2\int_0^1 (2t-1)\cos n\pi t dt \\ &= 2\left[\frac{(2t-1)}{n\pi}\sin n\pi t + \frac{2}{(n\pi)^2}\cos n\pi t\right]_0^1 \\ &= \frac{4}{(n\pi)^2}[(-1)^n - 1] \\ &= \begin{cases} 0, & n \text{ even} \\ -\dfrac{8}{(n\pi)^2}, & n \text{ odd} \end{cases} \end{aligned}$$

Thus, the half range Fourier cosine series expansion of $f(t)$ is

$$f(t) = -\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\pi t$$

Again plotting the graph should cause no problems.

■ **16(a)**

$$\begin{aligned} a_0 &= 2\int_0^1 (1-t^2)dt = 2\big[t - \frac{1}{3}t^3\big]_0^1 = \frac{4}{3} \\ a_n &= 2\int_0^1 (1-t^2)\cos 2n\pi t dt \\ &= 2\left[\frac{(1-t^2)}{2n\pi}\sin 2n\pi t - \frac{2t}{(2n\pi)^2}\cos 2n\pi t + \frac{2}{(2n\pi)^3}\sin 2n\pi t\right]_0^1 \\ &= -\frac{1}{(n\pi)^2} \end{aligned}$$

$$\begin{aligned} b_n &= 2\int_0^1 (1-t^2)\sin 2n\pi t dt \\ &= 2\left[-\frac{(1-t^2)}{2n\pi}\cos 2n\pi t - \frac{2t}{(2n\pi)^2}\sin 2n\pi t - \frac{2}{(2n\pi)^3}\cos 2n\pi t\right]_0^1 \\ &= \frac{1}{n\pi} \end{aligned}$$

Thus, the full-range Fourier series expansion for $f(t)$ is

$$f(t) = f_1(t) = \frac{2}{3} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2n\pi t + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi t$$

16(b) Half-range sine series expansion is

$$f_2(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$$

with

$$\begin{aligned} b_n &= 2\int_0^1 (1-t^2)\sin n\pi t dt \\ &= 2\left[-\frac{(1-t^2)}{n\pi}\cos n\pi t - \frac{2t}{(n\pi)^2}\sin n\pi t - \frac{2}{(n\pi)^3}\cos n\pi t\right]_0^1 \\ &= 2\left[-\frac{2}{(n\pi)^3}(-1)^n + \frac{1}{n\pi} + \frac{2}{(n\pi)^3}\right] \\ &= \begin{cases} \dfrac{2}{n\pi}, & n \text{ even} \\ 2\left[\dfrac{1}{n\pi} + \dfrac{4}{(n\pi)^3}\right], & n \text{ odd} \end{cases} \end{aligned}$$

Thus, half-range sine series expansion is

$$f_2(t) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi t + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)} + \frac{4}{\pi^2(2n-1)^3}\right]\sin(2n-1)\pi t$$

16(c) Half-range cosine series expansion is

$$f_3(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t$$

with

$$a_0 = 2\int_0^1 (1-t^2)dt = \frac{4}{3}$$

$$a_n = 2\int_0^1 (1-t^2)\cos n\pi t dt$$

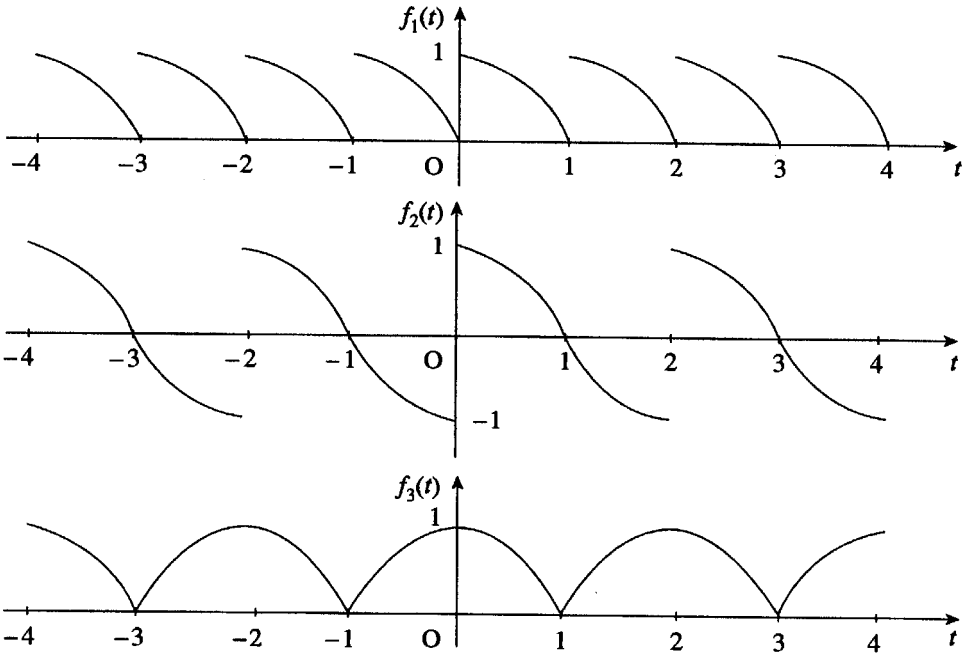
$$= 2\left[\frac{(1-t^2)}{n\pi}\sin n\pi t - \frac{2t}{(n\pi)^2}\cos n\pi t + \frac{2}{(n\pi)^3}\sin n\pi t\right]_0^1$$

$$= \frac{-4(-1)^n}{(n\pi)^2}$$

Thus, half-range cosine series expansion is

$$f_3(t) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos n\pi t$$

Graphs of the functions $f_1(t), f_2(t), f_3(t)$ for $-4 < t < 4$ are as follows:



■ **17** Fourier cosine series expansion is

$$f_1(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

with

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi}(\pi t - t^2)dt = \frac{1}{3}\pi^2 \\
a_n &= \frac{2}{\pi}\int_0^{\pi}(\pi t - t^2)\cos ntdt \\
&= \frac{2}{\pi}\left[\frac{(\pi t - t^2)}{n}\sin nt + \frac{(\pi - 2t)}{n^2}\cos nt + \frac{2}{n^3}\sin nt\right]_0^{\pi} \\
&= -\frac{2}{n^2}[1 + (-1)^n] \\
&= \begin{cases} 0, & n \text{ odd} \\ -\dfrac{4}{n^2}, & n \text{ even} \end{cases}
\end{aligned}$$

Thus, the Fourier cosine series expansion is

$$f_1(t) = \frac{1}{6}\pi^2 - \sum_{n=1}^{\infty}\frac{1}{n^2}\cos 2nt$$

Fourier sine series expansion is

$$f_2(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

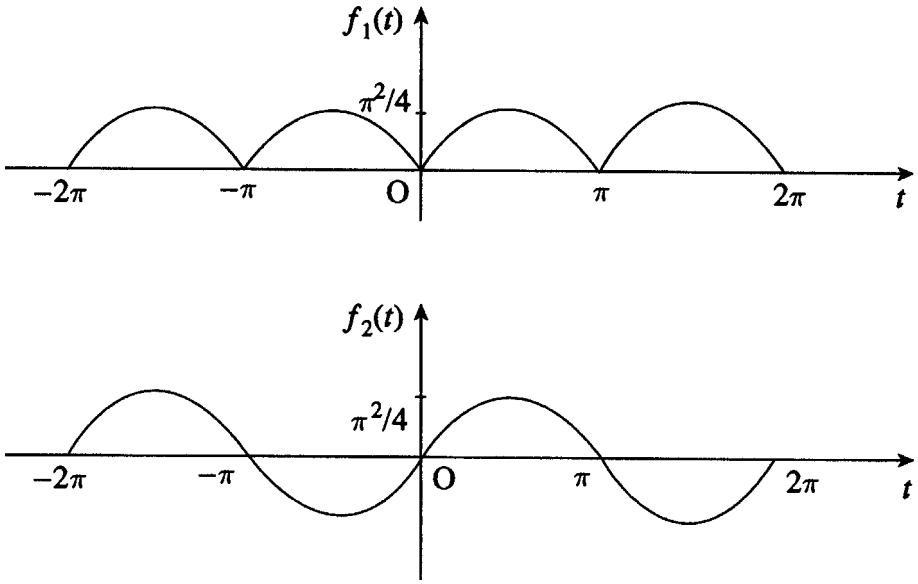
with

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi}(\pi t - t^2)\sin ntdt \\
&= \frac{2}{\pi}\left[-\frac{(\pi t - t^2)}{n}\cos nt + \frac{(\pi - 2t)}{n^2}\sin nt - \frac{2}{n^3}\cos nt\right]_0^{\pi} \\
&= \frac{4}{\pi n^3}[1 - (-1)^n] \\
&= \begin{cases} 0, & n \text{ even} \\ \dfrac{8}{\pi n^3}, & n \text{ odd} \end{cases}
\end{aligned}$$

Thus, the Fourier sine series expansion is

$$f_2(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)t$$

Graphs of the functions $f_1(t)$ and $f_2(t)$ for $-2\pi < t < 2\pi$ are



■ **18**

$$f(x) = \frac{2a}{\ell}x \ , \ 0 < x < \frac{\ell}{2}$$

$$f(x) = \frac{2a}{\ell}(\ell - x) \ , \ \frac{\ell}{2} < x < \ell$$

Fourier sine series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

with

$$\begin{aligned} b_n &= \frac{2a}{\ell}\cdot\frac{2}{\ell}\left[\int_0^{\ell/2} x\sin\frac{n\pi x}{\ell}dx + \int_{\ell/2}^{\ell}(\ell - x)\sin\frac{n\pi x}{\ell}dx\right] \\ &= \frac{4a}{\ell^2}\left\{\left[-\frac{\ell x}{n\pi}\cos\frac{n\pi x}{\ell} + \frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi x}{\ell}\right]_0^{\ell/2}\right. \\ &\qquad \left. + \left[-\frac{\ell}{n\pi}(\ell - x)\cos\frac{n\pi x}{\ell} - \frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi x}{\ell}\right]_{\ell/2}^{\ell}\right\} \\ &= \frac{4a}{\ell^2}\left[\frac{2\ell^2}{(n\pi)^2}\sin\frac{n\pi}{2}\right] = \frac{8a}{(n\pi)^2}\sin\frac{n\pi}{2} \\ &= \begin{cases} 0, & n \text{ even} \\ \dfrac{8a}{(n\pi)^2}, & n = 1, 5, 9, \ldots \\ -\dfrac{8a}{(n\pi)^2}, & n = 3, 7, \ldots \end{cases} \end{aligned}$$

Thus, the required Fourier sine series expansion is

$$f(x) = \frac{8a}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{\ell}$$

■ **19**

$$f(x) = \begin{cases} x, & 0 < x < \dfrac{\ell}{4} \\ \dfrac{\ell}{2} - x, & \dfrac{\ell}{4} < x < \dfrac{3\ell}{4} \\ x - \ell, & \dfrac{3\ell}{4} < x < \ell \end{cases}$$

Fourier sine series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

with

$$\begin{aligned} b_n &= \frac{2}{\ell}\left[\int_0^{\ell/4} x \sin\frac{n\pi x}{\ell}dx + \int_{\ell/4}^{3\ell/4} \left(\frac{\ell}{2} - x\right)\sin\frac{n\pi x}{\ell}dx + \int_{3\ell/4}^{\ell} (x-\ell)\sin\frac{n\pi x}{\ell}dx\right] \\ &= \frac{2}{\ell}\left\{\left[-\frac{\ell x}{n\pi}\cos\frac{n\pi x}{\ell} + \frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi x}{\ell}\right]_0^{\ell/4}\right. \\ &\quad + \left[-\frac{\ell}{n\pi}\left(\frac{\ell}{2} - x\right)\cos\frac{n\pi x}{\ell} - \frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi x}{\ell}\right]_{\ell/4}^{3\ell/4} \\ &\quad \left. + \left[-\frac{\ell}{n\pi}(x-\ell)\cos\frac{n\pi x}{\ell} + \frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi x}{\ell}\right]_{3\ell/4}^{\ell}\right\} \\ &= \left[\sin\frac{n\pi}{4} - \sin\frac{3n\pi}{4}\right] \\ &= -\frac{8\ell}{(n\pi)^2}\cos\frac{n\pi}{2}\sin\frac{n\pi}{4} \\ &= \begin{cases} 0, & n \text{ odd} \\ 0, & n = 4, 8, 12, \ldots \\ \dfrac{8\ell}{(n\pi)^2}, & n = 2, 10, 18, \ldots \\ -\dfrac{8\ell}{(n\pi)^2}, & n = 6, 14, 22, \ldots \end{cases} \end{aligned}$$

Thus, the required Fourier sine series expansion is

$$f(x) = \frac{2\ell}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin 2(2n-1)\frac{\pi x}{\ell}$$

■ **20** Fourier sine series expansion is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

with

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi/2} \sin t \sin nt dt \\ &= -\frac{1}{\pi}\int_0^{\pi/2} [\cos(n+1)t - \cos(n-1)t]dt \\ &= -\frac{1}{\pi}\left[\frac{1}{(n+1)}\sin(n+1)t - \frac{1}{(n-1)}\sin(n-1)t\right]_0^{\pi/2}, \ n \neq 1 \\ &= -\frac{1}{\pi}\left[\frac{1}{(n+1)}\sin(n+1)\frac{\pi}{2} - \frac{1}{(n-1)}\sin(n-1)\frac{\pi}{2}\right] \end{aligned}$$

Using the trigonometric expansions for $\sin(A+B)$ and $\sin(A-B)$ gives

$$\begin{aligned} b_n &= -\frac{2n}{\pi(n^2-1)}\cos\frac{n\pi}{2}, n \neq 1 \\ &= \begin{cases} 0, & n \text{ odd} \\ \dfrac{2n}{\pi(n^2-1)}, & n = 2, 6, \ldots \\ -\dfrac{2n}{\pi(n^2-1)}, & n = 4, 8, 10, \ldots \end{cases} \end{aligned}$$

In the case $n = 1$,

$$b_1 = \frac{2}{\pi}\int_0^{\pi/2} \sin^2 t dt = \frac{1}{\pi}\int_0^{\pi/2} (1 - \cos 2t)dt = \frac{1}{2}$$

Thus, the required Fourier sine series expansion is

$$f(t) = \frac{1}{2}\sin t + \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n \sin 2nt}{4n^2-1}$$

The corresponding plot presents no problem.

■ **21** Since $f(x)$ is an even function, the Fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

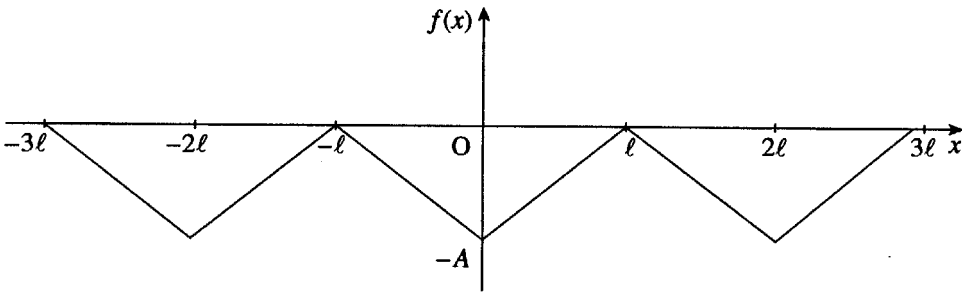
with

$$\begin{aligned} a_0 &= \frac{2}{\ell}\int_0^{\ell} \frac{A}{\ell}(x-\ell)dx, \text{ since } \mid x \mid = x \text{ for } x \geq 0 \\ &= \frac{2A}{\ell^2}\left[\frac{1}{2}x^2 - \ell x\right]_0^{\ell} = -A \\ a_n &= \frac{2}{\ell}\int_0^{\ell} \frac{A}{\ell}(x-\ell)\cos\frac{n\pi x}{\ell}dx \\ &= \frac{2A}{\ell^2}\left[\frac{\ell}{n\pi}(x-\ell)\sin\frac{n\pi x}{\ell} + \frac{\ell^2}{(n\pi)^2}\cos\frac{n\pi x}{\ell}\right]_0^{\ell} \\ &= \frac{2A}{(n\pi)^2}(\cos n\pi - 1) = \begin{cases} 0, & n \text{ even} \\ -\dfrac{4A}{(n\pi)^2}, & n \text{ odd} \end{cases} \end{aligned}$$

Thus, the Fourier series expansion is

$$f(t) = -\frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{\ell}$$

The graph represented by the series for $-3\ell \leq x \leq 3\ell$ is as follows:



■ **22** Fourier sine series expansion is

$$T(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

with

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L Kx(L-x)\sin\frac{n\pi x}{L}dx \\ &= \frac{2K}{L}\left[-\frac{Lx(L-x)}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{(n\pi)^2}(L-2x)\sin\frac{n\pi x}{L}\right. \\ &\qquad \left. - \frac{2L^3}{(n\pi)^3}\cos\frac{n\pi x}{L}\right]_0^L \\ &= \frac{4KL^2}{(n\pi)^3}(1-\cos n\pi) = \begin{cases} 0, & n \text{ even} \\ \dfrac{8KL^2}{(n\pi)^3}, & n \text{ odd} \end{cases} \end{aligned}$$

Thus, the Fourier sine series expansion is

$$T(x) = \frac{8KL^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}$$

■ **23**

$$\begin{aligned} a_0 &= \frac{2}{2}\left[\int_{-1}^0 1dt + \int_0^1 \cos\pi t dt\right] = [t]_{-1}^0 + \left[\frac{1}{\pi}\sin\pi t\right]_0^1 = 1 \\ a_n &= \int_{-1}^0 \cos n\pi t dt + \int_0^1 \cos\pi t\cos n\pi t dt \\ &= \left[\frac{1}{n\pi}\sin n\pi t\right]_{-1}^0 + \frac{1}{2}\int_0^1 \cos(n+1)\pi t + \cos(n-1)\pi t dt \\ &= \frac{1}{2}\left[\frac{1}{(n+1)\pi}\sin(n+1)\pi t + \frac{1}{(n-1)\pi}\sin(n-1)\pi t\right]_0^1, \quad n \neq 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{1}{2}\int_0^1 2\cos\frac{2}{\pi}t dt = \frac{1}{2}\int_0^1 (1+\cos 2\pi t)dt = \frac{1}{2} \\ b_n &= \int_{-1}^0 \sin n\pi t dt + \int_0^1 \cos\pi t\sin n\pi t dt \\ &= \left[-\frac{1}{n\pi}\cos n\pi t\right]_{-1}^0 + \frac{1}{2}\int_0^1 \sin(n+1)\pi t + \sin(n-1)\pi t dt \\ &= \frac{1}{n\pi}[(-1)^n - 1] + \frac{1}{2\pi}\left[-\frac{1}{(n+1)}\cos(n+1)\pi t - \frac{1}{(n-1)}\cos(n-1)\pi t\right]_0^1 \end{aligned}$$

$$= \frac{1}{n\pi}[(-1)^n - 1] + \frac{1}{2\pi}\left[\frac{2n}{(n^2-1)}\right][1 + \cos n\pi]$$

$$= \begin{cases} -\dfrac{2}{n\pi}, & n \text{ odd} \\ \dfrac{2n}{\pi(n^2-1)}, & n \text{ even} \end{cases}$$

Thus, the Fourier series expansion is

$$f(t) = \frac{1}{2} + \frac{1}{2}\cos \pi t - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin(2n-1)\pi t + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin 2n\pi t$$

Exercises 7.4.4

■ **24** Since $f(t)$ is an odd function, its Fourier expansion is of the form

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{T} t$$

with

$$b_n = \frac{2}{T}\int_0^T t \sin \frac{n\pi}{T} t$$

$$= \frac{2}{T}\left[-\frac{Tt}{n\pi}\cos \frac{n\pi}{T} t + \frac{T^2}{n^2\pi^2}\sin \frac{n\pi}{T} t\right]_0^T$$

$$= \frac{2}{T}\left[-\frac{T^2}{n\pi}\cos n\pi\right] = -\frac{2T}{n\pi}(-1)^n$$

Thus, the Fourier expansion is

$$f(t) = t = \frac{2T}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} \sin \frac{n\pi}{T} t$$

Integrating term by term gives

$$\frac{t^2}{2} = -\frac{2T^2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^{n+1} \cos \frac{n\pi}{T} t + \text{ const.}$$

Taking mean value over a period,

$$\frac{1}{2T}\int_{-T}^{T} \frac{t^2}{2} dt = -\frac{2T^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \int_{-T}^{T} \cos \frac{n\pi}{T} t dt + \frac{1}{2T}\int_{-T}^{T} (\text{const.}) dt$$

so that

$$\frac{T^2}{6} = 0 + \text{ const.}$$

giving (const.) $= T^2/6$

Thus,

$$g(t) = t^2 = \frac{T^2}{3} - \frac{4T^2}{\pi^2}\sum \frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi}{T}t$$

■ **25** $\pi^2 - t^2 = h(t) = \dfrac{2}{3}\pi^2 + 4\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n^2}\cos nt$

Since $h(t)$ is continuous within and at the end points of the interval $-\pi \le t \le \pi$, we may apply Theorem 4.4 to obtain the Fourier series of

$$f(t) = t,\ -\pi < t < \pi;\ f(t + 2\pi) = f(t)$$

Differentiating gives

$$-2t = -4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt$$

So that the Fourier series of $f(t)$ is

$$f(t) = t = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt$$

which confirms the series of Exercise 24 when $T = \pi$.

26(b) Derived series is

$$\sum_{n=1}^{\infty}\frac{4}{n}(-1)^{n+1}\sin nt - \sum_{n=1}^{n} 2(-1)^n\cos nt$$

This is not a Fourier expansion of $g(t)$ since $f(t)$ is discontinuous at the end points of $-\pi \le t \le \pi$.

26(c) Using the results of (a)

$$A_0 = \frac{1}{\pi}[f(\pi_-) - f(-\pi_+)] = \frac{1}{\pi}\cdot 2\pi = 2$$

$$A_n = (-1)^n A_0 + nb_n = (-1)^n 2 - \frac{n2}{n}(-1)^n = 2(-1)^n - 2(-1)^n = 0$$

$$B_n = -na_n = \frac{4}{n}(-1)^{n+1}$$

Thus, the Fourier expansion $g(t)$ is

$$\begin{aligned} g(t) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nt + \sum_{n=1}^{\infty} B_n \sin nt \\ &= 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt \end{aligned}$$

Using Euler's formulae

$$\begin{aligned} A_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} (2t+1)dt = \frac{1}{\pi}[t^2+t]_{-\pi}^{\pi} = 2 \\ A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} (2t+1)\cos nt \; dt \\ &= \frac{1}{\pi}\left[\frac{(2t+1)}{n}\sin nt + \frac{2}{n^2}\cos nt\right]_{-\pi}^{\pi} = 0 \end{aligned}$$

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} (2t+1)\sin nt \; dt \\ &= \frac{1}{\pi}\left[-\frac{(2t+1)}{n}\cos nt + \frac{2}{n^2}\sin nt\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi n}\left[-(2\pi+1)(-1)^n + (-2\pi+1)(-1)^n\right] \\ &= \frac{4}{n}(-1)^{n+1} \end{aligned}$$

thus confirming the values obtained using (a).

- **27(a)**

$$\begin{aligned} p_1(t) &= -1 & p_2(t) &= 1 \\ d_1 &= 2 & d_2 &= -2 \\ p_1^{(1)}(t) &= 0 & p_2^{(1)}(t) &= 0 \\ d_1^{(1)} &= 0 & d_2^{(1)} &= 0 \end{aligned}$$

$t_1 = 0,\ t_2 = \pi$ and since $\omega = 1$ using (4.39) gives

$$\begin{aligned} a_n &= \frac{1}{n\pi}\left[-\sum_{s=1}^{2} d_s \sin nt_s - \frac{1}{n}\sum_{s=1}^{2} d_s^{(1)} \cos nt_s\right] \\ &= \frac{1}{n\pi}\left[-2\sin 0 + 2\sin n\pi\right] = 0 \end{aligned}$$

$$b_n = \frac{1}{n\pi}\left[\sum_{s=1}^{2} d_s \cos nt_s - \frac{1}{n}\sum_{s=1}^{2} d_s^{(1)} \sin nt_s\right]$$
$$= \frac{1}{n\pi}\left[2\cos 0 - 2\cos\pi\right] = \frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{n\pi}, & n \text{ odd} \end{cases}$$
$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)dt + \int_{0}^{\pi} 1dt\right] = 0$$

Thus, Fourier series is

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)t$$

confirming (7.21).

27(b)

$$\begin{aligned} p_1(t) &= t, & d_1 &= -2\pi \\ p_1^{(1)}(t) &= 1, & d_1^{(1)} &= 0 \\ t_1 &= 0, & t_2 &= \pi, \quad \omega = 1 \end{aligned}$$

Thus, from (7.39)

$$a_n = \frac{1}{n\pi}\left[-d_1 \sin nt_1\right] = \frac{1}{n\pi}\left[2\pi \sin 0\right] = 0$$
$$b_n = \frac{1}{n\pi}\left[d_1 \cos nt_1\right] = \frac{1}{n\pi}\left[-2\pi\cos 0\right] = -\frac{2}{n}$$
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t \ at = 2\pi$$

Thus, Fourier series is

$$f(t) = \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nt$$

confirming the result obtained in Example 7.1.

27(c)

$$\begin{aligned} p_1(t) &= t & p_2(t) &= \tfrac{1}{2}\pi & p_3(t) &= \pi - \tfrac{1}{2}t \\ d_1 &= 0 & d_2 &= 0 & d_3 &= 0 \\ p_1^{(1)}(t) &= 1 & p_2^{(1)}(t) &= 0 & p_3^{(1)}(t) &= -\tfrac{1}{2} \\ d_1^{(1)} &= -1 & d_2^{(1)} &= -\tfrac{1}{2} & d_3^{(1)} &= \tfrac{3}{2} \end{aligned}$$

$$t_1 = \tfrac{\pi}{2},\ t_2 = \pi,\ t_3 = 2\pi,\ \omega = 1$$

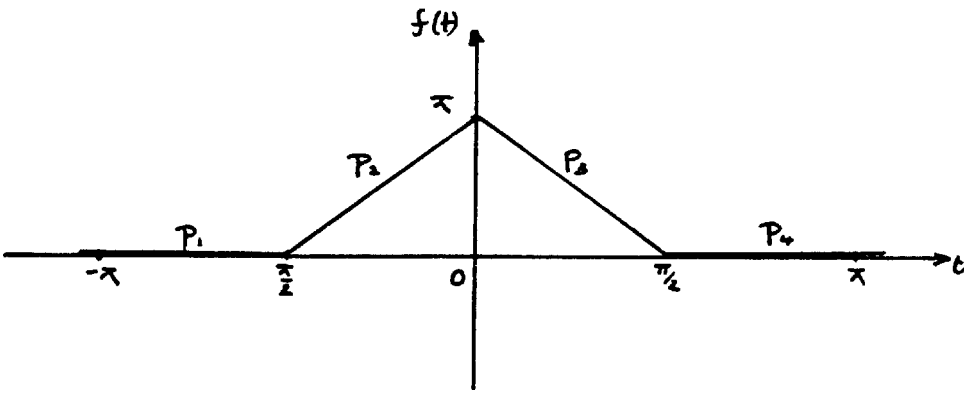
Thus, from (7.39)

$$\begin{aligned} a_n &= \frac{1}{n\pi}\left[-\sum_{s=1}^{3} d_s \sin nt_s - \frac{1}{n}\sum_{s=1}^{3} d_s^{(1)} \cos nt_s\right] \\ &= -\frac{1}{n^2\pi}\left[d_1^{(1)} \cos n\frac{\pi}{2} + d_2^{(1)} \cos n\pi + d_3^{(1)} \cos 2\pi n\right] \quad \text{since} \quad d_s = 0, s = 1, 2, 3 \\ &= -\frac{1}{n^2\pi}\left[-1 \cos \frac{n\pi}{2} - \frac{1}{2}\cos n\pi + \frac{3}{2}\cos 2n\pi\right] \\ &= -\frac{1}{n^2\pi}\left[-\cos \frac{n\pi}{2} - \frac{1}{2}(-1)^n + \frac{3}{2}\right] \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{n\pi}\left[\sum_{s=1}^{3} d_s \cos nt_s - \frac{1}{n}\sum_{s=1}^{3} d_s^{(1)} \sin nt_s\right] \\ &= -\frac{1}{n^2\pi}\left[-1 \sin \frac{n\pi}{2} - \frac{1}{2}\sin n\pi + \frac{3}{2}\sin 2n\pi\right] \\ &= \frac{1}{n^2\pi}\sin \frac{n\pi}{2} \\ a_0 &= \frac{1}{\pi}\left[\int_0^{\pi/2} t\ dt + \int_{\pi/2}^{\pi} \frac{\pi}{2} dt + \int_{\pi}^{2\pi} (\pi - \frac{1}{2}t) dt\right] = \frac{5}{8}\pi \end{aligned}$$

which agree with the Fourier coefficients of Example 7.3.

■ **28(a)** Graph of $f(t)$ for $-\pi < t < \pi$ as follows and is readily extended to $-4\pi < t < 4\pi$:



28(b)

$$\begin{array}{llll} p_1(t)=0 & p_2(t)=\pi+2t & p_3(t)=\pi-2t & p_4(t)=0 \\ d_1=0 & d_2=0 & d_3=0 & d_4=0 \\ p_1^{(1)}(t)=0 & p_2^{(1)}(t)=2 & p_3^{(1)}(t)=-2 & p_4^{(1)}(t)=0 \\ d_1^{(1)}=2 & d_2^{(1)}=-4 & d_3^{(1)}=2 & d_4^{(1)}=0 \\ p_1^{(2)}(t)=0 & p_2^{(2)}(t)=0 & p_3^{(2)}(t)=0 & p_4^{(2)}(t)=0 \\ d_1^{(2)}=0 & d_2^{(2)}=0 & d_3^{(2)}=0 & d_4^{(2)}=0 \end{array}$$

$$t_1=-\frac{\pi}{2},\ t_2=0,\ t_3=\frac{\pi}{2},\ t_4=\pi, \omega=1$$

Thus, from (7.39)

$$\begin{aligned} a_n &= \frac{1}{n\pi}\left[-\sum_{s=1}^{4} d_s \sin nt_s - \frac{1}{n}\sum_{s=1}^{4} d_s^{(1)} \cos nt_s + \frac{1}{n^2}\sum_{s=1}^{4} d_s^{(2)} \sin t_s\right] \\ &= -\frac{1}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 4\cos 0 + 2\cos\frac{n\pi}{2}\right] \\ &= -\frac{4}{n^2\pi}\left[\cos\frac{n\pi}{2} - 1\right] \\ b_n &= \frac{1}{n\pi}\left[\sum_{s=1}^{4} d_s \cos nt_s - \frac{1}{n}\sum_{s=1}^{4} d_s^{(1)} \sin nt_s - \frac{1}{n^2}\sum_{s=1}^{4} d_s^{(2)} \cos nt_s\right] \\ &= -\frac{1}{n^2\pi}\left[-2\sin\frac{n\pi}{2} - 4\sin 0 + 2\sin\frac{n\pi}{2}\right] = 0 \\ a_0 &= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} 0dt + \int_{-\pi/2}^{0} (\pi+2t)dt + \int_{0}^{\pi/2} (\pi-2t)dt + \int_{\pi/2}^{\pi} 0dt\right] \\ &= \frac{\pi}{2} \end{aligned}$$

Thus, Fourier series is

$$f(t) = \frac{\pi}{4} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos nt$$

■ **29(a)**

$$\begin{aligned} p_1(t) &= 1 & p_2(t) &= t^2 \\ d_1 &= 0 & d_2 &= -\pi^2 \\ p_1^{(1)}(t) &= 0 & p_2^{(1)}(t) &= 2t \\ d_1^{(1)} &= 0 & d_2^{(1)} &= -2\pi \\ p_1^{(2)}(t) &= 0 & p_2^{(2)}(t) &= 2 \\ d_1^{(2)} &= 2 & d_2^{(2)} &= -2 \\ p_1^{(3)}(t) &= 0 & p_2^{(3)}(t) &= 0 \\ d_1^{(3)} &= 0 & d_2^{(3)} &= 0 \end{aligned}$$

$$t_1 = 0,\ t_2 = \pi,\ \omega = 1$$

Thus, from (7.39)

$$\begin{aligned} a_n &= \frac{1}{n\pi}\left[-\sum_{s=1}^{2} d_s \sin nt_s - \frac{1}{n}\sum_{s=1}^{2} d_s^{(1)} \cos nt_s \right. \\ &\quad \left. + \frac{1}{n^2}\sum_{s=1}^{2} d_s^{(2)} \sin nt_s\right] \\ &= \frac{1}{n\pi}\left[\pi^2 \sin n\pi + \frac{2\pi}{n}\cos n\pi \right. \\ &\quad \left. - \frac{2}{n^2}\sin n\pi + \frac{2}{n^2}\sin 0\right] \\ &= \frac{2}{\pi^2}(-1)^n \\ b_n &= \frac{1}{n\pi}\left[-\pi^2 \cos n\pi + \frac{2\pi}{n}\sin n\pi \right. \\ &\quad \left. - \frac{2}{n^2}\cos 0 + \frac{2}{n^2}\cos n\pi\right] \\ &= \frac{1}{\pi}\left[-\frac{\pi^2}{n}(-1)^n - \frac{2}{n^3} + \frac{2}{n^3}(-1)^n\right] \\ a_0 &= \frac{1}{\pi}\int_0^{\pi} t^2 dt = \frac{\pi^2}{3} \end{aligned}$$

From which the Fourier series may be readily written down.

29(b)

$$
\begin{array}{lll}
p_1(t) = 2 & p_2(t) = t^3 & p_3(t) = -2 \\
d_1 = -(2 + \frac{\pi^3}{8}) & d_2 = -(2 + \frac{\pi^3}{8}) & d_3 = 4 \\
p_1^{(1)}(t) = 0 & p_2^{(1)}(t) = 3t^2 & p_3^{(1)}(t) = 0 \\
d_1^{(1)} = \frac{3\pi^2}{4} & d_2^{(1)} = -\frac{3\pi^2}{4} & d_3^{(1)} = 0 \\
p_1^{(2)}(t) = 0 & p_2^{(2)}(t) = 6t & p_3^{(2)}(t) = 0 \\
d_1^{(2)} = -3\pi & d_2^{(2)} = -3\pi & d_3^{(2)} = 0 \\
p_1^{(3)}(t) = 0 & p_2^{(3)} = 6 & p_3^{(3)}(t) = 0 \\
d_1^{(3)} = 6 & d_2^{(3)} = -6 & d_3^{(3)} = 0 \\
p_1^{(4)}(t) = 0 & p_2^{(4)}(t) = 0 & p_3^{(4)}(t) = 0 \\
d_1^{(4)} = 0 & d_2^{(4)} = 0 & d_3^{(4)} = 0
\end{array}
$$

$$t_1 = -\frac{\pi}{2},\ t_2 = \frac{\pi}{2},\ t_3 = \pi,\ \omega = 1$$

Thus, from (7.39)

$$
\begin{aligned}
a_n &= \frac{1}{n\pi}\Big[-(2+\frac{\pi^3}{8})\sin\frac{n\pi}{2} + (2+\frac{\pi^3}{8})\sin\frac{n\pi}{2} - 4\sin n\pi - \frac{3\pi^2}{4n}\cos\frac{n\pi}{2} \\
&\quad + \frac{3\pi^2}{4n}\cos\frac{n\pi}{2} + \frac{3\pi}{n^2}\sin\frac{n\pi}{2} - \frac{3\pi}{n^2}\sin\frac{n\pi}{2} + \frac{6}{n^3}\cos\frac{n\pi}{2} - \frac{6}{n^3}\cos\frac{n\pi}{2}\Big] \\
&= 0 \quad \text{(which is readily confirmed since odd function)} \\
b_n &= \frac{1}{n\pi}\Big[-(2+\frac{\pi^3}{8})\cos\frac{n\pi}{2} - (2+\frac{\pi^3}{8})\cos\frac{n\pi}{2} + 4\cos n\pi + \frac{3\pi^2}{4n}\sin\frac{n\pi}{2} \\
&\quad + \frac{3\pi^2}{4n}\sin\frac{n\pi}{2} + \frac{3\pi}{n^2}\cos\frac{n\pi}{2} + \frac{3\pi}{n^2}\cos\frac{n\pi}{2} - \frac{6}{n^3}\sin\frac{n\pi}{2} - \frac{6}{n^3}\cos\frac{n\pi}{2}\Big] \\
&= \frac{4}{n\pi}(\cos n\pi - \cos\frac{n\pi}{2}) + 2\left(\frac{3\pi}{4n^2}\sin\frac{n\pi}{2} - \frac{\pi^2}{8n}\cos\frac{n\pi}{2}\right. \\
&\qquad \left. + \frac{3}{n^3}\cos\frac{n\pi}{2} - \frac{6}{\pi n^4}\sin\frac{n\pi}{2}\right) \\
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = 0 \quad \text{since } f(t) \quad \text{is even function}
\end{aligned}
$$

Thus, Fourier series may be written down.

29(c)

$$\begin{aligned} p_1(t) &= t & p_2(t) &= 1-t \\ d_1 &= -1 & d_2 &= 2 \\ p_1^{(1)}(t) &= 1 & p_2^{(1)}(t) &= -1 \\ d_1^{(1)} &= -2 & d_2^{(1)} &= 2 \\ p_1^{(2)}(t) &= 0 & p_2^{(2)}(t) &= 0 \\ d_1^{(2)} &= 0 & d_2^{(2)} &= 0 \end{aligned}$$

$$t_1 = 1, t_2 = 2, \omega = \pi$$

Thus, from (7.39)

$$\begin{aligned} a_n &= \frac{1}{n\pi}\left[-\sum_{s=1}^{2} d_s \sin n\pi t_s - \frac{1}{n\pi}\sum_{s=1}^{2} d_s^{(1)} \cos n\pi t_s\right] \\ &= \frac{1}{n\pi}\left[1\sin n\pi - 2\sin 2n\pi - \frac{1}{n\pi}(-2\cos n\pi + 2\cos 2n\pi)\right] \\ &= \frac{2}{n^2\pi^2}[(-1)^n - 1] = \begin{cases} 0, & n \text{ even} \\ \dfrac{-4}{n^2\pi^2}, & n \text{ odd} \end{cases} \\ b_n &= \frac{1}{n\pi}\left[\sum_{s=1}^{2} d_s \cos n\pi t_s - \frac{1}{n\pi}\sum_{s=1}^{2} d_s^{(1)} \sin n\pi t_s\right] \\ &= \frac{1}{n\pi}\left[-\cos n\pi + \cos 2n\pi - 0\right] \\ &= \frac{1}{n\pi}\left[1 - (-1)^n\right] = \begin{cases} 0, & n \text{ even} \\ \dfrac{2}{n\pi}, & n \text{ odd} \end{cases} \\ a_0 &= \frac{2}{2}\int_0^2 f(t)dt = \left[\int_0^1 t dt + \int_1^2 (1-t)dt\right] = 0 \end{aligned}$$

The Fourier series is

$$f(t) = -\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi t}{(2n-1)^2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)}$$

29(d)

$$\begin{aligned} p_1(t) &= \frac{1}{2} + t & p_2(t) &= \frac{1}{2} - t \\ d_1 &= 0 & d_2 &= 0 \\ p_1^{(1)}(t) &= 1 & p_2^{(1)}(t) &= -1 \\ d_1^{(1)} &= -2 & d_2^{(1)} &= 2 \\ p_1^{(2)}(t) &= 0 & p_2^{(2)}(t) &= 0 \\ d_1^{(2)} &= 0 & d_2^{(0)} &= 0 \end{aligned}$$

$$t_1 = 0,\ t_2 = \tfrac{1}{2}, \omega = 2\pi$$

Thus, from (7.39)

$$\begin{aligned} a_n &= \frac{1}{n\pi}\left[-\sum_{s=1}^{2} d_s \sin 2\pi n t_s - \frac{1}{2n\pi}\sum_{s=1}^{2} d_s^{(1)} \cos 2n\pi t_s\right] \\ &= \frac{1}{n\pi}\left[-\frac{1}{2n\pi}[-2\cos 0 + 2\cos n\pi]\right] \\ &= -\frac{1}{(n\pi)^2}(\cos n\pi - 1) = \begin{cases} 0, & n \text{ even} \\ \dfrac{2}{(n\pi)^2}, & n \text{ odd} \end{cases} \\ b_n &= \frac{1}{n\pi}\left[\sum_{s=1}^{2} d_s \cos 2n\pi t_s - \frac{1}{2n\pi}\sum_{s=1}^{2} d_s^{(1)} \sin 2n\pi t_s\right] = 0 \\ a_0 &= 2\left[\int_{-\frac{1}{2}}^{0} (\frac{1}{2} + t)dt + \int_{0}^{\frac{1}{2}} (\frac{1}{2} - t)dt\right] = \frac{1}{2} \end{aligned}$$

Thus, Fourier expansion is

$$f(t) = \frac{1}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos 2(2n-1)\pi t$$

Exercises 7.5.2

■ **30** Fourier expansion to the voltage $e(t)$ is

$$e(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t,\ \omega = 100\pi$$

where

$$a_0 = 100\int_0^{\frac{1}{100}} dt = 10$$

$$a_n = 100\int_0^{\frac{1}{100}} 10\cos 100n\pi t dt = 100\Big[\frac{100\sin 100n\pi t}{100n\pi}\Big]_0^{\frac{1}{100}} = 0$$

$$b_n = 100\int_0^{\frac{1}{100}} 10\sin 100n\pi t dt = 100\Big[-10\frac{\cos 100n\pi t}{100n\pi}\Big]_0^{\frac{1}{100}}$$

$$= \frac{10}{n\pi}[1-(-1)^n] = \begin{cases} 0, & n\ even \\ \dfrac{20}{n\pi}, & n\ odd \end{cases}$$

Thus, Fourier expansion is

$$e(t) = 5 + \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)100\pi t$$

$$= 5 + \sum_{n=1}^{\infty} u_n(t),\ \text{where } u_n(t) = \frac{20}{\pi(2n-1)}\sin(2n-1)100\pi t$$

By Kirchhoff's second law, charge on the capacitor is given by

$$0.02\frac{d^2q}{dt^2} + 300\frac{dq}{dt} + 250000q = e(t)$$

System transfer function is $G(s) = \frac{1}{0.02s^2+300s+250000}$

$$\text{giving } \mid G(j\omega)\mid = \frac{1}{\sqrt{(250000-0.02\omega^2)^2+(300\omega)^2}}$$

$$argG(j\omega) = -\tan^{-1}\left[\frac{300\omega}{250000-0.02\omega^2}\right]$$

From (7.42), the steady-state response to the nth harmonic $u_n(t)$ is

$$q_{ssn}(t) = \frac{20}{\pi(2n-1)}\mid G(j(2n-1)100\pi)\mid \sin[(2n-1)100\pi t + argG(j(2n-1)100\pi)]$$

So steady-state current response $i_{ssn}(t)$ to nth harmonic is

$$i_{ssn}(t) = 2000\mid G(j(2n-1)100\pi)\mid \cos[(2n-1)100\pi t + argG(j(2n-1)100\pi)]$$

Note that the d.c. term in $e(t)$ gives no contribution to current steady-state response, which becomes

$$i_{ss} = \sum_{n=1}^{\infty} i_{ssn}(t)$$

Evaluating the first few terms gives

$$i_{ss} \simeq 0.008\cos(100\pi t - 1.96) + 0.005\cos(300\pi t - 0.33)$$

■ **31** Since the applied force represents an odd function, its Fourier expansion is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$$

where

$$b_n = \frac{4}{2}\int_0^1 100\sin n\pi t dt = 200\left[-\frac{1}{n\pi}\cos n\pi t\right]_0^1$$
$$= \frac{200}{n\pi}(1-(-1)^n) = \begin{cases} 0, & n \text{ even} \\ \dfrac{400}{n\pi}, & n \text{ odd} \end{cases}$$

Thus, Fourier expansion is

$$f(t) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)t = \sum_{n=1}^{\infty} u_n(t)$$

where $u_n(t) = \dfrac{400}{\pi}\dfrac{\sin(2n-1)t}{(2n-1)}$

From Newton's law, the displacement $x(t)$ of the mass is given by

$$10\frac{d^2x}{dt^2} + 0.5\frac{dx}{dt} + 1000 = f(t)$$

The transfer function is $G(s) = \dfrac{1}{10s^2 + 0.5s + 1000}$

$$\text{so that } G(j\omega) = \frac{1}{-10\omega^2 + 0.5j\omega + 1000} = \frac{1000 - 10\omega^2}{D} - j\frac{0.5\omega}{D}$$

$$\text{giving } \mid G(j\omega) \mid = \frac{1}{\sqrt{D}} = \frac{1}{\sqrt{(1000-10\omega^2)^2 + 0.25\omega^2}}$$
$$arg G(j\omega) = -\tan^{-1}\left[\frac{0.5\omega}{1000 - 10\omega^2}\right]$$

Thus, from (7.42) the steady-state response to the nth harmonic $u_n(t)$ is

$$x_{ssn} = \frac{400}{\pi(2n-1)} \mid G(j(2n-1)\pi) \mid \sin[(2n-1)\pi t + argG(j(2n-1)\pi)]$$

and steady-state response to $f(t)$ is $x_{ss}(t) = \sum_{n=1}^{\infty} x_{ssn}(t)$
Evaluating the first few terms gives

$$x_{ss}(t) \simeq 0.14\sin(\pi t - 0.1) + 0.379\sin(3\pi t - 2.415) + 0.017\sin(5\pi t - 2.83)$$

■ **32** Since the applied force represents an odd function, its Fourier expansion is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t,\ \omega = 2\pi$$

where

$$\begin{aligned} b_n &= \frac{4}{1}\int_0^{\frac{1}{2}} 100t\sin 2n\pi t dt \\ &= 400\left[-\frac{t}{2n\pi}\cos 2n\pi t + \frac{1}{(2n\pi)^2}\sin 2n\pi t\right]_0^{\frac{1}{2}} \\ &= -\frac{100}{n\pi}\cos n\pi = \frac{100}{n\pi}(-1)^{n+1} \end{aligned}$$

Thus, Fourier expansion is

$$f(t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin 2n\pi t\ = \sum_{n=1}^{\infty} u_n$$

$$\text{where } u_n(t) = \frac{100(-1)^n}{\pi n}\sin 2n\pi t$$

From Newton's law, the displacement $x(t)$ of the mass is given by

$$20\frac{d^2x}{dt^2} + 0.02\frac{dx}{dt} + 80x = f(t)$$

Transfer function is $G(s) = \dfrac{1}{20s^2 + 0.02s + 80}$

giving

$$| G(j\omega) | = \frac{1}{\sqrt{(80-20\omega^2)^2+(0.02\omega)^2}}\ ,\ argG(j\omega) = -\tan^{-1}\left[\frac{0.02\omega}{80-20\omega^2}\right]$$

Then from (7.42), the steady-state response to the nth harmonic $u_n(t)$ is

$$x_{ssn}(t) = \frac{100(-1)^n}{n\pi} \mid G(j2n\pi) \mid \sin[2n\pi t + argG(jn\pi)]$$

and the steady-state response to $f(t)$ is

$$x_{ss}(t) = \sum_{n=1}^{\infty} x_{ssn}(t)$$

Evaluating the first few terms gives

$$x_{ss}(t) \simeq 0.044\sin(2\pi t - 3.13) - 0.0052\sin(4\pi t - 3.14)$$

■ **33** Taking $A = 100$ and $\omega = 50\pi$ in Exercise 11, gives the Fourier expansion of the applied voltage $e(t)$ as

$$\begin{aligned} e(t) &= \frac{100}{\pi} + 50\sin 50\pi t - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{\cos 100n\pi t}{4n^2-1} \\ &= u_0 + u_s - \sum_{n=1}^{\infty} u_n(t) \end{aligned}$$

By Kirchhoff's second law, the charge $q(t)$ on the capacitor is given by

$$0.4\frac{d^2q}{dt^2} + 100\frac{dq}{dt} + 10^5 q = e(t)$$

System transfer function is $G(s) = \dfrac{1}{0.4s^2+100s+10^5}$ giving

$$| G(j\omega) | = \frac{1}{\sqrt{[(10^5-0.4\omega^2)^2+(100\omega)^2]}}\ ,\ argG(j\omega) = -\tan^{-1}\left[\frac{100\omega}{10^5-0.4\omega^2}\right]$$

From (7.42), the steady-state response to $u_s = 50\sin 50\pi t$ is

$$\begin{aligned} q_{sss}(t) &= 50 \mid G(j50\pi) \mid \sin(50\pi t + argG(j50\pi)) \\ &= 0.005\sin(50\pi t - 0.17) \end{aligned}$$

and the steady-state response to $u_n = \dfrac{200}{\pi}\cos\dfrac{100n\pi t}{4n^2-1}$ is

$$q_{ssn}(t) = \frac{200}{\pi}\frac{1}{4n^2-1} \mid G(j100n\pi) \mid \cos[100n\pi t + argG(j100n\pi)]$$

Since the d.c. term u_0 does not contribute to the steady-state current, this is given by

$$i_{ss} = 0.785\cos(50\pi t - 0.17) - \sum_{n=1}^{\infty}\frac{2\times 10^4 n}{4n^2-1} \mid G(j100n\pi) \mid \sin[100n\pi t + argG(j100n\pi)]$$

or

$$i_{ss} \simeq 0.785\cos(50\pi t - 0.17) - 0.1\sin(100\pi t - 0.48)$$

Exercises 7.6.5

■ **34** Since $T = 2\pi$ complex form of the Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnt}$$

with

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-jnt}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-jnt}dt \\ &= \frac{1}{2\pi}\left[-\frac{t^2}{jn}e^{-jnt} - \frac{2t}{(jn)^2}e^{-jnt} - \frac{2}{(jn)^3}e^{-jnt}\right]_{-\pi}^{\pi},\ n\neq 0 \\ &= \frac{1}{2\pi}\left[(\frac{j\pi^2}{n}e^{-jn\pi} + \frac{2\pi}{n^2}e^{-jn\pi} - \frac{2j}{n^3}e^{-jn\pi})\right. \\ &\qquad\left. - (\frac{j\pi^2}{n}e^{jn\pi} - \frac{2\pi}{n^2}e^{jn\pi} - \frac{2j}{n^3}e^{jn\pi})\right] \end{aligned}$$

Since $e^{-jn\pi} = e^{jn\pi} = \cos n\pi$

$$c_n = \frac{2}{n^2}\cos n\pi = \frac{2}{n^2}(-1)^n,\ n\neq 0$$

When $n = 0$, $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 dt = \dfrac{\pi^2}{3}$

Thus, complex form of the Fourier series is

$$f(t) = \frac{\pi^2}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{2}{n^2}(-1)^n e^{jnt}$$

Using (7.56),

$$a_0 = 2c_0 = \frac{2\pi^2}{3}$$
$$a_n - jb_n = \frac{4}{n^2}(-1)^n,\ a_n + jb_n = \frac{4}{n^2}(-1)^n$$
$$\text{giving } b_n = 0 \ \text{ and } a_n = \tfrac{4}{n^2}(-1)^n$$

thus confirming the series obtained in Example 7.5.

■ **35** Since $T = 4$, the complex form of the Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{jn\pi}{2}t}$$

with

$$c_n = \frac{1}{4}\int_{-2}^{2} f(t)e^{-\frac{jn\pi}{2}t}dt = \frac{1}{4}\int_{0}^{2} e^{-\frac{jn\pi}{2}t}dt$$
$$= \frac{1}{4}\left[-\frac{2}{jn\pi}e^{-\frac{jn\pi}{2}t}\right]_0^2,\ n \neq 0$$
$$= \frac{j}{2n\pi}[(-1)^n - 1],\ n \neq 0$$
$$c_0 = \frac{1}{4}\int_0^2 1dt = \frac{1}{2}$$

Thus, the complex form of the Fourier series is

$$f(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j}{2n\pi}[(-1)^n - 1]e^{\frac{jn\pi}{2}t}$$

Using (7.56),

$$a_0 = 2c_0 = 1$$
$$a_{n-j}b_n = \frac{j}{n\pi}[(-1)^n - 1]$$
$$a_{n_j} + b_n = \frac{j}{n\pi}[1 - (-1)^n]$$
$$\text{giving } a_n = 0\ ,\ b_n = \tfrac{1}{n\pi}[1-(-1)^n] = \begin{cases} 0, & n \text{ even} \\ \dfrac{2}{n\pi}, & n \text{ odd} \end{cases}$$

thus agreeing with series obtained in Example 7.7.

■ **36(a)**

$$c_n = \frac{1}{2\pi}\left[\int_{-\pi}^{0} \pi e^{-jnt}dt + \int_{0}^{\pi} te^{-jnt}dt\right]$$
$$= \frac{1}{2\pi}\left[\left[-\frac{\pi}{jn}e^{-jnt}\right]_{-\pi}^{0} + \left[-\frac{t}{jn}e^{-jnt} - \frac{1}{(jn)^2}e^{-jnt}\right]_{0}^{\pi}\right]$$
$$= \frac{1}{2\pi}\left[\frac{j\pi}{n} - \frac{1}{n^2}(1+(-1)^n)\right],\ n \neq 0$$
$$c_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} \pi dt + \int_{0}^{\pi} tdt\right] = \frac{3\pi}{4}$$

Thus, complex form of Fourier series is

$$f(t) = \frac{3\pi}{4} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{2\pi}\left[\frac{j\pi}{n} - \frac{1}{n^2}[1+(-1)^n]\right]e^{jnt}$$

36(b)

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt = \frac{a}{T}\int_0^{T/2} \sin\omega t e^{-jn\omega t}dt,\ T = \frac{2\pi}{\omega}$$
$$= \frac{a}{2jT}\int_0^{T/2} (e^{j\omega t} - e^{-j\omega t})e^{-jn\omega t}dt$$
$$= \frac{a}{2jT}\left[-\frac{e^{-j(n-1)\omega t}}{j(n-1)\omega} + \frac{e^{-j(n+1)\omega t}}{j(n+1)\omega}\right]_0^{T/2}$$
$$= \frac{a}{4\pi}\left[\frac{e^{-jn\omega t}e^{j\omega t}}{n-1} - \frac{e^{-jn\omega t}e^{-j\omega t}}{n+1}\right]_0^{T/2}$$
$$= \frac{a}{4\pi}\left\{\left[\frac{e^{-jn\pi}e^{j\pi}}{n-1} - \frac{e^{-jn\pi}e^{-j\pi}}{n+1}\right] - \left[\frac{1}{n-1} - \frac{1}{n+1}\right]\right\}$$

Since $e^{j\pi} = e^{-j\pi} = -1,\ e^{-jn\pi} = (-1)^n$

$$c_n = \frac{a}{4\pi}\left[-\left[\frac{1}{n-1} - \frac{1}{n+1}\right](-1)^n - \left[\frac{2}{n^2-1}\right]\right]$$
$$= -\frac{a}{2\pi(n^2-1)}[1+(-1)^n],\ n \neq \pm 1$$

$$c_{\pm 1} = \frac{a}{T}\int_0^{T/2} \sin\omega t(\cos\omega t \mp j\sin\omega t)dt$$
$$= \frac{a}{T}\left[-\frac{1}{2\omega}\cos 2\omega t \mp \frac{j}{2}\left(t - \frac{\sin 2\omega t}{2\omega}\right)\right]_0^{T/2} = \mp ja/2$$

Thus, complex form of Fourier series is

$$f(t) = \frac{a}{2}\sin\omega t - \sum_{\substack{n=-\infty \\ n\neq\pm1}}^{\infty} \frac{a}{2\pi(n^2-1)}[1+(-1)^n]e^{jn\omega t}$$

36(c)

$$\begin{aligned} c_n &= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 2e^{-jnt}dt + \int_{0}^{\pi} 1e^{-jnt}dt\right] \\ &= \frac{1}{2\pi}\left\{\left[-\frac{2}{jn}e^{-jnt}\right]_{-\pi}^{0} + \left[-\frac{1}{jn}e^{-jnt}\right]_{0}^{\pi}\right\} \\ &= -\frac{1}{2jn\pi}\left[2 - 2e^{jn\pi} + e^{-jn\pi} - 1\right] \\ &= \frac{j}{2n\pi}[1-(-1)^n],\ n\neq 0 \\ c_0 &= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 2dt + \int_{0}^{\pi} 1dt\right] = 3/2 \end{aligned}$$

Thus, complex form of Fourier series is

$$f(t) = \frac{3}{2} + \sum_{\substack{n=-\infty \\ n\neq0}}^{\infty} \frac{j}{2n\pi}[1-(-1)^n]e^{jnt}$$

36(d)

$$\begin{aligned} c_n &= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\sin te^{-jnt}dt + \int_{0}^{\pi} \sin te^{-jnt}dt\right] \\ &= \frac{1}{4nj}\left[\int_{-\pi}^{0} -(e^{jt}-e^{-jt})e^{-jnt}dt + \int_{0}^{\pi}(e^{jt}-e^{-jt})e^{-jnt}dt\right] \\ &= \frac{1}{4\pi j}\left\{\int_{-\pi}^{0}\left[-e^{-j(n-1)t} + e^{-j(n+1)t}\right]dt + \int_{0}^{\pi}\left[e^{-j(n-1)t} - e^{-j(n+1)t}\right]dt\right\} \\ &= \frac{1}{4\pi j}\left\{\left[-\frac{e^{-j(n-1)t}}{-j(n-1)} + \frac{e^{-j(n+1)t}}{-j(n+1)}\right]_{-\pi}^{0} + \left[\frac{e^{-j(n-1)t}}{-j(n-1)} - \frac{e^{-j(n+1)t}}{-j(n+1)}\right]_{0}^{\pi}\right\} \\ &= \frac{1}{4\pi}\left[-\frac{4}{n^2-1} - \frac{(-1)^n}{n-1} + \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \frac{(-1)^n}{n+1}\right] \\ &= -\frac{1}{\pi(n^2-1)}[1+(-1)^n],\ n\neq\pm1 \end{aligned}$$

By direct calculation $c_{\pm 1} = 0$. Thus, complex form of Fourier series is

$$\begin{aligned} f(t) &= \sum_{\substack{n=-\infty \\ n\neq \pm 1}}^{\infty} \frac{1}{\pi(1-n^2)}[1+(-1)^n]e^{jnt} \\ &= \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)}e^{2jnt} \end{aligned}$$

By noting that $|\sin t|$ is periodic with period π, we could have obtained the series from

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2nt}$$

with

$$\begin{aligned} c_n &= \frac{1}{\pi}\int_0^{\pi} \sin t e^{-j2nt} dt \\ &= \frac{1}{2\pi j}\int_0^{\pi} e^{-j(2n-1)t} - e^{-j(2n+1)t} dt \\ &= \frac{1}{2\pi}\left[\frac{e^{-j2nt}e^{jt}}{2n-1} - \frac{e^{-j2nt}e^{-jt}}{2n+1}\right]_0^{\pi} \\ &= -\frac{2}{\pi(4n^2-1)} \end{aligned}$$

Giving $f(t) = \frac{2}{\pi}\sum\limits_{n=-\infty}^{\infty} \frac{1}{(1-4n^2)}e^{j2nt}$

■ **37**

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi} dt = 1 \\ a_n &= \frac{1}{\pi}\int_0^{\pi} \cos nt dt = \frac{1}{\pi}\left[\frac{1}{\pi}\sin nt\right]_0^{\pi} = 0 \\ b_n &= \frac{1}{\pi}\int_0^{\pi} \sin nt dt = \frac{1}{\pi}\left[-\frac{1}{n}\cos nt\right]_0^{\pi} \\ &= \frac{1}{\pi n}(1-\cos n\pi) = \begin{cases} 0, & n \text{ even} \\ \dfrac{2}{\pi n}, & n \text{ odd} \end{cases} \end{aligned}$$

Thus, by Parseval's theorem

$$\frac{1}{2\pi}\int_0^{\pi} 1^2 dt = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$$

$$\text{or} \qquad \frac{1}{2} = \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{4}{\pi^2(2n-1)^2}$$

giving $\sum\limits_{n=1}^{\infty} \dfrac{1}{(2n-1)^2} = \dfrac{1}{8}\pi^2$

■ **38(a)** Fourier expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

with $\omega = \dfrac{2\pi}{T} = 100\pi$ and

$$a_0 = \frac{2}{T}\int_0^T f(1)dt = 100\int_0^{\frac{1}{50}} 500\pi t dt = 10\pi$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos 100n\pi t dt = 100\int_0^{\frac{1}{50}} 500\pi t \cos 100n\pi t dt$$

$$= 100.500\pi\left[\frac{1}{100n\pi} t\sin 100n\pi t + \frac{1}{(100n\pi)^2}\cos 100n\pi t\right]_0^{\frac{1}{50}} = 0$$

$$b_n = 100\int_0^{\frac{1}{50}} 500\pi t \sin 100n\pi t dt$$

$$= 100.500\pi\left[-\frac{t}{100n\pi}\cos 100n\pi t + \frac{1}{(100n\pi)^2}\sin 100n\pi t\right]_0^{\frac{1}{50}}$$

$$= -\frac{10}{n}\cos 2n\pi \quad = \quad -\frac{10}{n}$$

Thus, Fourier series expansion is

$$f(t) = 5\pi - 10\sum_{n=1}^{\infty} \frac{1}{n}\sin 100n\pi t$$

■ **38(b)** From (7.66), RMS value given by

$$f_{\text{RMS}}^2 = \frac{1}{T}\int_0^T [f(t)]^2 dt = 50\int_0^{\frac{1}{50}} (500\pi t)^2 dt$$
$$= \frac{100}{3}\pi^2 \simeq 328.987$$
$$\therefore \quad f_{\text{RMS}} = 18.14$$

$$\text{Using } \frac{1}{T}\int_0^T [f(t)]^2 dt = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

estimates using

(i) First four terms : $\frac{1}{4}a_0^2 + \frac{1}{2}(b_1^2 + b_2^2 + b_3^2) \simeq 314.79$
Thus $f_{\text{RMS}} \simeq 17.74$

(ii) First eight terms : $\frac{1}{4}a_0^2 + \frac{1}{2}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2) \simeq 322.32$
Thus $f_{\text{RMS}} \simeq 17.95$

■ **38(c)** True RMS value given by

$$f_{\text{RMS}}^2 = \frac{1}{T}\int_0^T [f(t)]^2 dt = 50\int_0^{\frac{1}{50}} (500\pi t)^2 dt$$
$$= \frac{100}{3}\pi^2 \simeq 328.987$$
$$\therefore \quad f_{\text{RMS}} = 18.14$$

$$\%\ \text{Error} = \frac{\text{Actual - Estimate}}{\text{Actual}} \times 100$$

giving the estimated percentage error in estimates (i) and (ii) as 2.20% and 1.05%, respectively.

■ **39(a)**

$$c_n = \frac{1}{5}\int_0^{5/4} 60e^{-j\frac{2n\pi}{5}t} dt$$
$$= 12\big[-\frac{5}{j2n\pi}e^{-j\frac{2n\pi}{5}t}\big]_0^{5/4}$$
$$= \frac{30}{jn\pi}[1 - e^{-\frac{jn\pi}{2}}], n \neq 0$$
$$c_0 = \frac{1}{5}\cdot 60\cdot \frac{5}{4} = 15$$

First five non-zero terms are

$$\begin{aligned} c_0 &= 15 & c_1 &= \frac{30}{j\pi}(1+j) = \frac{30}{\pi}(1-j) \\ c_2 &= \frac{30}{j\pi} = -\frac{30}{\pi}j & c_3 &= \frac{10}{j\pi}(1-j) = \frac{10}{\pi}(-1-j) \\ c_4 &= 0 & c_5 &= \frac{6}{j\pi}(1+j) = \frac{6}{\pi}(1-j) \end{aligned}$$

39(b) Power associated with the five non-zero terms are

$$\begin{aligned} P_0 &= \frac{15^2}{15} = 15\text{W} \\ P_1 &= \frac{1}{15}[2 \mid c_1 \mid^2] = \frac{2}{15}(13.50)^2 = 24.30\text{W} \\ P_2 &= \frac{1}{15}[2 \mid c_2 \mid^2] = \frac{2}{15}(9.55)^2 = 12.16\text{W} \\ P_3 &= \frac{1}{15}[2 \mid c_3 \mid^2] = \frac{2}{15}(4.50)^2 = 2.70\text{W} \\ P_4 &= 0 \\ P_5 &= \frac{1}{15}[2 \mid c_5 \mid^2] = \frac{2}{15}(2.70)^2 = 0.97\text{W} \end{aligned}$$

Total power delivered by the first five terms is

$$P = P_0 + P_1 + P_2 + P_3 + P_5 = 55.13\text{W}$$

39(c) Total power delivered by $15\,\Omega$ resistor is

$$P = \frac{1}{15}\Big[\frac{1}{5}\int_0^{5/4} 60^2 dt\Big] = \frac{1}{15}\cdot\frac{1}{5}\cdot 60^2\cdot\frac{5}{4} = 60\text{W}$$

39(d) The % of total power delivered by the first five non-zero terms is

$$\frac{55.13}{60} \times 100 = 91.9\%$$

Exercises 7.7.4

- **40**

$$\text{MSE} = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi}[f(t)]^2 dt - \sum_{n=1}^{\infty}\pi b_n^2\right]$$

Based on one term

$$(\text{MSE})_1 = \frac{1}{2\pi}\left[2\pi - \pi(\frac{4}{\pi})^2\right] = 0.19$$

Based on two terms

$$(\text{MSE})_2 = \frac{1}{2\pi}\left[2\pi - \pi(\frac{4}{\pi})^2 - \pi(\frac{4}{3\pi})^2\right] = 0.10$$

Based on three terms

$$(\text{MSE})_3 = \frac{1}{2\pi}\left[2\pi - \pi(\frac{4}{\pi})^2 - \pi(\frac{4}{3\pi})^2 - \pi(\frac{4}{5\pi})^2\right] = 0.0675$$

- **41(a)** From given formula,

$$P_0(t) = 1$$
$$P_1(t) = \frac{1}{2}\frac{d}{dt}(t^2-1) = t$$
$$P_2(t) = \frac{1}{8}\frac{d^2}{dt^2}(t^2-1)^2 = \frac{1}{2}(3t^2-1)$$

or from given recurrence relationship

$$2P_2(t) = 3tP_1(t) - P_0(t) = 3t^2 - 1$$

Also from the relationship

$$3P_3(t) = 5tP_2(t) - 2P_1(t) = \frac{5t}{2}(3t^2-1) - 2t$$

giving $\quad P_3(t) = \frac{1}{2}(5t^3 - 3t)$

41(b)

$$\int_{-1}^{1} P_m(t)P_n(t)dt = \frac{1}{2^{m+n}m!n!}\int_{-1}^{1} D^m(t^2-1)^m D^n(t^2-1)^n dt, \quad D \equiv \frac{d}{dt}$$
$$= \frac{1}{2^{m+n}m!n!} \quad I_{m,n}$$

Integrating by parts m times

$$I_{m,n} = (-1)\int_{-1}^{1} D^{m-1}(t^2-1)^m D^{n+1}(t^2-1)^n dt$$
$$\vdots$$
$$= (-1)^m \int_{-1}^{1} D^0(t^2-1)^m D^{n+m}(t^2-1)^n dt$$

If $m \neq n$ suppose $m > n$, then $m+n > 2n$ which implies that

$$D^{n+m}(t^2-1)^n = 0$$

so that $I_{m,n} = 0$
If $m = n$ then

$$I_{m,n} = I_{n,n} = (-1)^n \int_{-1}^{1} (t^2-1)^n D^{2n}(t^2-1)^n dt$$
$$= (2n)!(-1)^n \int_{-1}^{1} (t^2-1)^n dt$$
$$= 2(2n)! \int_{0}^{1} (1-t^2)^n dt$$

Making the substitution $t = \sin\theta$ then gives

$$I_{n,n} = 2(2n)! \int_0^{\pi/2} \cos^{2n+1}\theta d\theta = 2(2n)! \frac{2}{2n+1} \cdots \frac{2}{3}$$
$$= \frac{2^{2n+1}}{2n+1}(n!)^2$$

and the result follows.

41(c) $f(t) = c_0P_0(t) + c_1P_1(t) + c_2P_2(t) + \ldots$
Multiplying by $P_0(t)$

$$\int_{-1}^{1} f(t)P_0(t)dt = c_0 \int_{-1}^{1} P_0^2(t)dt = 2c_0$$

giving

$$\int_{-1}^{1} (-1)1dt + \int_{0}^{1} (1)1dt = 0 \ = 2c_0 \text{ so that } c_0 = 0$$

Multiplying by $P_1(t)$,

$$\int_{-1}^{1} f(t)P_1(t)dt = c_1 \int_{-1}^{1} P_1^2(t)dt = \frac{2}{3}a_1$$

giving

$$\int_{-1}^{0}(-1)tdt + \int_{0}^{1}(1)tdt = 1 = \frac{2}{3}c_1, \text{ so that } c_1 = \frac{3}{2}$$

Likewise,

$$\int_{-1}^{1} f(t)P_2(t)dt = c_2 \int_{-1}^{1} P_2^2(t)dt = \frac{2}{5}c_2$$

giving

$$\frac{1}{2}\int_{-1}^{0}(-1)(3t^2-1)dt + \frac{1}{2}\int_{1}^{0}(1)(3t^2-1)dt = 0 = \frac{2}{5}c_2, \text{ so that } c_2 = 0$$

and

$$\int_{-1}^{1} f(t)P_3(t)dt = c_3 \int_{-1}^{1} P_3^2(t)dt = \frac{2}{7}c_3$$

giving

$$\frac{1}{2}\int_{-1}^{0}(-1)(5t^3-3t)dt + \frac{1}{2}\int_{0}^{1}(1)(5t^3-3t)dt = -\frac{1}{4} = \frac{2}{7}c_3, \text{ so that } c_3 = -\frac{7}{8}$$

■ **42** Taking

$$f(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) + \ldots$$

and adopting the same approach as in 41(c) gives

$$\int_{-1}^{1} f(x)P_0(x)dx = c_0 \int_{-1}^{1} P_0^2(x)dx = 2c_0$$

giving

$$\int_{0}^{1} xdx = \frac{1}{2} = 2c_0, \text{ so that } c_0 = \frac{1}{4}$$

$$\int_{-1}^{1} f(x)P_1(x)dx = c_1 \int_{-1}^{1} P_1^2(x)dx = \frac{2}{3}c_1$$

giving

$$\int_0^1 x^2 dx = \frac{1}{3} = \frac{2}{3}c_1, \text{ so that } c_1 = \frac{1}{2}$$

$$\int_{-1}^1 f(x)P_2(x)dx = c_2 \int_{-1}^1 P_2^2(x)dx = \frac{2}{5}c_2$$

giving

$$\frac{1}{2}\int_0^1 x(3x^2-1)dx = \frac{1}{8} = \frac{2}{5}c_2, \text{ so that } c_2 = \frac{5}{16}$$

$$\int_{-1}^1 f(x)P_3(x)dx = c_3 \int_{-1}^1 P_3^2(x)dx = \frac{2}{7}c_3$$

giving

$$\frac{1}{2}\int_0^1 x(5x^3-3x)dx = 0 = \frac{2}{7}c_3, \text{ so that } c_3 = 0$$

■ **43(a)**

$$L_0(t) = e^t(t^0 e^{-t}) = 1$$
$$L_1(t) = e^t(-te^{-t} + e^{-t}) = 1 - t$$

Using the recurrence relation,

$$L_2(t) = (3-t)L_1(t) - L_0(t) = t^2 - 4t + 2$$

$$\begin{aligned} L_3(t) &= (5-t)L_2(t) - 4L_1(t) \\ &= (5-t)(t^2-4t+2) - 4(1-t) \\ &= 6 - 18t + 9t^2 - t^3 \end{aligned}$$

43(b) This involves evaluating the integral $\int_0^\infty e^{-t}L_m(t)L_n(t)dt$ for the 10 combinations of m and n.

43(c) If $f(t) = \sum_{r=0}^{\infty} c_r L_r(t)$ to determine c_n, multiply throughout by $e^{-t}L_n(t)$ and integrate over $(0, \infty)$

$$\int_0^\infty e^{-t}L_n(t)f(t)dt = \int_0^\infty \sum_{r=0}^{\infty} c_r e^{-t}L_r(t)L_n(t)dt$$

Using the orthogonality property then gives

$$\int_0^\infty e^{-t}L_n(t)f(t)dt = c_n\int_0^\infty e^{-t}L_n(t)L_n(t)dt$$
$$= c_n(n!)^2$$
$$\text{giving } c_n = \frac{1}{(n!)^2}\int_0^\infty e^{-t}L_n(t)f(t)dt,\ n = 0, 1, 2, \ldots$$

■ **44(a)** By direct use of formula,

$$H_0(t) = (-1)^0 e^{t^2/2}e^{-t^2/2} = 1$$
$$H_1(t) = (-1)e^{t^2/2}\frac{d}{dt}e^{-t^2/2} = t$$

Using recurrence relation,

$$H_n(t) = tH_{n-1}(t) - (n-1)H_{n-2}(t)$$

$$H_2(t) = t.t - 1.1 = t^2 - 1$$
$$H_3(t) = t(t^2-1) - 2(t) = t^3 - 3t$$
$$H_4(t) = t(t^3-3t) - 3(t^2-1) = t^4 - 6t^2 + 3$$

44(b) This involves evaluating the integral $\int_{-\infty}^{\infty} e^{-t^2/2}H_n(t)H_m(t)dt$ for the 10 combinations of n and m.

44(c) If $f(t) = \sum_{r=0}^{\infty} c_r H_r(t)$ to determine c_n, multiply throughout by $e^{-t^2/2}H_n(t)$ and integrate over $(-\infty, \infty)$ giving

$$\int_{-\infty}^{\infty} e^{-t^2/2}H_n(t)f(t)dt = \int_{-\infty}^{\infty}\sum_{r=0}^{\infty} c_r e^{-t^2/2}H_n(t)H_r(t)dt$$
$$= c_n\int_{-\infty}^{\infty} e^{-t^2/2}H_n(t)H_n(t)dt$$
$$= c_n\sqrt{(2\pi)}n!$$

so that

$$c_n = \frac{1}{n!\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-t^2/2} f(t) H_n(t) dt$$

■ **45(a)** Directly from the formula,

$$T_0(t) = \cos 0 = 1$$
$$T_1(t) = \cos(\cos^{-1} t) = t$$

then from the recurrence relationship

$$T_2(t) = 2t(t) - 1 = 2t^2 - 1$$
$$T_3(t) = 2t(2t^2 - 1) - t = 4t^3 - 3t$$
$$T_4(t) = 2t(4t^3 - 3t) - (2t^2 - 1) = 8t^4 - 8t^2 + 1$$
$$T_5(t) = 2t(8t^4 - 8t^2 + 1) - (4t^3 - 3t) = 16t^5 - 20t^3 + 5t$$

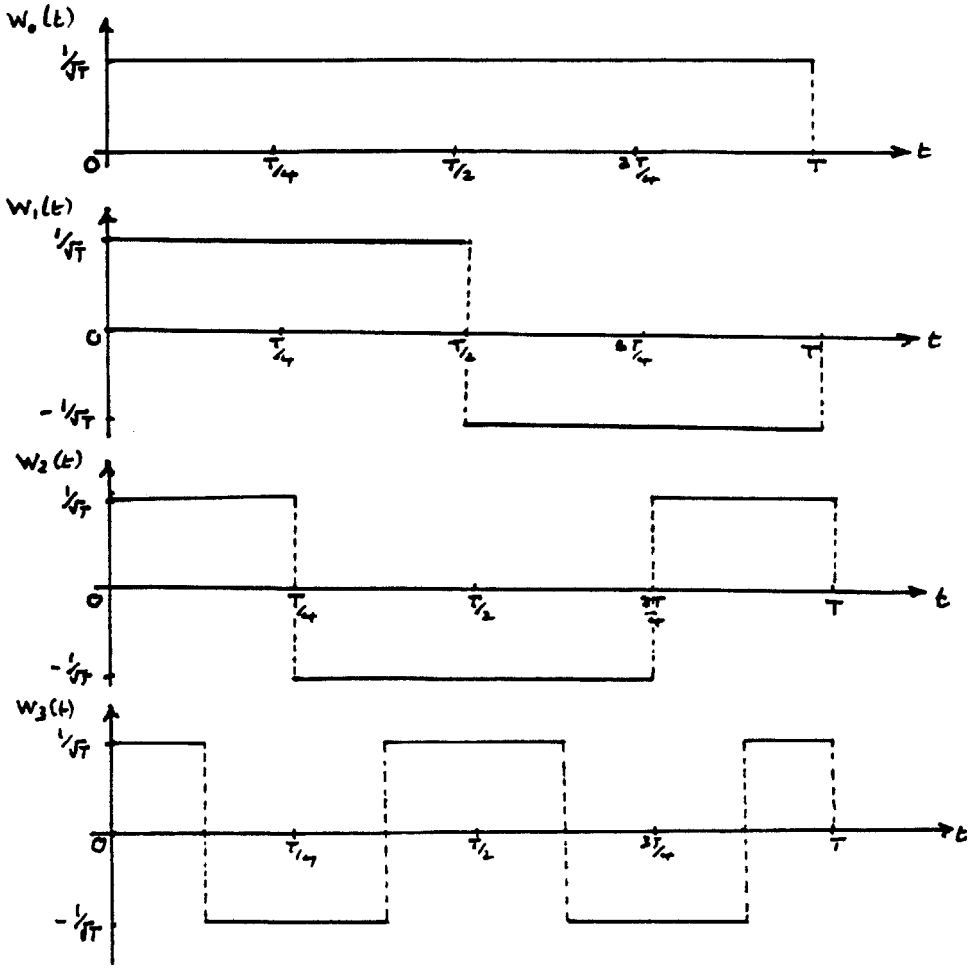
45(b) Evaluate the integral $\int_{-1}^{1} \frac{T_n(t) T_m(t)}{\sqrt{(1-t^2)}} dt$ for the 10 combinations of n and m.

45(c) If $f(t) = \sum_{r=0}^{\infty} c_r T_r(t)$ to obtain c_n, multiply throughout by $c_n T_n(t)/\sqrt{(1-t^2)}$ and integrate over $(-1, 1)$ giving

$$\int_{-1}^{1} \frac{T_n(t) f(t)}{\sqrt{(1-t^2)}} dt = \int_{-1}^{1} \sum_{r=0}^{\infty} \frac{c_r T_n(t) T_r(t)}{\sqrt{(1-t)^2}} dt$$
$$= c_n \int_{-1}^{1} \frac{T_n(t) T_n(t)}{\sqrt{(1-t^2)}} dt \quad T_n = 0, 1, 2, 3, \ldots$$
$$= \begin{cases} c_0 \pi, & n = 0 \\ c_n \frac{\pi}{2}, & n \neq 0 \end{cases}$$

Hence the required results.

■ **46(a)**



To show that they are orthonormal on $(0, T)$, evaluate the integral $\int_0^T W_n(t)$ $W_m(t)dt$ for the 10 combinations of n and m. For example,

$$\int_0^T W_0(t)W_0(t)dt = \int_0^T \frac{1}{T}at = 1$$

and it is readily seen that this extends to $\int_0^T W_n^2(t)dt = 1$

$$\int_0^T W_1(t)W_2(t)dt = \int_0^{T/4} \frac{1}{T}dt + \int_{T/4}^{T/2} \frac{(-1)}{T}dt + \int_{T/2}^{3T/4} \frac{1}{T}dt + \int_{3T/4}^{T} \frac{(-1)}{T}dt = 0$$

46(b) $f(t) = c_0W_0(t) + c_1W_1(t) + c_2W_2(t) + \ldots$ where $f(t)$ is the square wave of Exercise 40. In this case $T = 2\pi$. Multiplying throughout by the appropriate Walsh function and integrating over $(0, 2\pi)$ gives

$$\int_0^{2\pi} W_0(t)f(t)dt = c_0 \int_0^{2\pi} W_0^2(t)dt = c_0,\ W_0(t) = \frac{1}{\sqrt{2\pi}}$$

giving

$$c_0 = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} 1f(t)dt = \frac{1}{\sqrt{2\pi}} \Big[\int_0^{\pi} dt - \int_{\pi}^{2\pi} dt \Big] = 0$$

$$\int_0^{2\pi} W_1(t)f(t)dt = c_1 \int_0^{2\pi} W_1^2(t)dt = c_1,\ W_1(t) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & 0 < t < \pi \\ -\frac{1}{\sqrt{2\pi}}, & \pi < t < 2\pi \end{cases}$$

giving

$$c_1 = \frac{1}{\sqrt{2\pi}} \Big[\int_0^{\pi} dt + \int_{\pi}^{2\pi} (-1)(-1)dt \Big] = \sqrt{2\pi}$$

$$\int_0^{2\pi} W_2(t)f(t)dt = c_2,\ W_2(t) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & 0 < t < \frac{\pi}{2}, \frac{3}{2}\pi < t < 2\pi \\ -\frac{1}{\sqrt{2\pi}}, & \frac{\pi}{2} < t < \frac{3}{2}\pi \end{cases}$$

giving

$$c_2 = \frac{1}{\sqrt{2\pi}} \Big[\int_0^{\pi/2} (1)(1)dt + \int_{\frac{\pi}{2}}^{\pi} (1)(-1)dt + \int_{\pi}^{\frac{3\pi}{2}} (-1)(-1)dt + \int_{\frac{3\pi}{2}}^{2\pi} (-1)(1)dt \Big] = 0$$

Mean square error based on three terms is

$$\frac{1}{2\pi} \Big[\int_0^{2\pi} [f(t)]^2 dt - \sum_{n=0}^{3} c_n^2 \Big] = \frac{1}{2\pi} \Big[\int_0^{2\pi} dt - (\sqrt{2\pi})^2 \Big] = 0$$

This is zero in this case simply because the series based on three terms is exact as $W_2(t)$ exactly 'matches' the given square wave $f(t)$.

Review Exercises 7.9

- **1**

$$a_0 = \frac{1}{\pi}\int_0^\pi t^2 dt = \frac{1}{\pi}\left[\frac{1}{3}t^3\right]_0^\pi = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^\pi t^2\cos nt dt = \frac{1}{\pi}\left[\frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt\right]_0^\pi$$

$$= \frac{2}{n^2}\cos n\pi = 2$$

$$b_n = \frac{1}{\pi}\int_0^\pi t^2\sin nt dt = \frac{1}{\pi}\left[-\frac{t^2}{n}\cos nt + \frac{2t}{n^2}\sin nt + \frac{2}{n^3}\cos nt\right]_0^\pi$$

$$= \frac{1}{\pi}\left\{\frac{2}{n^3}[(-1)^n - 1] - \frac{\pi^2}{n}(-1)^n\right\}$$

$$= \begin{cases} -\dfrac{\pi}{n}, & n \text{ even} \\ \dfrac{1}{\pi}\left[-\dfrac{4}{n^3} + \dfrac{\pi^2}{n}\right], & n \text{ odd} \end{cases}$$

Thus, Fourier series expansion is

$$f(t) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2}(-1)^n\cos nt + \sum_{n=1}^{\infty}\left[\frac{\pi}{2n-1} - \frac{4}{\pi(2n-1)^3}\right]\sin(2n-1)t - \sum_{n=1}^{\infty}\frac{\pi}{2n}\sin 2nt$$

Taking $t = \pi$ when the series converges to $\pi^2/2$ gives

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2}(-1)^n(-1)^n = \sum_{n=1}^{\infty}\frac{2}{n^2}$$

that is, $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

■ **2**

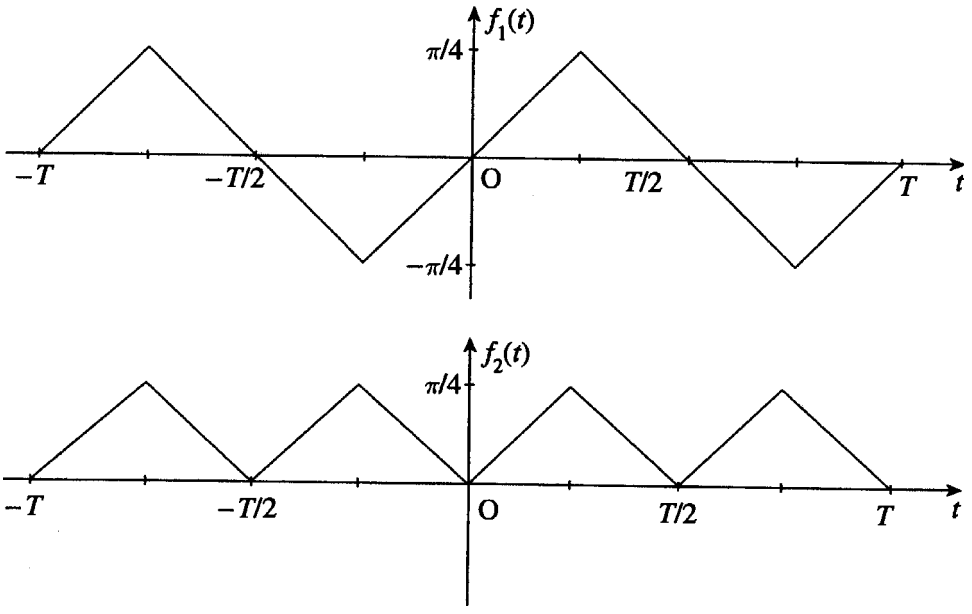
$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/3} \frac{2}{3}t dt + \frac{1}{3}\int_{\pi/3}^{\pi}(\pi - t)dt\right]$$
$$= \frac{2}{\pi}\left\{\left[\frac{1}{3}t^2\right]_0^{\pi/3} + \frac{1}{3}\left[-\frac{1}{2}(\pi - t)^2\right]_{\pi/3}^{\pi}\right\} = \frac{2\pi}{9}$$
$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/3} \frac{2}{3}t\cos nt dt + \frac{1}{3}\int_{\pi/3}^{\pi}(\pi - t)\cos nt dt\right]$$
$$= \frac{2}{\pi}\left\{\left[\frac{2t}{3n}\sin nt + \frac{2}{3n^2}\cos nt\right]_0^{\pi/3} + \frac{1}{3}\left[\frac{(\pi - t)}{n}\sin nt - \frac{1}{n^2}\cos nt\right]_{\pi/3}^{\pi}\right\}$$
$$= \frac{2}{\pi}\left[\frac{1}{n^2}\cos\frac{n\pi}{3} - \frac{1}{3n^2}[2 + \cos n\pi]\right]$$

Thus, the Fourier expansion of the even function is

$$f(t) = \frac{\pi}{9} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\frac{n\pi}{3} - \frac{1}{3}(2 + (-1)^n)\right]\cos nt$$

At $t = \frac{1}{3}\pi$ the series converges to $\frac{2}{9}\pi$.

■ **3** Sketches of odd function $f_1(t)$ and even function $f_2(t)$, having period T and equal to $f(t), a \le t \le \frac{1}{2}T$, are plotted for $-T \le t \le T$ below:



3(a) Half-range Fourier sine series is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi t}{T}$$

with

$$\begin{aligned} b_n &= \frac{4}{T}\left[\int_0^{\pi/4} t \sin \frac{2n\pi t}{T} dt + \int_{\pi/4}^{\pi/2} \left(\frac{1}{2}T - t\right) \sin \frac{2n\pi t}{T} dt\right] \\ &= \frac{4}{T}\left\{\left[-\frac{Tt}{2n\pi} \cos \frac{2n\pi t}{T} + \frac{T^2}{(2n\pi)^2} \sin \frac{2n\pi t}{T}\right]_0^{t/4}\right. \\ &\left. + \left[-\frac{T}{2n\pi}\left(\frac{1}{2}T - t\right) \cos \frac{2n\pi t}{T} - \frac{T^2}{(2n\pi)^2} \sin \frac{2n\pi t}{T}\right]_{T/4}^{T/2}\right\} \\ &= \frac{8T}{(2n\pi)^2} \sin \frac{n\pi}{2} = \begin{cases} 0, & n \text{ even} \\ \dfrac{2T}{n^2\pi^2}, & n = 1, 5, 9, \ldots \\ -\dfrac{2T}{n^2\pi^2}, & n = 3, 7, 11, \ldots \end{cases} \end{aligned}$$

Thus, Fourier sine series expansion is

$$f(t) = \frac{2T}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{2(2n-1)\pi t}{T}$$

3(b) From the sketch of $f_1(t)$, the series converges to $-T/4$ at $t = -\frac{1}{4}T$.

3(c) Taking $t = \frac{1}{4}T$, then $\sin \frac{2(2n-1)}{T}\pi t = (-1)^{n+1}$ giving

$$\frac{1}{4}T = \frac{2T}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}$$

so that the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}$ is $\frac{\pi^2}{8}$.

■ **4**

$$\begin{aligned} g(-x)[c + f(-x)] &= cg(-x) + g(-x)f(-x) \\ &= -cg(x) - cg(x)f(x) \quad \text{from the given information} \\ &= -g(x)[c + f(x)] \end{aligned}$$

Thus, the product is an odd function.
Since $y = \theta$ is an odd function and $y = \theta^2$ is an even function, it follows from the above that $F(\theta)$ is an odd function. Thus, it has a Fourier series of the form

$$F(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

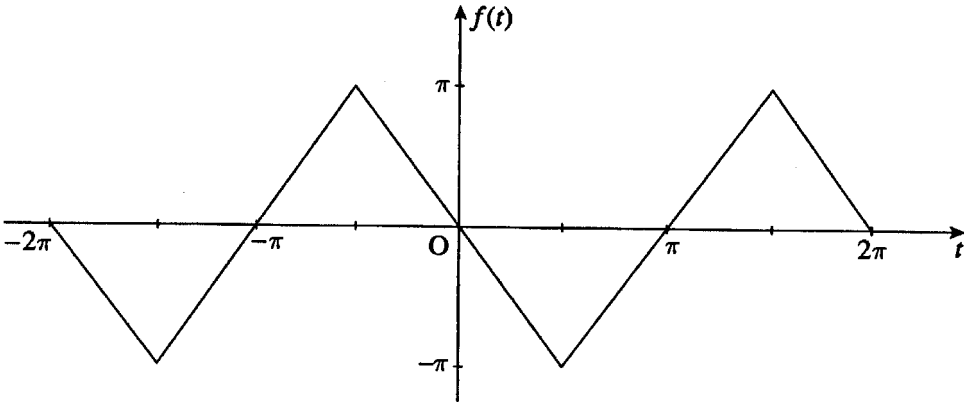
with

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{12}\theta(\pi^2 - \theta^2)\sin n\theta d\theta \\ &= \frac{1}{6\pi}\left\{\pi^2\left[-\frac{\theta}{n}\cos n\theta + \frac{1}{n^2}\sin n\theta\right]_0^{\pi}\right. \\ &\qquad \left. - \left[-\frac{\theta^3}{n}\cos n\theta + \frac{3\theta^2}{n^2}\sin n\theta - \frac{6\theta}{n^3}\cos n\theta + \frac{6}{n^4}\sin n\theta\right]_0^{\pi}\right\} \\ &= \frac{1}{6\pi}\left[-\frac{6\pi}{n^3}\cos n\pi\right] = \frac{1}{n^3}(-1)^{n+1} \end{aligned}$$

Thus, the Fourier expansion is

$$F(\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}\sin n\theta$$

■ **5**



Clearly, $f(t)$ is an odd function so it has a Fourier expansion of the form

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

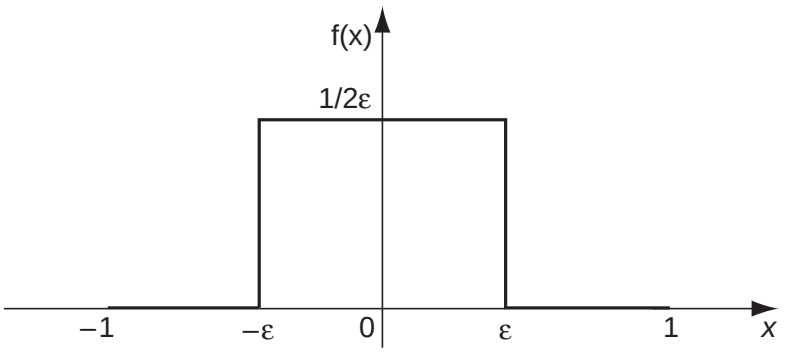
with

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[\int_0^{\pi/2} -t\sin ntdt + \int_{\pi/2}^{\pi}(t-\pi)\sin ntdt\right] \\ &= \frac{2}{\pi}\left\{\left[\frac{t}{n}\cos nt - \frac{1}{n^2}\sin nt\right]_0^{\pi/2} + \left[-\frac{(t-\pi)}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_{\pi/2}^{\pi}\right\} \\ &= \frac{2}{\pi}\left[-\frac{2}{n^2}\sin\frac{n\pi}{2}\right] \end{aligned}$$

Thus, Fourier expansion is

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin(2n-1)t$$

■ **6**



Since $f(x)$ is an even function, over the interval $-1 \le x \le 1$, it may be represented within this range by the Fourier cosine expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

with

$$a_0 = \frac{2}{1}\int_0^{\epsilon} \frac{1}{2\epsilon}dx = 2\left[\frac{1}{2\epsilon}x\right]_0^{\epsilon} = 1$$

$$a_n = \frac{2}{2\epsilon}\int_0^{\epsilon} \cos n\pi x dx = \frac{1}{\epsilon}\left[\frac{1}{n\pi}\sin n\pi x\right]_0^{\epsilon}$$

$$= \frac{1}{n\pi\epsilon}\sin n\pi\epsilon$$

Thus, Fourier expansion is

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin n\pi\epsilon}{n\pi\epsilon}\cos n\pi x$$

valid in the interval $-1 \leq x \leq 1$.

■ **7** Half-range Fourier sine expansion is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi} \left(1 - \frac{t}{\pi}\right)^2 \sin nt dt$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\left(1 - \frac{t}{\pi}\right)^2 \cos nt - \frac{2}{n^2\pi}\left(1 - \frac{t}{\pi}\right)\sin nt + \frac{2}{n^3\pi^2}\cos nt\right]_0^{\pi}$$

$$= \frac{2}{n\pi} + \frac{4}{n^3\pi^3}[(-1)^n - 1]$$

Thus, Fourier expansion is

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - \frac{2}{n^2\pi^2}[1 - (-1)^n]\sin nt\right]$$

■ **8** Half-range Fourier sine expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx dx$$

with

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx dx\right] \\ &= \frac{2}{\pi}\left\{\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi/2} + \left[-\frac{(\pi - x)}{n}\cos nx - \frac{1}{n^2}\sin nx\right]_{\pi/2}^{\pi}\right\} \\ &= \frac{4}{n^2\pi}\sin\frac{n\pi}{2} \end{aligned}$$

Thus, half-range Fourier sine expansion is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin(2n-1)x$$

Half-range Fourier cosine expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

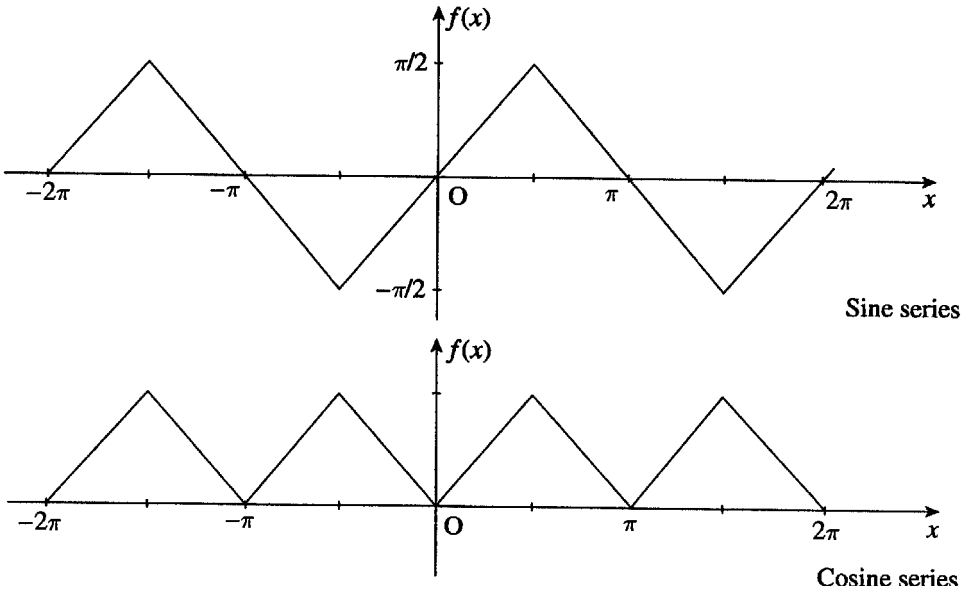
with

$$\begin{aligned} a_0 &= \frac{2}{\pi}\left[\int_0^{\pi/2} x dx + \int_{\pi/2}^{\pi}(\pi - x)dx\right] = \frac{\pi}{2} \\ a_n &= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx dx\right] \\ &= \frac{2}{\pi}\left\{\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi/2} + \left[\frac{\pi - x}{n}\sin nx - \frac{1}{n^2}\cos nx\right]_{\pi/2}^{\pi}\right\} \\ &= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{2}{n^2} - \frac{2}{n^2}\cos n\pi\right] \\ &= \begin{cases} 0, & n \text{ odd} \\ \dfrac{4}{\pi n^2}\left[(-1)^{n/2} - 1\right], & n \text{ even} \end{cases} \end{aligned}$$

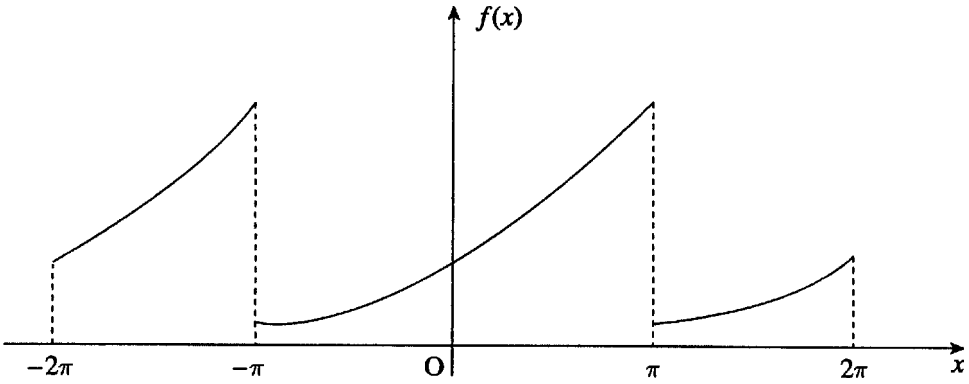
Thus, Fourier cosine expansion is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2(2n-1)x}{(2n-1)^2}$$

Sketches of the functions represented by the two Fourier series are



■ **9**



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi}[e^{\pi} - \pi^{-\pi}] = \frac{2}{\pi}\sinh\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx dx = \frac{n^2}{n^2+1}\cdot\frac{1}{\pi}\left[\frac{1}{n}e^x \sin nx + \frac{1}{n^2}e^x \cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{\pi(n^2+1)}[e^{\pi} - e^{-\pi}] = \frac{2(-1)^n}{\pi(n^2+1)}\sinh\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx dx = \frac{n^2}{\pi(n^2+1)}\left[-\frac{e^x}{n}\cos nx + \frac{e^x}{n^2}\sin nx\right]_{-\pi}^{\pi}$$

$$= -\frac{n(-1)^n}{\pi(n^2+1)}\sinh\pi$$

Thus, Fourier expansion is

$$f(x) = \frac{1}{\pi}\sinh\pi + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\sinh\pi\cos nx - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\sinh\pi\cos nx$$
$$= \frac{2}{\pi^2}\sinh\pi\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin x)\right]$$

■ **10(a)** Half-range Fourier sine expansion is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi - t)\sin nt dt$$
$$= \frac{2}{\pi}\left[-\frac{(\pi-t)}{n}\cos nt - \frac{1}{n^2}\sin nt\right]_0^{\pi} = \frac{2}{n}$$

Thus, Fourier sine expansion is

$$f(t) = \sum_{n=1}^{\infty}\frac{2}{n}\sin nt$$

10(b) Half-range Fourier cosine expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

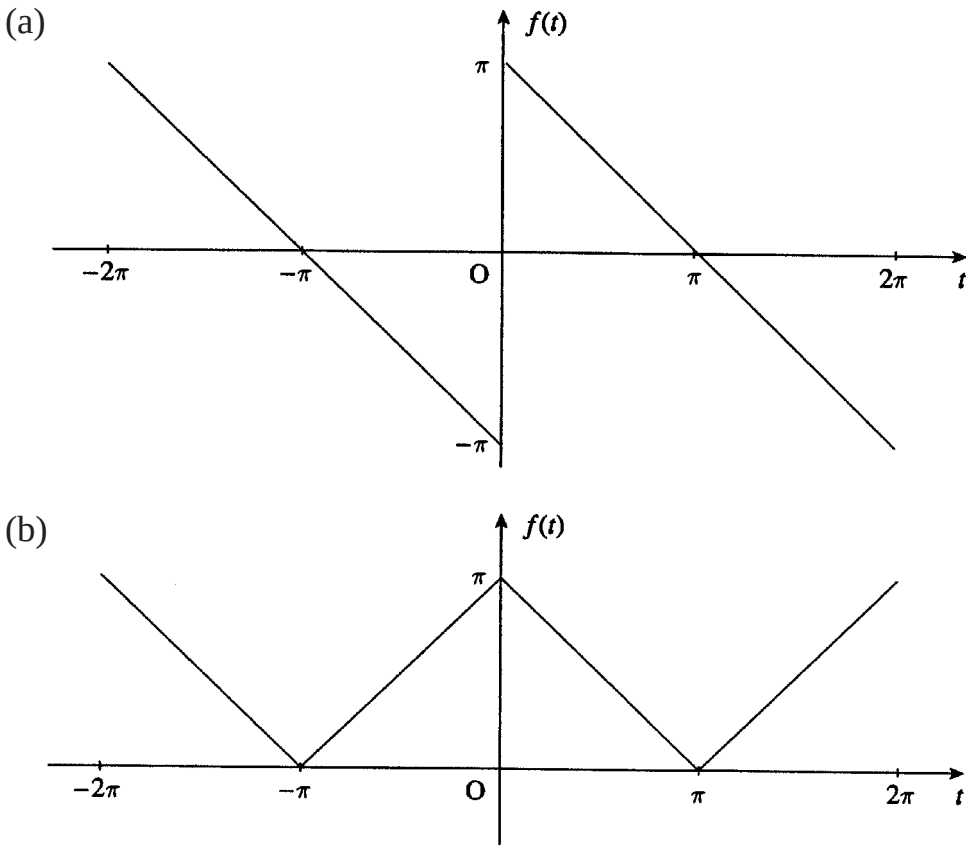
with

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - t)dt = \pi$$
$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - t)\cos nt dt = \frac{2}{\pi}\left[\frac{(\pi-t)}{n}\sin nt - \frac{1}{n^2}\cos nt\right]_0^{\pi}$$
$$= \frac{2}{\pi n^2}[1-(-1)^n] = \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{\pi n^2}, & n \text{ odd} \end{cases}$$

Thus, Fourier cosine expansion is

$$f(t) = \frac{1}{2}\pi + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)t$$

Graphs of the functions represented by the two series are



■ **11** Since $f(t)$ is an even function, it has a Fourier series expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} -tdt + \int_{0}^{\pi} tdt\right] = \pi$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -t\cos ntdt + \int_{0}^{\pi} t\cos ntdt\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{t}{n}\sin nt - \frac{\cos nt}{n^2}\right]_{-\pi}^{0} + \left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_{0}^{\pi}\right]$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1) = \begin{cases} 0, & n \text{ even} \\ -\frac{4}{\pi n^2}, & n \text{ odd} \end{cases}$$

Thus, the Fourier expansion of $f(t)$ is

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)t$$

Since $\dfrac{dx}{dt} + x = f(t)$ is linear, response is sum individual responses.
Steady-state response corresponds to the Particular Integral (PI). For $f_0(t) = \dfrac{\pi}{2}$ steady-state response is $x_0(t) = \dfrac{\pi}{2}$.
When $f(t) = \cos\omega t$, then steady-state response is of the form $x = A\cos\omega t + B\sin\omega t$. Substituting back and comparing coefficients of $\sin\omega t$ and $\cos\omega t$ gives

$$A = \frac{1}{1+\omega^2} \quad , \quad B = \frac{\omega}{1+\omega^2}$$

Taking $\omega = (2n-1)$, then required steady-state response is

$$x = \frac{1}{2}\pi - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[\frac{\cos(2n-1)t + (2n-1)\sin(2n-1)t}{1+(2n-1)^2}\right]$$

■ **12** Since $f(t)$ is an even function, Fourier series expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(t)dt = \frac{1}{\pi}\left[\int_0^{\pi}\frac{t}{\pi}dt + \int_{\pi}^{2\pi}\frac{(2\pi - t)}{\pi}\,dt\right] \\
&= \frac{1}{\pi^2}\left[\left[\frac{1}{2}t^2\right]_0^{\pi} + \left[2\pi t - \frac{1}{2}t^2\right]_{\pi}^{2\pi}\right] = 1 \\
a_n &= \frac{1}{\pi^2}\left[\int_0^{\pi} t\cos ntdt + \int_{\pi}^{2\pi}(2\pi - t)\cos ntdt\right] \\
&= \frac{1}{\pi^2}\left[\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_0^{\pi} + \left[\frac{(2\pi - t)}{n}\sin nt - \frac{1}{n^2}\cos nt\right]_{\pi}^{2\pi}\right] \\
&= \frac{2}{\pi^2 n^2}(\cos n\pi - 1) = \begin{cases} -\dfrac{4}{n^2\pi^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}
\end{aligned}$$

Thus, Fourier series expansion is

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)t}{(2n+1)^2}$$

It can be shown by direct substitution that this satisfies the given differential equation. Alternatively, we solve the differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = \frac{1}{2} - \sum_{n=0}^{\infty}\alpha_n\cos\omega_n t,\ \omega \text{ not integer}$$

Solving the unforced system gives the complementary function as

$$y_1 = A\cos\omega t + B\sin\omega t$$

The particular integral is the sum of the PI's for the individual terms in $f(t)$. In the case of the $\frac{1}{2}$ on the RHS response is

$$y_2 = \frac{1}{2\omega^2}$$

For the term $\alpha_n\cos\omega_n t$ the PI is of the form

$$y_{\alpha_n} = C\cos\omega_n t + D\sin\omega_n t$$

Substituting in $\dfrac{d^2y}{dt^2} + \omega^2 y = \alpha_n \cos\omega_n t$ and comparing coefficients gives $C = \alpha_n/(\omega^2 - \omega_n^2),\ D = 0$, so that

$$y_{\alpha_n} = \frac{\alpha_n}{\omega^2 - \omega_n^2}\cos\omega_n t$$

Thus, the solution of the differential equation is

$$y = A\cos\omega t + B\sin\omega t + \frac{1}{2\omega^2} - \sum_{n=0}^{\infty}\frac{\alpha_n}{\omega^2 - \omega_n^2}\cos\omega_n t$$

From the given initial condition, $y = dy/dt = 0$ at $t = 0$, so that

$$B = 0 \text{ and } A = -\frac{1}{2\omega^2} + \sum_{n=0}^{\infty}\frac{\alpha_n}{\omega^2 - \omega_n^2}$$

giving on taking $\alpha_n = -4/[\pi^2(2n+1)^2],\ \omega_n = (2n+1)$

$$y = \frac{1}{2\omega^2}(1 - \cos\omega t) - \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)t - \cos\omega t}{(2n+1)^2[\omega^2 - (2n+1)^2]}$$

■ **13(a)** Since $f(t)$ is an even function, Fourier expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} -t dt + \int_{0}^{\pi} t dt\right] = \pi \\
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} -t\cos nt dt + \int_{0}^{\pi} t\cos nt dt\right] \\
&= \frac{1}{\pi}\left[\left[-\frac{t}{n}\sin nt - \frac{\cos nt}{n^2}\right]_{-\pi}^{0} + \left[\frac{t}{n}\sin nt + \frac{\cos nt}{n^2}\right]_{0}^{\pi}\right] \\
&= \frac{2}{\pi n^2}(\cos n\pi - 1) = \begin{cases} 0, & n \text{ even} \\ -\dfrac{4}{\pi n^2}, & n \text{ odd} \end{cases}
\end{aligned}$$

Thus, Fourier expansion $f(t)$ is

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)t$$

Since $b_n = 0$, Parseval's theorem gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(t)]^2 dt = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

$$\text{i.e. } \frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{1}{2}\cdot\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

$$\text{or, rearranging, } \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

13(b) Differentiating formally term by term, we obtain the Fourier expansion of the square wave at

$$g(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)t$$

Check. Since $g(t)$ is an odd function it has Fourier expansion

$$g(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

where

$$\begin{aligned} b_n &= \frac{1}{\pi}\Big[\int_{-\pi}^{0} -\sin nt dt + \int_{0}^{\pi}\sin nt dt\Big] \\ &= \frac{1}{\pi}\left[\Big[\frac{1}{n}\cos nt\Big]_{-\pi}^{0} + \Big[-\frac{1}{n}\cos nt\Big]_{0}^{\pi}\right] \\ &= \frac{2}{n\pi}[1-\cos n\pi] = \begin{cases} \dfrac{4}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \end{aligned}$$

confirming the Fourier expansion as

$$g(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)t$$

■ **14** Complex form of the Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnt}$$

where

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\frac{t}{2} e^{-jnt}dt \\ &= \frac{1}{4\pi j}\int_{-\pi}^{\pi} \left[e^{\frac{1}{2}jt} - e^{-\frac{1}{2}jt}\right] e^{-jnt}dt \\ &= \frac{1}{4\pi j}\int_{-\pi}^{\pi} \left[e^{-j(n-\frac{1}{2})t} - e^{-j(n+\frac{1}{2})t}\right] dt \\ &= \frac{1}{4\pi j}\left[\frac{e^{-j(n-\frac{1}{2})t}}{-j(n-\frac{1}{2})} - \frac{e^{-j(n+\frac{1}{2})t}}{-j(n+\frac{1}{2})}\right]_{-\pi}^{\pi} \end{aligned}$$

Using the results $e^{jn\pi} = \cos n\pi + j\sin n\pi = (-1)^n = e^{-jn\pi}$

$$e^{\frac{1}{2}j\pi} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j,\ e^{-j\frac{\pi}{2}} = -j$$

gives

$$\begin{aligned} c_n &= \frac{1}{4\pi}\left[\frac{j}{(n-\frac{1}{2})} + \frac{j}{(n+\frac{1}{2})} + \frac{j}{(n-\frac{1}{2})} + \frac{j}{(n+\frac{1}{2})}\right](-1)^n \\ &= \frac{j(-1)^n}{\pi}\left[\frac{4n}{4n^2-1}\right] \end{aligned}$$

Thus, the complex form of the Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{4nj(-1)^n}{\pi(4n^2-1)} e^{jnt}$$

■ **15(a)** Following the same procedure as in Review Exercise 11 gives

$$\begin{aligned} a_0 &= \frac{20}{\pi} \\ a_n &= \begin{cases} 0, & n \text{ odd}, \quad n \neq 1 \\ \dfrac{-20}{\pi(n^2-1)}, & n \text{ even} \end{cases} \\ a_1 &= 0 \\ b_n &= 0,\ n \neq 1 \\ b_1 &= 5 \end{aligned}$$

so that the Fourier representation is

$$v(t) = \frac{10}{\pi} + 5\sin\omega_0 t - \frac{20}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2n\omega_0 t}{4n^2-1},\ \omega_0 = \frac{2\pi}{T}$$

15(b)

$$\text{Total power } = \frac{1}{T}\int_0^{T/2} 100.\sin^2\omega_0 t dt$$
$$= \frac{50}{T}\int_0^{T/2}(1-\cos 2\omega_0 t)dt = 25$$

Thus, total average power delivered to 10Ω resistor is

$$P_{av} = \frac{25}{10} = 2.5\text{W}$$

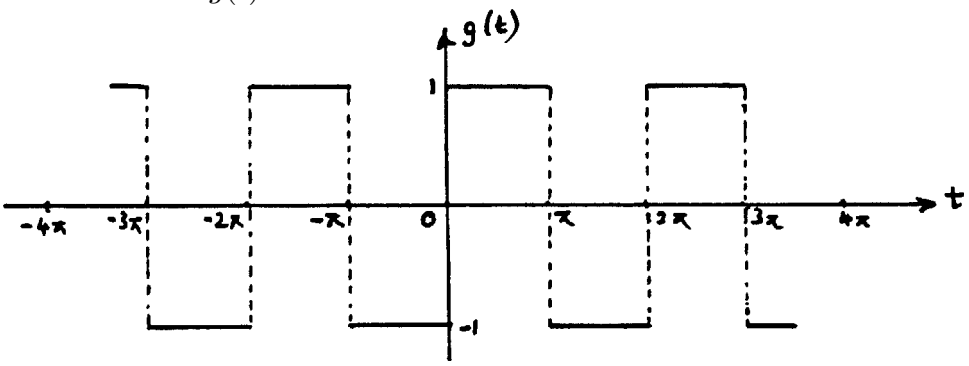
Coefficient second harmonic in series expansion $v(t)$ is $a_2 = \dfrac{-20}{3\pi}$.
When applied to 10Ω resistor power associated with this harmonic is

$$\frac{1}{2}\left(\frac{20}{3\pi}\right)^2\frac{1}{10} = \frac{20}{9\pi^2}\text{W}$$

Thus, the percentage of the total power carried by the second harmonic is

$$\frac{100}{P_{av}}\cdot\frac{20}{9\pi^2} = \frac{800}{9\pi^2} \simeq 9.01$$

■ **16(a)** A sketch of $g(t)$ is



16(b) Over the period $-\pi < t < \pi$ $g(t)$ is defined by

$$g(t) = -1, \quad -\pi < t < 0$$
$$g(t) = 1, \quad 0 < t < \pi$$

Since $g(t)$ is an odd function, it has a Fourier series expansion of the form

$$g(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

$$\text{with} \quad b_n = \frac{2}{\pi}\int_0^{\pi} 1.\sin nt dt$$
$$= \frac{2}{\pi}\Big[-\frac{1}{n}\cos nt\Big]_0^{\pi} = \frac{2}{n\pi}[1-(-1)^n] = \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{n\pi}, & n \text{ odd} \end{cases}$$

Thus, the Fourier expansion of $g(t)$ is

$$g(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)}$$

giving the Fourier expansion of $f(t) = 1 + g(t)$ as

$$f(t) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)}$$

■ **17** Complex form of Fourier expansion is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnt}$$

$$\text{where} \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jnt}dt = \frac{1}{2\pi}\int_0^{2\pi} te^{-jnt}dt$$
$$= \frac{1}{2\pi}\left[-\frac{t}{jn}e^{-jnt} + \frac{1}{n^2}e^{-jnt}\right]_0^{2\pi}$$

Using the results $e^{-j2n\pi} = \cos 2n\pi - j\sin 2n\pi = 1,\ e^o = 1$, we have

$$c_n = \frac{1}{2\pi}\Big[-\frac{2\pi}{jn}\Big] = -\frac{1}{jn} = \frac{j}{n}\ ,\ n \neq 0$$

When $n = 0, \quad c_0 = \dfrac{1}{2\pi}\int_0^{2\pi} t dt = \pi$
Hence, complex Fourier series is

$$f(t) = \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{j}{n}e^{jnt}$$

■ **18(a)** Since $v(t)$ is an odd function, its Fourier expansion is of the form

$$v(t) = \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi t}{T}$$

$$\text{with} \quad b_n = \frac{4}{T}\int_0^{T/2} 1.\sin \frac{2n\pi t}{T} dt$$

$$= \frac{4}{T}\left[-\frac{T}{2n\pi}\cos \frac{2n\pi t}{T}\right]_0^{T/2} = \frac{2}{n\pi}[1 - \cos n\pi]$$

$$\text{that is,} \quad b_n = \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{n\pi}, & n \text{ odd} \end{cases}$$

Thus, Fourier expansion is

$$v(t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin \frac{2(2n-1)\pi t}{T}$$

18(b) Response $i_\omega(t)$ of the circuit is given by

$$\frac{di_\omega(t)}{dt} + i_\omega(t) = v_\omega(t) = \sin \omega t$$

Taking Laplace transforms with $i_\omega(0) = 0$ gives

$$I_\omega(s) = \frac{\omega}{(s+1)(s^2+\omega^2)}$$

$$= \frac{\omega}{\omega^2+1}\cdot\frac{1}{(s+1)} - \frac{\omega}{\omega^2+1}\cdot\frac{s}{s^2+\omega^2} + \frac{1}{\omega^2+1}\cdot\frac{2}{s^2+\omega^2}$$

which, on taking inverse transforms, gives the response as

$$i_\omega(t) = \frac{\omega}{\omega^2+1}e^{-t} - \frac{\omega}{\omega^2+1}\cos \omega t + \frac{1}{\omega^2+1}\sin \omega t$$

Since the first term decays to zero, the steady-state response is

$$i_{\omega ss} = \frac{1}{\omega^2+1}(\sin \omega t - \omega \cos \omega t)$$

As the system is linear steady-state response $i_s(t)$ to the square wave $v(t)$ is

$$i_s(t) = \sum_{n=1}^{\infty} i_{\omega_n}(t)$$

where $i_{\omega_n}(t)$ is the steady-state response to $v_{\omega_n}(t)$ with

$$\omega_n = 2(2n-1)\pi/T$$

Thus,

$$i_s(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\big[\frac{1}{\omega_n^2+1}\big](\sin\omega_n t - \omega_n\cos\omega_n t)$$

- **19(a)**

$$\begin{aligned}\cos^n\theta &= \big[\frac{1}{2}(e^{j\theta}+e^{-j\theta})\big]^n \\ &= \frac{1}{2^n}\big[e^{nj\theta}+\binom{n}{1}e^{(n-2)j\theta}+\ldots+e^{-nj\theta}\big] \\ &= \frac{1}{2^n}\big[(e^{nj\theta}+e^{-nj\theta})+\binom{n}{1}(e^{(n-2)j\theta}+e^{-(n-2)j\theta})+\ldots\big]\end{aligned}$$

Hence,

$$\cos^{2\kappa}\theta = \frac{1}{2^{2\kappa}}\big[2\cos 2\kappa\theta+\binom{2\kappa}{1}2\cos(2\kappa-2)\theta+\ldots+\binom{2\kappa}{\kappa-1}2\cos 2\theta+\binom{2\kappa}{\kappa}\big]$$

Putting $\cos\theta = t$,

$$\begin{aligned}t^{2\kappa} &= \tfrac{1}{2^{\kappa-1}}\big[T_{2\kappa}(t)+\binom{2\kappa}{1}T_{2\kappa-2}(t)+\ldots+\binom{2\kappa}{\kappa-1}T_2(t)+\frac{1}{2}\binom{2\kappa}{\kappa}T_0(t)\big] \\ t^{2\kappa-1} &= \frac{1}{2^{\kappa}}\big[T_{2\kappa+1}(t)+\binom{2\kappa+1}{1}T_{2\kappa-1}(t)+\ldots+\binom{2\kappa+1}{\kappa}T_1(t)\big]\end{aligned}$$

Note that $T_0(t)$ may be omitted.

19(b) $\cos n\theta + \cos(n-2)\theta = 2\cos(n-1)\theta\cos\theta$ Hence, putting $\theta = \cos^{-1} t$

$$T_n(t) + T_{n-2}(t) = 2tT_{n-1}(t)$$

19(c)

$$\begin{aligned} T_0(t) &= \cos(0.\cos^{-1} t) = \cos 0 = 1 \\ T_1(t) &= \cos(1.\cos^{-1} t) = \cos(\cos^{-1} t) = t \\ T_2(t) &= 2tT_1(t) - T_0(t) = 2t^2 - 1 \\ T_3(t) &= 2t(2t^2-1) - t = 4t^3 - 3t \end{aligned}$$

19(d)

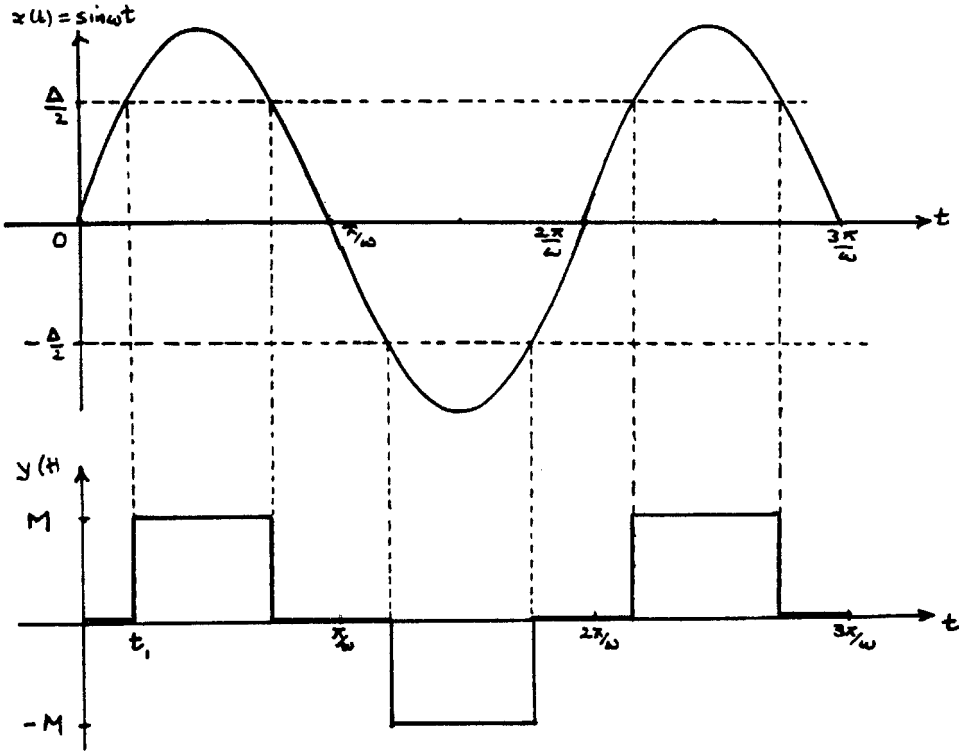
$$\begin{aligned} t^5 - 5t^4 + 7t^3 + 6t - 8 = {} & \frac{1}{2^4}(T_5(t) + 5T_3(t) + 10T_1(t)) \\ & - \frac{5}{2^3}(T_4(t) + 4T_2(t) + 3) + \frac{7}{2^2}(T_3(t) + 3T_1(t)) \\ & + 6T_1(t) - 8 \\ = {} & \frac{1}{16}T_5(t) - \frac{5}{8}T_4(t) + \frac{33}{16}T_3(t) \\ & - \frac{5}{2}T_2(t) + \frac{95}{8}T_1(t) - \frac{79}{8}T_0(t) \end{aligned}$$

19(e) The required cubic polynomial is obtained by omitting the first two terms. It is therefore,

$$\begin{aligned} & \frac{33}{16}(4t^3 - 3t) - \frac{5}{2}(2t^2 - 1) + \frac{95}{8}t - \frac{79}{8} \\ \text{or} \quad & \frac{33}{4}t^3 - 5t^2 + \frac{91}{16}t - \frac{59}{8} \end{aligned}$$

Since $\mid T_n(t) \mid \leq 1$ over $(-1, 1)$, the error can nowhere exceed $\frac{1}{16} + \frac{5}{8} = \frac{11}{16}$ in absolute value. An error of this magnitude occurs at $t = -1$, since $T_n(-1) = \cos n\pi = (-1)^n$.

■ **20**



If the input is $x(t) = X_i \sin \omega t$, then the input and output $y(t)$ waveforms to the non-linear element are shown in the figure. Clearly, the output waveform is an odd function of period π/ω and over the interval $0 < t < \pi/\omega$,

$$y(t) = \begin{cases} 0, & 0 < t < t_1 \\ M, & t_1 < t < \frac{\pi}{\omega} - t_1 \\ 0, & \frac{\pi}{\omega} - t_1 < t < \frac{\pi}{\omega} \end{cases}$$

The amplitude of the fundamental harmonic is

$$\begin{aligned} b_1 &= \frac{2}{\pi/\omega} \int_0^{\pi/\omega} y(t) \sin \omega t dt \\ &= \frac{2\omega}{\pi} \int_{t_1}^{\pi/\omega - t_1} M \sin \omega t dt \\ &= -\frac{2M}{\pi} [\cos(\pi - \omega t_1) - \cos \omega t_1] \\ &= \frac{4M}{\pi} \cos \omega t_1 \end{aligned}$$

Since $\sin \omega t_1 = \frac{\Delta}{2X_i}$, we obtain $\cos \omega t_1 = \sqrt{1 - \left(\frac{\Delta}{2X_i}\right)^2}$

Thus, the required describing function is

$$N(X_i) = \frac{4M}{\pi X_i}\left[1 - \left(\frac{\Delta}{2X_i}\right)^2\right]^{\frac{1}{2}}$$

Limit cycle will occur if $N(X_i) \geq -\dfrac{1}{KG(j\omega)}$.

$N(X_i)$ will have a maximum value when $\frac{dN}{dX_i} = 0$; that is, when $X_i = \Delta/\sqrt{2}$. Maximum value is $N(X_i)_{\max} = \frac{4M}{\pi\Delta}$. Since this is real, we are only interested in the real values of $1/(KG(j\omega))$.

In this case,

$$\begin{aligned}\frac{1}{KG(j\omega)} &= \frac{1}{K}j\omega(T_1 j\omega + 1)(T_2 j\omega + 1)\\ &= \frac{1}{K}[-T_1T_2 j\omega^3 - (T_1 + T_2)\omega^2 + j\omega]\end{aligned}$$

and for this to be real

$$-T_1T_2\omega^3 + \omega = 0 \quad \text{giving} \quad \omega^2 = 1/(T_1T_2)$$

At this frequency,

$$\text{magnitude} \quad \frac{1}{KG(j\omega)} = -\left(\frac{T_1 + T_2}{K}\right)\omega^2 = -\frac{(T_1 + T_2)}{KT_1T_2}$$

and the required result follows, namely that limit cycles will not occur if

$$\Delta > \frac{4MK}{\pi} \cdot \frac{T_1T_2}{T_1 + T_2}$$

8

The Fourier Transform

Exercises 8.2.4

■ **1**

$$\begin{aligned} F(j\omega) &= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\,\mathrm{d}t + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\,\mathrm{d}t \\ &= \frac{2a}{a^2+\omega^2} \end{aligned}$$

■ **2**

$$\begin{aligned} F(j\omega) &= \int_{-T}^{0} Ae^{-j\omega t}\,\mathrm{d}t + \int_{0}^{T} -Ae^{-j\omega t}\,\mathrm{d}t \\ &= \int_{0}^{T} 2jA\sin\omega t\,\mathrm{d}t \\ &= \frac{2jA}{\omega}(1-\cos\omega T) \\ &= \frac{4jA}{\omega}\sin^2\frac{\omega T}{2} \\ &= j\omega AT^2\,\mathrm{sinc}^2\left(\frac{\omega T}{2}\right) \end{aligned}$$

■ **3**

$$\begin{aligned} F(j\omega) &= \int_{-T}^{0} \left(\frac{At}{T}+A\right)e^{-j\omega t}\,\mathrm{d}t + \int_{0}^{T} \left(-\frac{At}{T}+A\right)e^{-j\omega t}\,\mathrm{d}t \\ &= 2\int_{0}^{T} \left(-\frac{At}{T}+A\right)\cos\omega t\,\mathrm{d}t \\ &= AT\mathrm{sinc}^2\left(\frac{\omega T}{2}\right) \end{aligned}$$

Exercise 2 is $T\times$ derivative of Exercise 3, so result 2 follows as $(j\omega \times T)$ $\times$ result 3.

Sketch is readily drawn.

■ **4**

$$F(j\omega) = \int_{-2}^{2} 2Ke^{-j\omega t}\,\mathrm{d}t = 8K\,\mathrm{sinc}(2\omega)$$
$$G(j\omega) = \int_{-1}^{1} Ke^{-j\omega t}\,\mathrm{d}t = 2K\,\mathrm{sinc}(\omega)$$
$$H(j\omega) = F(j\omega) - G(j\omega) = 2K(4\,\mathrm{sinc}(2\omega) - \mathrm{sinc}(\omega))$$

■ **5**

$$F(j\omega) = \int_{-2}^{-1} e^{-j\omega t}\,\mathrm{d}t + \int_{-1}^{1} e^{-j\omega t}\,\mathrm{d}t + \int_{1}^{2} -e^{-j\omega t}\,\mathrm{d}t$$
$$= \frac{1}{j\omega}\left[2(e^{j\omega} - e^{-j\omega}) - (e^{2j\omega} - e^{-2j\omega})\right]$$
$$= 4\,\mathrm{sinc}(\omega) - 2\,\mathrm{sinc}(2\omega)$$

■ **6**

$$F(j\omega) = \frac{1}{2j}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} (e^{jat} - e^{-jat})e^{-j\omega t}\,\mathrm{d}t$$

$$\bar{f}(a) = \frac{1}{2j}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} e^{jat}e^{-j\omega t}\,\mathrm{d}t = \frac{1}{2j}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} e^{j(a-\omega)t}\,\mathrm{d}t$$
$$= \frac{\sin\omega\frac{\pi}{a}}{j(a-\omega)}$$
$$F(j\omega) = \bar{f}(a) + \bar{f}(-a) = \frac{2j\omega}{\omega^2 - a^2}\sin\omega\frac{\pi}{a}$$

■ **7**

$$F(j\omega) = \int_0^\infty e^{-at}.\sin\omega_0 t.e^{-j\omega t}\,\mathrm{d}t$$
$$= \bar{f}(\omega_0) - \bar{f}(-\omega_0)$$
$$\text{where}\quad \bar{f}(\omega_0) = \frac{1}{2j}\int_0^\infty e^{(-a+j(\omega_0-\omega)t)}\,\mathrm{d}t$$
$$= \frac{1}{2j}\left(\frac{1}{a - j(\omega_0 - \omega)}\right) = \frac{1}{2j}\left(\frac{1}{(a+j\omega) - j\omega_0}\right)$$
$$\therefore F(j\omega) = \frac{\omega_0}{(a+j\omega)^2 + \omega_0^2}$$

■ **8**

$$F_c(x) = \frac{1}{4}\int_0^a (e^{jt} + e^{-jt})(e^{jxt} + e^{-jxt})\,\mathrm{d}t$$

$$\text{define} \quad g(x,b) = \int_0^a e^{j(b+x)t}\,\mathrm{d}t$$

$$= \frac{1}{j(b+x)}[e^{j(b+x)a} - 1]$$

$$F_c(x) = \frac{1}{4}[g(x,1) + g(x,-1) + g(-x,1) + g(-x,-1)]$$

$$= \frac{1}{2}\left[\frac{\sin(1+x)a}{1+x} + \frac{\sin(1-x)a}{1-x}\right]$$

■ **9** Consider $F(x) = \int_0^a 1.e^{jxt}\,\mathrm{d}t$

$$= \frac{-j}{x}(\cos ax + j\sin ax - 1)$$

$$F_c(x) = \mathrm{Re}\,F(x) = \frac{\sin ax}{x}$$

$$F_s(x) = \mathrm{Im}\,F(x) = \frac{1-\cos ax}{x}$$

■ **10** Consider $F(x) = \int_0^\infty e^{-at}e^{jxt}\,\mathrm{d}t$

$$= \frac{a+jx}{a^2+x^2}$$

$$F_c(x) = \mathrm{Re}\,F(x) = \frac{a}{a^2+x^2}$$

$$F_s(x) = \mathrm{Im}\,F(x) = \frac{x}{a^2+x^2}$$

Exercises 8.3.6

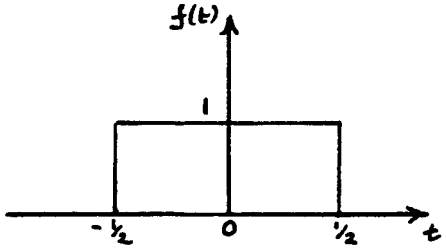
■ **11** Obvious

■ **12** $(j\omega)^2Y(j\omega) + 3j\omega Y(j\omega) + Y(j\omega) = U(j\omega)$

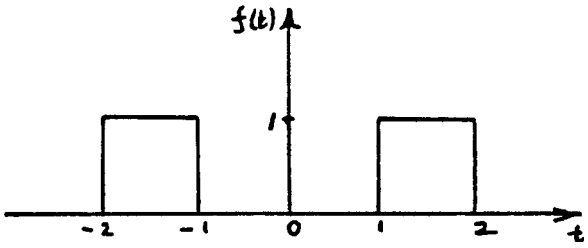
$$Y(j\omega) = \frac{1}{(1-\omega^2)+3j\omega}U(j\omega)$$

$$H(j\omega) = \frac{1}{(1-\omega^2)+3j\omega}$$

■ **13**



$$\rightarrow \operatorname{sinc}\frac{\omega}{2}$$



$$\begin{aligned}
&\rightarrow \left(e^{-i\omega 3/2} + e^{i\omega 3/2}\right)\operatorname{sinc}\frac{\omega}{2} \\
&= \frac{2}{\omega}(\sin(2\omega) - \sin(\omega)) \\
&= 4\operatorname{sinc}(2\omega) - 2\operatorname{sinc}(\omega)
\end{aligned}$$

■ **14**

$$\begin{aligned}
F(j\omega) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(\omega_0 t) e^{-i\omega t}\,\mathrm{d}t \\
&= \frac{1}{\omega_0 - \omega}\sin(\omega_0 - \omega)\frac{T}{2} + \frac{1}{\omega_0 + \omega}\sin(\omega_0 + \omega)\frac{T}{2} \qquad \omega \neq \pm\omega_0 \\
&= \frac{T}{2}\left[\frac{\sin(\omega_0 - \omega)\frac{T}{2}}{(\omega_0 - \omega)\frac{T}{2}} + \frac{\sin(\omega_0 + \omega)\frac{T}{2}}{(\omega_0 + \omega)\frac{T}{2}}\right]
\end{aligned}$$

Evaluating at $\omega = \pm\omega_0 \Rightarrow$

$$F(j\omega) = \frac{T}{2}\left[\operatorname{sinc}(\omega_0 - \omega)\frac{T}{2} + \operatorname{sinc}(\omega_0 + \omega)\frac{T}{2}\right]$$

■ **15**

$$\begin{aligned}
F(j\omega) &= \int_0^T \cos\omega_0 t.e^{-j\omega t}\,\mathrm{d}t \\
&= \frac{1}{2}[\bar{f}(\omega_0) + \bar{f}(-\omega_0)]
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad \bar{f}(\omega_0) &= \int_0^T e^{j(\omega_0 - \omega)t}\,\mathrm{d}t \\
&= \frac{1}{j(\omega_0 - \omega)}[e^{j(\omega_0 - \omega)T} - 1] \qquad \omega \neq \omega_0
\end{aligned}$$

$$\begin{aligned} F(j\omega) &= \frac{1}{2}\left[\frac{1}{j(\omega_0-\omega)}(e^{j(\omega_0-\omega)T}-1)\right. && \omega \neq \omega_0 \\ &\quad \left. -\frac{1}{j(\omega_0-\omega)}(e^{-j(\omega_0+\omega)T}-1)\right] \\ &= e^{-j\omega T/2}\left[\frac{e^{j\omega_0 T/2}}{\omega_0-\omega}\sin(\omega_0-\omega)\frac{T}{2}\right. \\ &\quad \left. +\frac{e^{-j\omega_0 T/2}}{\omega_0+\omega}\sin(\omega_0+\omega)\frac{T}{2}\right] && \omega \neq \pm\omega_0 \end{aligned}$$

Checking at $\omega = \pm\omega_0$ gives

$$F(j\omega) = \frac{T}{2}e^{-j\omega T/2}\left[e^{j\omega_0 T/2}\operatorname{sinc}(\omega_0-\omega)\frac{T}{2} + e^{-j\omega_0 T/2}\operatorname{sinc}(\omega_0+\omega)\frac{T}{2}\right]$$

- **16**

$$\begin{aligned} F(j\omega) &= \int_{-1}^{1} \sin 2t.e^{-j\omega t}\,\mathrm{d}t \\ &= \frac{1}{2j}\int_{-1}^{1} e^{-j(\omega-2)t} - e^{-j(\omega+2)t}\,\mathrm{d}t \\ \bar{f}(a) &= \int_{-1}^{1} e^{-j(\omega-a)t}\,\mathrm{d}t = 2\operatorname{sinc}(\omega-a) \\ F(j\omega) &= \frac{1}{2j}\bar{f}(a) - \frac{1}{2j}\bar{f}(-a),\ a=2 \\ &= j[\operatorname{sinc}(\omega+2) - \operatorname{sinc}(\omega-2)] \end{aligned}$$

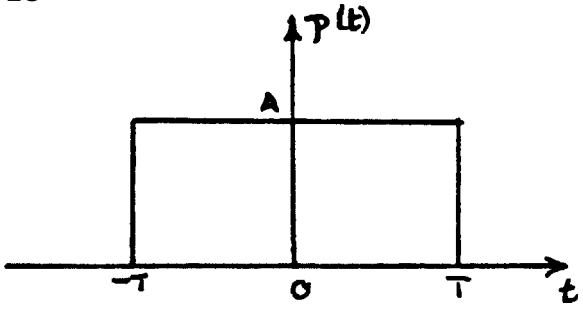
Exercises 8.4.3

- **17**

$$\begin{aligned} \text{I}\quad H(s) &= \frac{1}{s^2+3s+2} \qquad h(t) = (e^{-t}-e^{-2t})\xi(t) \\ H(j\omega) &= \int_0^\infty (e^{-t}-e^{-2t})e^{-j\omega t}\,\mathrm{d}t = \frac{1}{1+j\omega} - \frac{1}{2+j\omega} \\ &= \frac{1}{2-\omega^2+3j\omega} \text{ as required.} \\ \text{II}\quad H(s) &= \frac{s+2}{s^2+s+1} \quad h(t) = e^{-1/2t}\left(\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)\xi(t) \end{aligned}$$

$$\text{Consider} \quad G(\omega_0) = \int_0^\infty e^{-(1/2t j\omega - j\omega_0)t}\,\mathrm{d}t$$
$$= \frac{1}{\frac{1}{2} + j(\omega - \omega_0)}$$
$$H(j\omega) = \frac{1}{2}G(\omega_0) + \frac{1}{2}G(-\omega_0) + \frac{\sqrt{3}}{2j}(G(\omega_0) - G(-\omega_0)), \omega_0 = \frac{\sqrt{3}}{2}$$
$$\text{So} \quad H(j\omega) = \frac{2 + 4j\omega}{4 + 4j\omega - 4\omega^2} + \frac{6}{4 + 4j\omega - 4\omega^2}$$
$$= \frac{2 + j\omega}{1 - \omega^2 + j\omega}$$

■ **18**



$$P(j\omega) = 2AT\text{sinc}\ \omega T$$

$$\text{So} \quad F(j\omega) = (e^{-j\omega\tau} + e^{i\omega\tau})P(j\omega)$$
$$= 4AT\cos\omega\tau\text{sinc}\,\omega T$$

■ **19** $\quad G(s') = \dfrac{(s')^2}{(s')^2 + \sqrt{2}s' + 1} \qquad G(j\omega) = \dfrac{-\omega^2}{1 - \omega^2 + \sqrt{2}j\omega}$

$$= \frac{1}{\frac{1}{\omega^2} - 1 + \sqrt{2}\frac{j}{\omega}}$$

Thus, $|\,G(j\omega)\,| \to 0$ as $\omega \to 0$
and $|\,G(j\omega)\,| \to 1$ as $\omega \to \infty$
High-pass filter.

■ **20** $\quad g(t) = e^{-a|t|} \longrightarrow G(j\omega) = \dfrac{2a}{a^2 + \omega^2}$

$$f(jt) = \frac{1}{2}G(jt) \longrightarrow \pi g(-\omega) = \pi e^{-a|\omega|}$$

■ **21** $\mathcal{F}\{f(t)\cos\omega_0 t\} = \frac{1}{2}F(j(\omega-\omega_0)) + \frac{1}{2}F(j(\omega+\omega_0))$

$$\begin{aligned} F(j\omega) &= 2T\operatorname{sinc}\omega T \\ \therefore\ \mathcal{F}\{P_T(t)\cos\omega_0 t\} &\\ &= T\big[\operatorname{sinc}(\omega-\omega_0)T + \operatorname{sinc}(\omega+\omega_0)T\big] \end{aligned}$$

Exercises 8.5.3

■ **22**

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\delta(\omega+\omega_0)e^{j\omega t}d\omega \\ &\qquad = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) \\ &\qquad = \cos\omega_0 t \end{aligned}$$

■ **23**

$$\begin{aligned} \mathcal{F}\{e^{\pm j\omega_0 t}\} &= 2\pi\delta(\omega \mp \omega_0) \\ \therefore\ \mathcal{F}\{\sin\omega_0 t\} &= \frac{1}{2j}\{2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\} \\ &= j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)] \\ \frac{1}{2\pi}\int_{-\infty}^{\infty} &j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]e^{j\omega t}d\omega \\ &= \frac{j}{2}[e^{-j\omega_0 t} - e^{+j\omega_0 t}] = \sin\omega_0 t \end{aligned}$$

■ **24** $G(j\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,\mathrm{d}t;\ G(jt) = \int_{-\infty}^{\infty} g(\omega)e^{-j\omega t}d\omega$

So

$$\begin{aligned} &\int_{-\infty}^{\infty} f(t)G(jt)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty} g(\omega)e^{-j\omega t}d\omega\right)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} g(\omega)\left(\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,\mathrm{d}t\right)d\omega \\ &= \int_{-\infty}^{\infty} g(\omega)F(j\omega)d\omega = \int_{-\infty}^{\infty} g(t)F(jt)\,\mathrm{d}t \end{aligned}$$

■ **25** Write result 24 as

$$\int_{-\infty}^{\infty} f(\omega)\mathcal{F}\{g(t)\}d\omega = \int_{-\infty}^{\infty} \mathcal{F}\{f(t)\}g(\omega)d\omega$$

$$\text{so } \int_{-\infty}^{\infty} f(\omega)\mathcal{F}\{G(jt)\}d\omega = \int_{-\infty}^{\infty} \mathcal{F}\{f(t)\}G(j\omega)d\omega$$

$$\text{Now } \left.\begin{array}{ll} g(t) & \to G(j\omega) \\ G(jt) & \to 2\pi g(-\omega) \\ G(-jt) & \to 2\pi g(\omega) \end{array}\right\} \text{ symmetry}$$

$$\text{Thus, } \int_{-\infty}^{\infty} f(\omega).2\pi g(\omega)d\omega = \int_{-\infty}^{\infty} F(j\omega)G(-j\omega)d\omega$$

$$\text{or } \int_{-\infty}^{\infty} f(t)g(t)\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)G(-j\omega)d\omega$$

■ **26** $\mathcal{F}\{H(t)\sin\omega_0 t\}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi j\big[\delta(\omega-u+\omega_0)-\delta(\omega-u-\omega_0)\big]\left[\pi\delta(u)+\frac{1}{ju}\right]du$$

$$= \frac{j}{2}\big[\pi\delta(\omega+\omega_0)-\pi\delta(\omega-\omega_0)\big]+\frac{1}{2}\left[\frac{1}{\omega+\omega_0}-\frac{1}{\omega-\omega_0}\right]$$

$$= \frac{\pi j}{2}\big[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\big]-\ \frac{\omega_0}{\omega^2-\omega_0^2}$$

■ **27**

$$a_n = \frac{A}{T}\int_{-d/2}^{d/2} e^{-jn\omega_0 t}\,dt = \frac{Ad}{T}\operatorname{sinc}\frac{n\omega_0 d}{2}, \quad \omega_0 = 2\pi/T$$

$$f(t) = \frac{Ad}{T}\sum_{n=-\infty}^{\infty}\operatorname{sinc}\frac{n\omega_0 d}{2}e^{jn\omega_0 t},$$

$$F(j\omega) = \frac{2\pi Ad}{T}\sum_{n=-\infty}^{\infty}\operatorname{sinc}\frac{n\omega_0 d}{2}\delta(\omega-n\omega_0)$$

Exercises 8.6.6

■ **28**

$$T=1, \quad N=4, \quad \Delta\omega = 2\pi/(4\times 1) = \frac{\pi}{2}$$

$$G_0 = \sum_{n=0}^{3} g_n e^{-\jmath \times n \times 0 \times \pi/2} = 2$$

$$G_1 = \sum_{n=0}^{3} g_n e^{-\jmath \times n \times 1 \times \pi/2} = 0$$

$$G_2 = \sum_{n=0}^{3} g_n e^{-\jmath \times n \times 2 \times \pi/2} = 2$$

$$G_3 = \sum_{n=0}^{3} g_n e^{-\jmath \times n \times 3 \times \pi/2} = 0$$

$$G = \{2, 0, 2, 0\}$$

- **29**

$$N = 4, W^n = e^{-\jmath n\pi/2}$$

$$g'_n = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$G' = \begin{bmatrix} G_{00} \\ G_{10} \\ G_{01} \\ G_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -\jmath \\ 0 & 0 & 1 & \jmath \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Bit reversal gives

$$G = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

- **30** Computer experiment.

- **31** Follows by direct substitution.

Exercises 8.9.3

- **32** We have $\theta_c = \frac{\pi}{2}$ so

$$D(e^{j\theta}) = \begin{cases} 1, & |\theta| \leq \frac{\pi}{2} \\ 0, & |\theta| > \frac{\pi}{2} \end{cases}$$

The filter coefficients are given by

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D(e^{j\theta})e^{jn\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{jn\theta}\,d\theta \\ &= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right), n \neq 0 \\ &= \frac{1}{2}\text{sinc}\left(\frac{n\pi}{2}\right) \end{aligned}$$

Hence,

$$h_{\pm 5} = 0.06366, h_{\pm 4} = 0, h_{\pm 3} = -0.10610$$
$$h_{\pm 2} = 0, h_{\pm 1} = 0.31831, h_0 = 0.5$$

Thus, the non-causal transfer function is

$$\begin{aligned} \tilde{D}(z) = 0.06366z^{-5} - 0.1066z^{-3} + 0.31832z^{-1} + 0.5 \\ + 0.31831z - 0.10610z^3 + 0.06366z^5 \end{aligned}$$

and the causal version is

$$\begin{aligned} D(z) = 0.06366 - 0.10660z^{-2} + 0.31831z^{-4} + 0.5z^{-5} \\ + 0.31831z^{-6} - 0.010660z^{-8} + 0.06366z^{-10} \end{aligned}$$

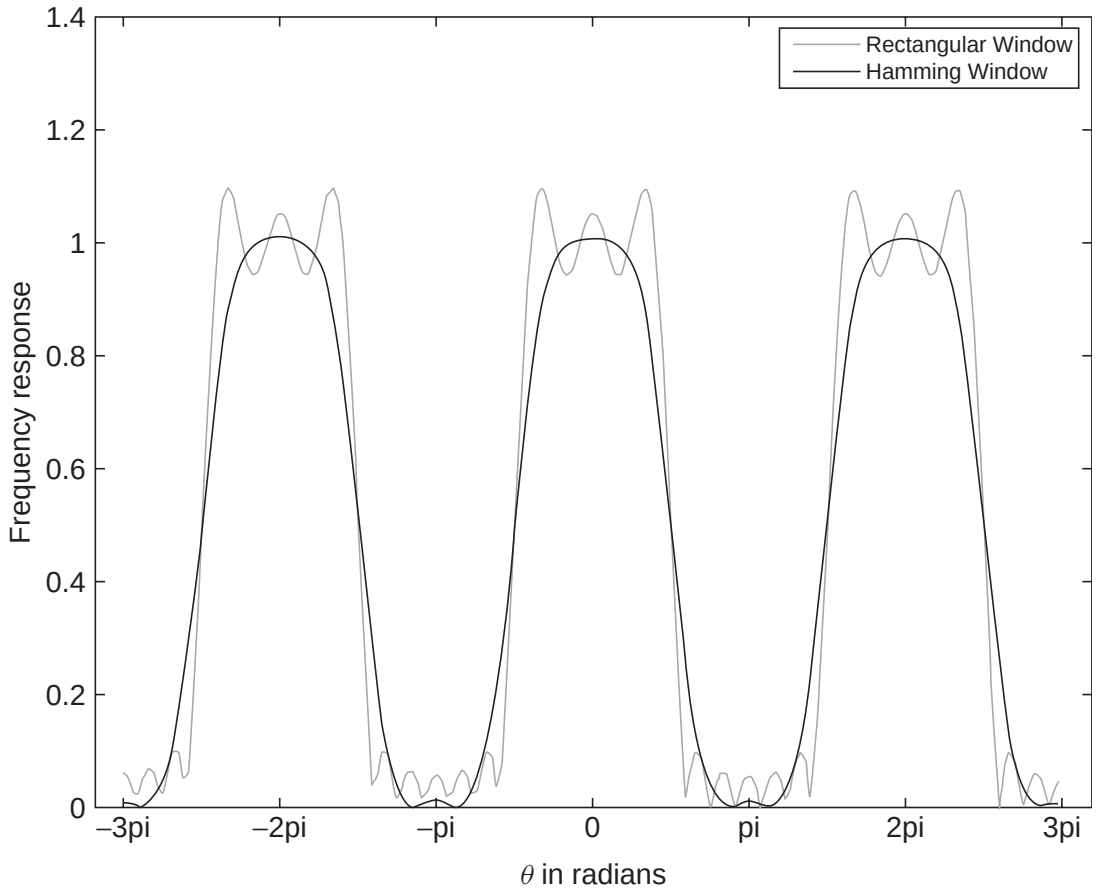
- **33** The Hamming window coefficients are given by

$$w_H(k) = 0.54 + 0.46\cos\left(\frac{\pi k}{5}\right), |k| \leq 5$$

Note that $w_H(\pm 4)$and$w_H(\pm 2)$are not needed.
Now $w_H(\pm 5) = 0.08000, w_H(\pm 3) = 0.39785, w_H(\pm 1) = 0.91215, w_H\ (0) = 1$ and the causal transfer function is found by multiplying the filter coefficients by the appropriate w_H giving

$$D(z) = 0.00509(1 + z^{-10}) - 0.04221(z^{-2} + z^{-8}) + 0.29035(z^{-4} + z^{-6}) + 0.5z^{-5}$$

Plots of the frequency responses for both Exercises 32 and 33 are given in the following figure.



Review exercises 8.10

■ **1**

$$F_S(x) = \int_0^1 t \sin xt \, \mathrm{d}t + \int_1^2 \sin xt \, \mathrm{d}t = \frac{\sin x}{x^2} - \frac{\cos 2x}{x}$$

■ **2**

$$f(t) = -\frac{\pi}{2} H(-t-2) + (H(t+2) - H(t-2)) \frac{\pi t}{4} + \frac{\pi}{2} H(t-2)$$

$$\mathcal{F}\{H(t)\} = \frac{1}{\jmath\,\omega} + \pi\delta(\omega)$$

$$\mathcal{F}\{H(t-2)\} = e^{-2\mathrm{j}\omega}\left[\frac{1}{\mathrm{j}\omega} + \pi\delta(\omega)\right]$$

$$\mathcal{F}\{H(-t-2)\} = e^{2\mathrm{j}\omega}\left[\frac{-1}{\mathrm{j}\omega} + \pi\delta(-\omega)\right] = e^{2\mathrm{j}\omega}\left[\frac{-1}{\mathrm{j}\omega} + \pi\delta(\omega)\right]$$

$$\mathcal{F}\{f(t)\} = \mathcal{F}\{-\frac{\pi}{2}H(-t-2)\} + \frac{\pi}{4}\int_{-2}^{2} te^{-\mathrm{j}\omega t}\,\mathrm{d}t + \mathcal{F}\left\{\frac{\pi}{2}H(t-2)\right\}$$

$$= \frac{-\pi\mathrm{j}}{\omega}\operatorname{sinc} 2\omega$$

■ **3**

$$\mathcal{F}\{H(t+T/2) - H(t-T/2)\} = T\operatorname{sinc}\frac{\omega T}{2}$$

$$\mathcal{F}\{\cos\omega_0 t\} = \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

Using convolution,

$$\mathcal{F}\{f(t)\} = \frac{\pi}{2\pi}\int_{-\infty}^{\infty} T\operatorname{sinc}\frac{T}{2}(\omega-u)\left(\delta(u+\omega_0) + \delta(u-\omega_0)\right)du$$

$$= \frac{T}{2}\left[\operatorname{sinc}(\omega-\omega_0)\frac{T}{2} + \operatorname{sinc}(\omega+\omega_0)\frac{T}{2}\right]$$

■ **4**

$$\mathcal{F}\{\cos\omega_0 t H(t)\} = \frac{1}{2\pi}\left[\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right] * \left[\pi\delta(\omega) + \frac{1}{\mathrm{j}\omega}\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\pi\delta(\omega-u-\omega_0) + \pi\delta(\omega-u+\omega_0)\right\}\left[\pi\delta(u) + \frac{1}{\mathrm{j}u}\right]du$$

$$= \frac{\pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right] + \frac{\mathrm{j}\omega}{\omega_0^2-\omega^2}$$

■ **5**

$$\mathcal{F}\{f(t)\cos\omega_c t\cos\omega_c t\}$$

$$= \frac{F(\mathrm{j}\omega+\mathrm{j}\omega_c) + F(\mathrm{j}\omega-\mathrm{j}\omega_c)}{2} * \pi\left[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)\right]$$

$$= \frac{1}{4}\int_{-\infty}^{\infty}\left[F(\mathrm{j}(u+\omega_c)) + F(\mathrm{j}(u-\omega_c))\right]\times$$

$$\left[\delta(\omega-u-\omega_c) + \delta(\omega-u+\omega_c)\right]du$$

$$= \frac{1}{2}F(\mathrm{j}\omega) + \frac{1}{4}\left[F(\mathrm{j}\omega+2\mathrm{j}\omega_c) + F(\mathrm{j}\omega-2\mathrm{j}\omega_c)\right]$$

Or write as

$$f(t)\frac{1}{2}(1+\cos 2\omega_c t)$$

etc.

- **6**

$$H(t+1)-H(t-1) \leftrightarrow 2\,\mathrm{sinc}\,\omega$$

By symmetry,

$$2\,\mathrm{sinc}\,t \leftrightarrow 2\pi(H(-\omega+1)-H(-\omega-1)) = 2\pi(H(\omega+1)-H(\omega-1))$$

- **7(a)** Simple poles at $s=a$ and $s=b$. Residue at $s=a$ is $e^{at}/(a-b)$, at $s=b$ it is $e^{bt}/(b-a)$, thus

$$f(t) = \frac{1}{a-b}\left(e^{at}-e^{bt}\right)H(t)$$

- **7(b)** Double pole at $s=2$, residue is

$$\lim_{s\to 2}\frac{\mathrm{d}}{\mathrm{d}s}\left((s-2)^2\frac{e^s t}{(s-2)^2}\right) = te^{2t}$$

So $f(t) = te^{2t}H(t)$

- **7(c)** Simple pole at $s=1$, residue e^{-t}, double pole at $s=0$, residue

$$\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{e^s t}{s+1}\right) = (t-1)H(t)$$

Thus, $f(t) = (t-1+e^{-t})H(t)$

- **8(a)**

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\,\mathrm{d}\tau$$

Thus,

$$-\sin\omega_0 t = \int_{-\infty}^{\infty} h(t-\tau)\cos\omega_0\tau d\tau = f(t), \text{ say}$$

If $u(\tau) = \cos\omega_0(\tau + \pi/4)$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\cos\omega_0(\tau+\pi/4)\ \mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} h(t-(\tau-\pi/4))\cos\omega_0\tau\,\mathrm{d}\tau = f(t+\pi/4)$$

$$= -\sin\omega_0(t+\pi/4)$$

- **8(b)** Since $\sin\omega_0 t = \cos\omega_0(t-\pi/2\omega_0)$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\sin\omega_0 t\ \mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} h(t-\tau)\cos\omega_0(\tau-\pi/2\omega_0)\ \mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} h(t-(\tau+\pi/2\omega_0))\cos\omega_0\tau\,\mathrm{d}\tau$$

$$= f(t-\pi/2\omega_0) = -\sin(\omega_0 t-\pi/2) = \cos\omega_0 t$$

- **8(c)**

$$e^{\mathrm{j}\omega_0 t} = \cos\omega_0 t + \mathrm{j}\sin\omega_0 t$$

This is transformed from above to

$$-\sin\omega_0 t + \mathrm{j}\cos\omega_0 t = \mathrm{j}\ e^{\mathrm{j}\omega_0 t}$$

- **8(d)** Proceed as above using

$$e^{-\mathrm{j}\omega_0 t} = \cos\omega_0 t - \mathrm{j}\sin\omega_0 t$$

- **9**

$$\mathcal{F}(\mathrm{sgn}(t)) = \mathcal{F}(f(t)) = F(\mathrm{j}\omega) = \frac{2}{\mathrm{j}\omega},\ \text{obvious}$$

Symmetry,

$$F(\mathrm{j}t) = \frac{2}{\mathrm{j}t} \leftrightarrow 2/\pi f(-\omega) = 2\pi\mathrm{sgn}(-\omega)$$

That is,

$$\frac{1}{\mathrm{j}t} \leftrightarrow -\pi\mathrm{sgn}(\omega)$$

or

$$-\left(\frac{1}{\pi t}\right) \leftrightarrow \text{jsgn}(\omega)$$

- **10**

$$g(t) = -\frac{1}{\pi t} * f(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau}\,\mathrm{d}\tau = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - t}\,\mathrm{d}\tau = F_{\text{Hi}}(t)$$

so

$$g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,\mathrm{d}t = F_{\text{Hi}}(x)$$

So from Review Exercise 9

$$F_{\text{Hi}}(\text{j}\omega) = \text{jsgn}(\omega) \times F(\text{j}\omega)$$

so

$$|F_{\text{Hi}}(\text{j}\omega)| = |\text{jsgn}(\omega)|\,|F(\text{j}\omega)| = |F(\text{j}\omega)|$$

and

$$\arg(F_{\text{Hi}}(\text{j}\omega)) = \arg(F(\text{j}\omega)) + \pi/2,\ \omega \geq 0$$

Similarly

$$\arg(F_{\text{Hi}}(\text{j}\omega)) = \arg(F(\text{j}\omega)) - \pi/2,\ \omega < 0$$

- **11** First part, elementary algebra.

$$F_{\text{Hi}}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t}{(t^2+a^2)(t-x)}\,\mathrm{d}t$$

$$= \frac{1}{\pi}\frac{1}{x^2+a^2}\int_{-\infty}^{\infty}\left[\frac{a^2}{t^2+a^2} + \frac{t}{t-x} - \frac{xt}{t^2+a^2}\right]\mathrm{d}t$$

$$= \frac{a}{x^2+a^2}$$

- **12(a)**

$$\mathcal{H}\{f(t)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,\mathrm{d}t = F_{\text{Hi}}(x)$$

$$\mathcal{H}\{f(a+t)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(a+t)}{t-x}\,\mathrm{d}t$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-(a+x)}\,\mathrm{d}t = F_{\text{Hi}}(a+x)$$

- **12(b)**

$$\mathcal{H}\{f(at)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(at)}{t-x}\,\mathrm{d}t$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-ax}\,\mathrm{d}t = F_{\mathrm{Hi}}(ax),\ a>0$$

- **12(c)**

$$\mathcal{H}\{f(-at)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(-at)}{t-x}\,\mathrm{d}t$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t+ax}\,\mathrm{d}t = -F_{\mathrm{Hi}}(-ax),\ a>0$$

- **12(d)**

$$\mathcal{H}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f'(t)}{t-x}\,\mathrm{d}t$$

$$= \frac{1}{\pi}\left[\frac{f(t)}{t-x}\bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{f(t)}{(t-x)^2}\,\mathrm{d}t\right]$$

Provided $\lim_{|t|\to\infty} f(t)/t = 0$, then

$$\mathcal{H}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{(t-x)^2}\,\mathrm{d}t = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,\mathrm{d}t$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}F_{\mathrm{Hi}}(x)$$

- **12(e)**

$$\frac{x}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,\mathrm{d}t + \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{tf(t)}{t-x}\,\mathrm{d}t$$

$$= \mathcal{H}\{tf(t)\}$$

- **13** From Review Exercise 10

$$F_{\mathrm{Hi}}(t) = -\frac{1}{\pi t} * f(t)$$

So from Review Exercise 9,

$$\mathcal{F}\{F_{\mathrm{Hi}}(t)\} = \mathrm{jsgn}\,(\omega) \times \mathcal{F}(\jmath\,\omega)$$

so

$$\mathcal{F}(\mathrm{j}\omega) = -\mathrm{j}\mathrm{sgn}\,(\omega) \times \mathcal{F}\{F_{\mathrm{Hi}}(t)\}$$

Thus,

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{\pi(t-\tau)} F_{\mathrm{Hi}}(\tau)\mathrm{d}\tau = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{(x-\tau)} F_{\mathrm{Hi}}(x)\mathrm{d}x$$

- **14**

$$f_a(t) = f(t) - \mathrm{j}F_{\mathrm{Hi}}(t)$$

$$\mathcal{F}\{f_a(t)\} = F(\mathrm{j}\omega) - \mathrm{j}(\mathrm{jsgn}\,(\omega))F(\mathrm{j}\omega) = F(\mathrm{j}\omega) + \mathrm{sgn}\,(\omega)F(\mathrm{j}\omega)$$

$$= \begin{cases} 2F(\mathrm{j}\omega), & \omega > 0 \\ 0, & \omega < 0 \end{cases}$$

- **15**

$$\mathcal{F}\{H(t)\} = \frac{1}{\mathrm{j}\omega} + \pi\delta(\omega) = F(\mathrm{j}w)$$

Symmetry,

$$F(\jmath\, t) = \frac{1}{\mathrm{j}t} + \pi\delta(t) \leftrightarrow 2\pi H(-\omega) = 2\pi[1 - H(\omega)]$$

$$= 2\pi[\mathcal{F}\delta(t) - H(\omega)]$$

or

$$H(\omega) \leftrightarrow \frac{\mathrm{j}}{2\pi t} + \frac{1}{2}\delta(t)$$

Thus,

$$\mathcal{F}^{-1}\{H(\omega)\} = \frac{\mathrm{j}}{2\pi t} + \frac{1}{2}\delta(t)$$

Then

$$\hat{f}(t) = 2\left[\frac{1}{2}\delta(t) + \frac{\mathrm{j}}{2\pi t}\right] * f(t) = f(t) - \mathrm{j}\left(-\frac{1}{\pi t}\right) * f(t)$$

$$= f(t) - \mathrm{j}F_{\mathrm{Hi}}(t)$$

When $f(t) = \cos\omega_0 t,\ \omega_0 > 0$, then

$$F(\mathrm{j}\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

so

$$\mathcal{F}\{\hat{f}(t)\} = 2\pi\delta(\omega - \omega_0)$$

whence

$$\hat{f}(t) = f(t) - \mathrm{j}F_{\mathrm{Hi}}(t) = e^{\mathrm{j}\omega_0 t} = \cos\omega_0 t + \mathrm{j}\sin\omega_0 t$$

and so

$$F_{\mathrm{Hi}}(t) = -\sin\omega_0 t$$

When $g(t) = \sin\omega_0 t,\ \omega_0 > 0$

$$G(\mathrm{j}\omega) = \mathrm{j}\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$$

and thus

$$\hat{g}(t) = -\mathrm{j}e^{\mathrm{j}\omega_0 t} = -\mathrm{j}(\cos\omega_0 t + \mathrm{j}\sin\omega_0 t)$$

so

$$\mathcal{H}\{\sin\omega_0 t\} = \cos\omega_0 t$$

■ **16** If $\bar{h}(t) = 0,\ t < 0$, then when $t < 0$

$$\bar{h}_e(t) = \frac{1}{2}\bar{h}(-t), \text{ and } \bar{h}_o(t) = -\frac{1}{2}\bar{h}(-t)$$

$$\text{that is, } \bar{h}_o(t) = -\bar{h}_e(t)$$

When $t > 0$, then

$$\bar{h}_e(t) = \frac{1}{2}\bar{h}(t), \text{ and } \bar{h}_o(t) = \frac{1}{2}\bar{h}(t)$$

$$\text{that is, } \bar{h}_o(t) = \bar{h}_e(t)$$

That is,

$$\bar{h}_o(t) = \mathrm{sgn}\,(t)\bar{h}_e(t)\ \forall t$$

Thus,

$$\bar{h}(t) = \bar{h}_e(t) + \mathrm{sgn}\,(t)\bar{h}_e(t)$$

When $h(t) = \sin t\, H(t)$,

$$\bar{h}_e(t) = \begin{cases} \frac{1}{2}\sin t, & t > 0 \\ -\frac{1}{2}\sin t, & t < 0 \end{cases}$$

and since

$$\mathrm{sgn}\,(t)\,\bar{h}_e(t) = \frac{1}{2}\sin t\ \forall t$$

the result is confirmed.

Then taking the FT of the result,

$$\bar{H}(\mathrm{j}\omega) = \bar{H}_e(\mathrm{j}\omega) + \mathcal{F}\{\mathrm{sgn}\,(t)\bar{h}_e(t)\}$$

$$= \bar{H}_e(\mathrm{j}\omega) + \frac{1}{2\pi}\left(\frac{2}{\mathrm{j}\omega} * \bar{H}_e(\mathrm{j}\omega)\right)$$

$$= \bar{H}_e(\mathrm{j}\omega) + \mathrm{j}\mathcal{H}\left\{\bar{H}_e(\mathrm{j}\omega)\right\}$$

When

$$\bar{H}(\mathrm{j}\omega) = \int_{-\infty}^{\infty} e^{-at}e^{-jwt}\,\mathrm{d}t = \frac{a}{a^2+\omega^2} - \jmath\,\frac{\omega}{a^2+\omega^2}$$

then

$$\mathcal{H}\left\{\frac{a}{a^2+\omega^2}\right\} = -\frac{\omega}{a^2+\omega^2}$$

or

$$\mathcal{H}\left\{\frac{a}{a^2+t^2}\right\} = -\frac{x}{a^2+x^2}$$

Finally,

$$\mathcal{H}\left\{\frac{at}{a^2+t^2}\right\} = -x\frac{x}{a^2+x^2} + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+t^2}\,\mathrm{d}t = \frac{a^2}{a^2+x^2}$$

So

$$\mathcal{H}\left\{\frac{t}{a^2+t^2}\right\} = \frac{a}{a^2+x^2}$$

- **17(a)**

$$F_H(s) = \int_0^{\infty} e^{-at}(\cos 2\pi st + \sin 2\pi st)\,\mathrm{d}t = \frac{a+2\pi s}{a^2+4\pi^2 s^2}$$

- **17(b)**

$$F_H(s) = \int_{-T}^{T}(\cos 2\pi st + \sin 2\pi st)\,\mathrm{d}t = \frac{1}{\pi s}\sin 2\pi st$$

- **18**

$$E(s) = \int_{-\infty}^{\infty} f(t)\cos 2\pi st\,\mathrm{d}t \quad O(s) = \int_{-\infty}^{\infty} f(t)\sin 2\pi st\,\mathrm{d}t$$

$$E(s) - \jmath\,O(s) = \int_{-\infty}^{\infty} f(t)e^{-\mathrm{j}2\pi st}\,\mathrm{d}t = F(\mathrm{j}s)$$

From Review Exercise 17(a)

$$F_H(s) = \frac{1 + \pi s}{2 + 2\pi^2 s^2}$$

whence

$$E(s) = \frac{1}{2 + 2\pi^2 s^2}, \quad O(s) = \frac{\pi s}{2 + 2\pi^2 s^2}$$

so

$$F(\jmath s) = \frac{1 - \mathrm{j}\pi s}{2 + 2\pi^2 s^2}$$

agreeing with the direct calculation,

$$F(\mathrm{j}s) = \int_0^\infty e^{-2t} e^{-\mathrm{j}2\pi st}\, \mathrm{d}t = \frac{1 - \mathrm{j}\pi s}{2 + 2\pi^2 s^2}$$

■ **19**

$$\mathsf{H}\{f(t-T)\} = \int_{-\infty}^{\infty} f(t-T)\cos 2\pi st\, \mathrm{d}t$$

$$= \int_{-\infty}^{\infty} f(\tau)\,[\cos 2\pi s\tau(\cos 2\pi sT + \sin 2\pi sT) +$$

$$\sin 2\pi s\tau(\cos 2\pi sT - \sin 2\pi sT)]\, \mathrm{d}t$$

$$= \cos 2\pi sTF_H(s) + \sin 2\pi sTF_H(-s)$$

■ **20** The Hartley transform follows at once since

$$F_H(s) = \Re\{F(\mathrm{j}s)\} - \Im\{F(\mathrm{j}s)\} = \frac{1}{2}\delta(s) + \frac{1}{s\pi}$$

From time shifting,

$$F_H(s) = \sin \pi s \left[\frac{1}{2}\delta(-s) - \frac{1}{s\pi}\right] + \cos \pi s \left[\frac{1}{2}\delta(s) + \frac{1}{s\pi}\right]$$

$$= \frac{1}{2}\delta(s) + \frac{\cos \pi s - \sin \pi s}{\pi s}$$

■ **21**

$$\mathrm{H}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\cos 2\pi st\, \mathrm{d}t = 1$$

From Review Exercise 18, it follows that the inversion integral for the Hartley transform is

$$f(t) = \int_{-\infty}^{\infty} F_H(s) \operatorname{cas} 2\pi st ds$$

and so the symmetry property is simply

$$f(t) \leftrightarrow F_H(s) \Longrightarrow F_H(t) \leftrightarrow f(s)$$

Thus,

$$\mathsf{H}\{1\} = \delta(s)$$

At once,

$$\mathsf{H}\{\delta(t-t_0)\} \int_{-\infty}^{\infty} \delta(t-t_0) \operatorname{cas} 2\pi st \, \mathrm{d}t = \operatorname{cas} 2\pi st_0$$

By symmetry,

$$\mathsf{H}\{\operatorname{cas} 2\pi s_0 t\} = \delta(s-s_0)$$

■ **22**

$$\frac{1}{2}F_H(s-s_0) + \frac{1}{2}F_H(s+s_0)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(t)\left\{\cos 2\pi(s-s_0)t + \sin 2\pi(s-s_0)t\right.$$

$$\left. + \cos 2\pi(s+s_0)t + \sin 2\pi(s+s_0)t\right\} \, \mathrm{d}t$$

$$= \int_{-\infty}^{\infty} f(t) \cos 2\pi s_0 t \left[\cos 2\pi st + \sin 2\pi st\right] \mathrm{d}t$$

$$= \mathsf{H}\{f(t)\cos 2\pi s_0 t\}$$

From Review Exercise 21, setting $f(t) = 1$

$$\mathsf{H}\{\cos 2\pi s_0 t\} = \frac{1}{2}(\delta(s-s_0) + \delta(s+s_0))$$

also

$$\mathsf{H}\{\sin 2\pi s_0 t\} = \mathsf{H}\{\operatorname{cas} 2\pi s_0 t\} - \mathsf{H}\{\cos 2\pi s_0 t\}$$

$$= \delta(s-s_0) - \frac{1}{2}(\delta(s-s_0) + \delta(s+s_0)) = \frac{1}{2}(\delta(s-s_0) - \delta(s+s_0))$$

■ **23**

$$\int_{-\infty}^{t} (1+\tau^2)^{-1}\,\mathrm{d}\tau = \tan^{-1} t + \frac{\pi}{2}$$

Thus,

$$\mathcal{F}\{\tan^{-1} t\} = \mathcal{F}\left\{\int_{-\infty}^{t} (1+\tau^2)^{-1}\,\mathrm{d}\tau\right\} - \mathcal{F}\left\{\frac{\pi}{2}\right\}$$

$$= \mathcal{F}\left\{\int_{-\infty}^{\infty} (1+\tau^2)^{-1} H(t-\tau)\mathrm{d}\tau\right\} - \mathcal{F}\left\{\frac{\pi}{2}\right\}$$

$$= \mathcal{F}\left\{\frac{1}{1+t^2} * H(t)\right\} - \mathcal{F}\left\{\frac{\pi}{2}\right\}$$

$$= \mathcal{F}\left\{\frac{1}{1+t^2}\right\} \times \left\{\frac{1}{\jmath\omega} + \pi\delta(\omega)\right\} - \frac{\pi}{2} \times 2\pi\delta(\omega)$$

But from Review Exercise 1

$$\mathcal{F}\left\{e^{-|t|}\right\} = \frac{2}{1+\omega^2}$$

and so by symmetry,

$$\mathcal{F}\left\{\frac{1}{1+t^2}\right\} = \pi e^{-|\omega|}$$

whence

$$\mathcal{F}\left\{\tan^{-1} t\right\} = \pi e^{-|\omega|} \times \left\{\frac{1}{\mathrm{j}\omega} + \pi\delta(\omega)\right\} - \frac{\pi}{2} \times 2\pi\delta(\omega)$$

and so

$$\mathcal{F}\left\{\tan^{-1} t\right\} = \frac{\pi e^{-|\omega|}}{\mathrm{j}\omega}$$

■ **24**

$$\frac{1}{2}\left[1 + \cos\omega_0 t\right] \leftrightarrow \frac{1}{2}\left[2\pi\delta(\omega) + \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right]$$

and

$$H(t+T/2) - H(t-T/2) \leftrightarrow 2T\,\mathrm{sinc}\,\omega$$

so

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} 2T\,\mathrm{sinc}\,(\omega-u)$$

$$\times \left[\pi\delta(u) + \frac{1}{2}\left(\delta(\omega-\omega_0) + \delta(u+\omega_0)\right)\right]\mathrm{d}u$$

$$= T\left[\operatorname{sinc}\omega + \frac{1}{2}\operatorname{sinc}(\omega-\omega_0) + \frac{1}{2}\operatorname{sinc}(\omega+\omega_0)\right]$$

■ **25**

$$H(\nu) = \frac{1}{4}\sum_{r=0}^{3} f(r)\operatorname{cas}\frac{2\pi\nu r}{4}$$

$$H(0) = \frac{1}{4}[f(0)+f(1)+f(2)+f(3)]$$

$$H(1) = \frac{1}{4}[f(0)+f(1)-f(2)-f(3)]$$

$$H(0) = \frac{1}{4}[f(0)-f(1)+f(2)-f(3)]$$

$$H(0) = \frac{1}{4}[f(0)-f(1)-f(2)+f(3)]$$

so

$$\mathbf{T} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

By elementary calculation, $\mathbf{T}^2 = 1/4\mathbf{T}$ and if $\mathbf{T}^{-1}$ exists, $\mathbf{T}^{-1} = 4\mathbf{T}$. Since $\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$, it does. Then

$$\mathbf{T}^{-1}\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

9

Partial Differential Equations

Exercises 9.2.6

- **1** Differentiating

$$\frac{\partial^2 u}{\partial t^2} = -a^2 \cos at \sin bx \text{ and} \frac{\partial^2 u}{\partial x^2} = -b^2 \cos at \sin bx$$

and hence $a^2 = c^2 b^2$

- **2** Since the function is a function of a single variable only, on differentiating $\frac{\partial^2 u}{\partial t^2} = \alpha^2 f''$ and $\frac{\partial^2 u}{\partial x^2} = f''$ and hence $\alpha^2 = c^2$.

- **3** Verified by differentiation.

- **4** Differentiating

$$Z_r = -\frac{1}{r^2}\cos(r-ct) - \frac{1}{r}\sin(r-ct)$$
$$Z_{rr} = \frac{2}{r^3}\cos(r-ct) + \frac{2}{r^2}\sin(r-ct) - \frac{1}{r}\cos(r-ct)$$
$$Z_{tt} = -\frac{c^2}{r}\cos(r-ct)$$

and it can be checked that the equation is satisfied.

- **5** Applying the given expression into the equation gives

$$\frac{\alpha}{\kappa} e^{\alpha t} V = e^{\alpha t} V'' \text{or } V'' = \frac{\alpha}{\kappa} V$$

and the solution clearly depends on the sign of α.

$$\alpha = 0 \Rightarrow V'' = 0 \quad \text{and hence } V = A + Bx$$
$$\alpha > 0 \Rightarrow V'' = a^2 V \quad \text{and hence } V = A\sinh ax + B\cosh ax$$

$$\text{where } a^2 = \frac{\alpha}{\kappa}$$

$$\alpha < 0 \Rightarrow V'' = -b^2 V \text{ and hence } V = A\cos bx + B\sin bx$$
$$\text{where } b^2 = -\frac{\alpha}{\kappa}$$

■ **6** Substituting the expression into the LHS of the equation,

$$\frac{\partial V}{\partial r} = nr^{n-1}(3\cos^2\theta - 1) \text{ and } \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = n(n+1)r^n(3\cos^2\theta - 1)$$

and in the RHS,

$$\frac{\partial V}{\partial \theta} = -r^n 6\cos\theta\sin\theta \text{ and } \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) = -r^n 6(-\sin^3\theta + 2\cos^2\theta\sin\theta)$$

Applying these expressions into the equation,

$$n(n+1)r^n(3\cos^2\theta - 1) - r^n 6(-\sin^2\theta + 2\cos^2\theta) = 0$$
$$\text{or } n(n+1)r^n(3\cos^2\theta - 1) - r^n 6(-1 + 3\cos^2\theta) = 0$$

and hence $n(n+1) - 6 = 0$ with roots -3 and 2.

■ **7** Now,

$$c^2\frac{\partial^2 u}{\partial x^2} = -c^2 m^2 e^{-kt}\cos mx\cos nt$$
$$\frac{\partial u}{\partial t} = -ke^{-kt}\cos mx\cos nt - ne^{-kt}\cos mx\sin nt$$

and

$$\frac{\partial^2 u}{\partial t^2} = k^2 e^{-kt}\cos mx\cos nt + 2kne^{-kt}\cos mx\sin nt - n^2 e^{-kt}\cos mx\cos nt$$

Thus,

$$\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t} = [k^2 - n^2 + 2k(-k)]e^{-kt}\cos mx\cos nt + [2kn + 2k(-n)]e^{-kt}\cos mx\sin nt$$

and comparing with the LHS gives $k^2 + n^2 = c^2 m^2$

■ **8** Differentiating

$$V_x = 3x^2 + ay^2 \text{ and } V_y = 2axy$$

and evaluating

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=3x^3+axy^2+2axy^2=3(x^3+axy^2)=3V$$

gives the required result.
Now $V_{xx}+V_{yy}=6x+2ax \Rightarrow rhs=0$ if $a=-3$
Putting $r^2=x^2+y^2$, first note that

$$2r\frac{\partial r}{\partial x}=2x \text{ and } 2r\frac{\partial r}{\partial y}=2y$$

so

$$u=r^3V \Rightarrow u_x=r^3V_x+3r^2\frac{x}{r}V=r^3V_x+3rxV$$

and differentiating again

$$u_{xx}=r^3V_{xx}+3r^2\frac{x}{r}V_x+3rxV_x+3rV+3\frac{x^2}{r}V$$

Similarly for u_{yy} and adding the two expressions and using the two previous results

$$u_{xx}+u_{yy}=r^3(V_{xx}+V_{yy})+6r(xV_x+yV_y)+6rV+3\frac{(x^2+y^2)}{r}V$$

the quoted answer is proved.

■ **9** Differentiating $\phi_{xx}=\Phi_{xx}e^{-kt/2}$ and

$$\phi_t=\Phi_t e^{-kt/2}-\frac{k}{2}\Phi e^{-kt/2} \qquad \phi_{tt}=\left(\Phi_{tt}-k\Phi_t+\frac{k^2}{4}\Phi\right)e^{-kt/2}$$

and substituting gives

$$0=\phi_{xx}-\frac{1}{c^2}(\phi_{tt}+k\phi_t)=\frac{1}{c^2}e^{-kt/2}\left(c^2\Phi_{xx}-\Phi_{tt}+k\Phi_t-\frac{k^2}{4}\Phi-k\Phi_t+\frac{k^2}{2}\Phi\right)$$

Neglecting terms in k^2, the RHS is just the wave equation for Φ.

■ **10(a)** With $r = g = 0$, the equations become

$$\begin{aligned} -I_x &= cv_t \\ -v_x &= LI_t \end{aligned} \Rightarrow -I_{xx} = cv_{xt} = c(-LI_t)_t = -cLI_{t_t}$$

and hence satisfy the wave equation.

10(b) When $L = 0$,

$$\begin{aligned} -I_x &= gv + cv_t \\ -v_x &= rI \end{aligned} \Rightarrow v_{xx} = r(gv + cv_t) = rgv + rcv_t$$

and the result is a heat conduction equation with an additional forcing term rgv. Applying $W = ve^{gt/c}$ it may be noted that

$$W_{xx} = v_{xx}e^{gt/c} \text{ and } W_t = \left(v_t + \frac{g}{c}v\right)e^{gt/c}$$

and hence comparing with the previous equation,

$$W_{xx} = (rc)W_t$$

which satisfies the usual heat conduction equation. The exponential damps the solution to zero over a long time.

10(c) First eliminate I

$$-v_{xx} = rI_x + LI_{xt} = r(-gv - cv_t) + L(-gv - cv_t)_t$$

$$v_{xx} = Lcv_{tt} + (rc + Lg)v_t + rgv$$

Apply in the expression for a

$$\frac{1}{Lc}v_{xx} = v_{tt} + 2av_t + \frac{rg}{Lc}v$$

and substitute $v = we^{-at}$

$$\frac{1}{Lc}w_{xx}e^{-at} = (w_{tt} - 2aw_t + a^2w)e^{-at} + 2a(w_t - aw)e^{-at} + \frac{rg}{Lc}we^{-at}$$

$$\frac{1}{Lc}w_{xx} = w_{tt} + \left(-a^2 + \frac{rg}{Lc}\right)w$$

But

$$\left(-a^2+\frac{rg}{Lc}\right)=\frac{rg}{Lc}-\frac{1}{4}\left(\frac{r^2}{L^2}+2\frac{rg}{Lc}+\frac{g^2}{c^2}\right)=-\frac{1}{4}\left(\frac{r^2}{L^2}-2\frac{rg}{Lc}+\frac{g^2}{c^2}\right)=0$$

from the condition $rc=gL$ and hence the variable w satisfies the wave equation. Such a transmission line is called a balanced line and transmits the signal exactly in shape, though damped by the exponential.

■ **11** Applying the expression into the equation

$$-a^2 f\sin(ay+b)=(f''-2af')\sin(ay+b)$$

so f must satisfy

$$f''-2af'+a^2 f=0$$

which is a second-order constant coefficient equation with equal roots a. Thus, $f=(A+Bx)e^{ax}$ and agrees with the given result.

■ **12** The given formula can be checked by differentiation.
The method of Section 9.2.5 solves the equations

$$\frac{dx}{x}=\frac{dy}{y}=\frac{df}{4x^2y^2}$$

$$\frac{dx}{x}=\frac{dy}{y}\overset{yields}{\longrightarrow}x=Ay$$

$$\frac{dy}{y}=\frac{df}{4x^2y^2}\overset{yields}{\longrightarrow}df=4A^2y^3dy\overset{yields}{\longrightarrow}f=A^2y^4+B$$

The arbitrary constants A and B can be isolated as

$$A=\frac{x}{y}\text{ and }B=f-x^2y^2$$

When $x=x(t),y=y(t)$ are given on a curve with $f=f(t)$ then $A(t)=\dfrac{x}{y}$ and hence $t=F\left(\dfrac{x}{y}\right)$ for some function F. Putting this into $B(t)$ gives

$B\left(F\left(\dfrac{x}{y}\right)\right) = g\left(\dfrac{x}{y}\right)$ for some function g and thus f is of the required form

$$f(x,y) = x^2y^2 + g\left(\frac{x}{y}\right)$$

The MAPLE code produces this solution also.
Given that $x = 1-t, y = t, f = t^2$

$$\frac{x}{y} = \frac{1-t}{t} \quad and \quad t^2 = (1-t)^2t^2 + g\left(\frac{x}{y}\right)$$

Eliminating t gives $g\left(\dfrac{x}{y}\right) = \dfrac{y^3(y+2x)}{(x+y)^4}$.
Use MAPLE to solve, as follows:

```
with (PDEtools):
Q12:=x*diff(u(x,y),x)+y*diff(u(x,y),y)-4*x^2*y^2;
sol:=pdsolve(Q12,u(x,y));

    # this instruction gives the solution
     sol:= u(x,y) = x^2 y^2+_F1(y/x)

eval (sol,{x=1-t,y=t,u(x,y)=t^2});
simplify (eval(%,t=1/(1+z)));
solve(%,_F1(1/z));

# gives the solution (1+2z)/(1+z)^4
```

■ **13** Write as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y} + u\right) = 0 \Rightarrow \dfrac{\partial u}{\partial y} + u = f(y)$
where f is an arbitrary function. Using an integrating factor e^y, this partial differential equation can now be written as

$$\frac{\partial}{\partial y}(ue^y) = e^y f(y)$$

which can be integrated to give

$$u = e^{-y}[H(x) + G(y)]$$

where $H(x)$ and $G(y)$ are arbitrary functions.

■ **14** The method of Section 9.2.5 solves the equations

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u}$$

These equations yield

$$\frac{dx}{x^2} = \frac{dy}{y^2} \overset{yields}{\longrightarrow} \frac{1}{x} = \frac{1}{y} - A$$

$$\frac{du}{u} = \left(\frac{2}{y} + \frac{A}{1-Ay}\right) dy \overset{yields}{\longrightarrow} u = \frac{By^2}{1-Ay} = Bxy$$

Hence from any starting curve, with parameter s,

$$A(s) = \frac{x-y}{xy} \text{ and } B(s) = \frac{u}{xy}$$

Eliminating s gives $u = xyF\left(\dfrac{x-y}{xy}\right)$, where F is an arbitrary function determined by the conditions on the starting curve.

MAPLE gives this general solution.

Putting in the data $x = s, y = 1, f = s^2$, the arbitrary function becomes $F(z) = \dfrac{1}{1-z}$ and the given result for u follows.

Exercises 9.3.4

■ **15** From the separated solutions (9.25) choose

$$u = \sin(\lambda x)\cos(\lambda ct)$$

Clearly, both initial conditions (a) and (b) are satisfied for $\lambda = 1$.

The d'Alembert solution is obtained from equation (9.19) as

$$u = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)]$$

which gives the same result when the sines are expanded.

■ **16** First note that $\sin x(1+\cos x) = \sin x + \frac{1}{2}\sin 2x$

The two initial conditions imply that the solution is of the form

$$u = A\sin x \sin ct + B\sin 2x \sin 2ct$$

and matching the conditions gives $A = 1/c$ and $B = 1/4c$.

■ **17** The MAPLE implementation is as follows:

```
f:=(x-c*t)/(1+(x-c*t)^2)+(x+c*t)/(1-(x+c*t)^2);

simplify (f);   # gives the simplification - nearly
simplify(diff(f,x,x)-diff(f,t,t)/c^2);   # gives zero as required
```

■ **18** Let $'$ denote differentiation with respect to $(ct-r)$ and 'dot' with respect to $(ct+r)$; then the terms of the spherically symmetric wave equation are

$$\frac{1}{c^2}u_{tt} = \frac{1}{r}[f''(ct-r) + \ddot{g}(ct+r)]$$

and

$$\begin{aligned} u_r &= -\frac{1}{r^2}[f(ct-r)+g(ct+r)] + \frac{1}{r}[-f'(ct-r)+\dot{g}(ct+r)] \\ u_{rr} &= \frac{2}{r^3}[f(ct-r)+g(ct+r)] - \frac{2}{r^2}[-f'(ct-r)+\dot{g}(ct+r)] \\ &\quad + \frac{1}{r}[f''(ct-r)+\ddot{g}(ct+r)] \end{aligned}$$

Collecting terms together

$$\begin{aligned} \frac{1}{c^2}u_{tt} - u_{rr} - \frac{2}{r}u_r =& \frac{1}{r^3}\left[r^2(f''+\ddot{g}) - 2(f+g) - 2r(f'-\dot{g})\right. \\ & \left. -r^2(f''+\ddot{g}) + 2(f+g) + 2r(f'-\dot{g})\right] = 0 \end{aligned}$$

so the equation is satisfied for *any* functions f and g. The two terms represent an outward spherical wave emanating from the origin and an inward wave converging into the origin. Note the singular behaviour at $r = 0$.

■ **19** The equation (9.28) is split by the trigonometric formula into two parts

$$\begin{aligned} \frac{2\pi^2 u}{(4l)} =& \sin\frac{\pi}{l}(x-ct) + \sin\frac{\pi}{l}(x+ct) - \frac{1}{9}\sin\frac{3\pi}{l}(x-ct) - \frac{1}{9}\sin\frac{3\pi}{l}(x+ct)\ldots \\ =& \left[\sin\frac{\pi}{l}(x-ct) - \frac{1}{9}\sin\frac{3\pi}{l}(x-ct) + \frac{1}{25}\sin\frac{5\pi}{l}(x-ct) + \ldots\right] \\ &+ \left[\sin\frac{\pi}{l}(x+ct) - \frac{1}{9}\sin\frac{3\pi}{l}(x+ct) + \frac{1}{25}\sin\frac{5\pi}{l}(x+ct) + \ldots\right] \end{aligned}$$

The two terms depend on $(x-ct)$ and $(x+ct)$ respectively and represent travelling waves in the $+x$ and $-x$ directions.

■ **20** The d'Alembert solution is obtained from equation (9.19) as

$$u = \frac{1}{2c}\int_{x-ct}^{x+ct} x\exp(-x^2)dx$$

which on integration gives the quoted result.

■ **21** Again the d'Alembert solution is obtained from equation (9.19) as

$$u = [F(x-ct) + F(x+ct)]/2$$

where F is the function given in the exercise.

■ **22** Try a solution of the form $u = f(x+ky)$, so the equation gives

$$3f'' + 6kf'' + k^2 f'' = 0 \Rightarrow 3 + 6k + k^2 = 0$$

which has solutions $k = -3 \pm \sqrt{6}$ and hence the characteristics are

$$x + (-3+\sqrt{6})y = \text{const}$$
$$x + (-3-\sqrt{6})y = \text{const}$$

■ **23** Substituting

$$6f'' - \lambda f'' - \lambda^2 f'' = 0 \Rightarrow \lambda = 2, -3$$

and hence a solution of the form

$$u = f(x+2t) + g(x-3t)$$

The initial conditions give

$$x^2 - 1 = f(x) + g(x) \text{ and } 2x = 2f'(x) - 3g'(x)$$

Integrating and solving for f and g produces the solution

$$u = \frac{1}{5}[4(x+2t)^2 + (x-3t)^2 - 5]$$

■ **24** Differentiating

$$\frac{\partial u}{\partial r} = -\frac{g}{r^2}\cos\omega t + \frac{g'}{r}\cos\omega t$$

and

$$\frac{\partial^2 u}{\partial r^2} = \left(\frac{2g}{r^3} - \frac{2g'}{r^2} + \frac{g''}{r}\right)\cos\omega t$$

Applying these expressions into the equation gives

$$\left(\frac{2g}{r^3} - \frac{2g'}{r^2} + \frac{g''}{r} + \frac{2}{r}\left[-\frac{g}{r^2} + \frac{g'}{r}\right]\right)\cos\omega t = -\frac{\omega^2}{c^2}\frac{g}{r}\cos\omega t$$

and cancelling produces the equation for g as

$$g'' + \frac{\omega^2}{c^2}g = 0$$

This simple harmonic equation has sine and cosine solutions which are written in the form

$$g = A\cos\frac{\omega}{c}(b-r) + B\sin\frac{\omega}{c}(b-r)$$

The second boundary condition is now satisfied by applying $A = 0$ and the first condition gives

$$u(a,t) = \beta\cos\omega t = \frac{B}{a}\sin\frac{\omega}{c}(b-a)\cos\omega t$$

and hence B is known and the required solution is

$$u(r,t) = \frac{a\beta\cos\omega t}{r}\frac{\sin\frac{\omega}{c}(b-r)}{\sin\frac{\omega}{c}(b-a)}$$

■ **25** This question is similar to Example 9.11 but initially the velocity is given and the displacement is zero.

On the initial line $t = 0$ the solution

$$u = f(x+t) + g(x-t)$$

satisfies condition (a) only if $f = -g$ and therefore the condition (b) gives

$$u_t(x,0) = 2f'(x) = \exp(-|x|)$$

Integrate to obtain f

$$f = \begin{cases} \frac{1}{2}e^{x} & \text{for } x < 0 \\ 1 - \frac{1}{2}e^{-x} & \text{for } x > 0 \end{cases}$$

where it has been arranged that the function goes to zero at infinity and matches at $x = 0$.

The numerical solution can now be computed from the values on the initial line given by f. The values at subsequent times $t = 0.5, 1.0, 1.5, 2.0, \ldots$ can be computed easily from

$$u(x, 0.5) = f(x + 0.5) - f(x - 0.5)$$

$$u(x, 1) = f(x + 1) - f(x - 1)$$

$$u(x, 1.5) = f(x + 1.5) - f(x - 1.5)$$

etc.

to give the quoted solution. On a spreadsheet, applying $f(x)$ into column B corresponding to values of $x = -3, -2.5, \ldots, 2.5, 3$, then a typical entry, which can be copied onto the other entries in the column,

in column D, D7 reads $+B8 - B6$
in column E, E7 reads $+B9 - B5$
in column F, F7 reads $+B10 - B4$
etc.

It is instructive to derive the exact solution and then compare with the numerical solution.

$$u(x,t) = \begin{cases} e^{x} \sinh\ t & \text{for } x < -t \\ 1 - e^{-t} \cosh\ x & \text{for } -t < x < t \\ e^{-x} \sinh\ t & \text{for } x > t \end{cases}$$

■ **26** From the possible separated solutions, the conditions (a) and (b) imply that

$$u = \cos \lambda ct \sin \lambda x$$

is the only one that satisfies these conditions. The condition (c) gives $\sin \lambda\pi = 0 \Rightarrow \lambda = N$ which is an integer. Thus, a superposition of these solutions gives

$$u(x,t) = \sum_{N=1}^{\infty} a_N \cos Nct \sin Nx$$

and the condition (d) gives the standard Fourier problem of evaluating the coefficients in

$$\pi x - x^2 = \sum_{N=1}^{\infty} a_N \sin Nx$$

The coefficients are obtained from the usual integral and the result follows by two integrations by parts

$$a_N = \frac{4}{\pi N^3}(1 - \cos N\pi)$$

■ **27** Taking the Laplace transform with respect to t, in equation (9.33) both $u(x,0)$ and $u_t(x,0)$ are zero from conditions (a) and (b); so the equation is

$$c^2 \frac{d^2U}{dx^2} = s^2U \text{ with solution } U = Ae^{sxlc} + Be^{-sxlc}$$

From condition (d), the constant $A = 0$ since the solution must be bounded for all $x > 0$. The condition (c) is transformed to

$$U(0,s) = \frac{a\omega}{s^2 + \omega^2}$$

and hence the solution for U takes the form

$$U(x,s) = \frac{a\omega}{s^2 + \omega^2} e^{-sxlc}$$

and the exponential just shifts the solution as

$$u(x,t) = a \sin\left[\omega\left(t - \frac{x}{c}\right)\right] H\left[\omega\left(t - \frac{x}{c}\right)\right]$$

It is easily checked that all the conditions are satisfied by the function. For $x > ct$ the wave has not reached this value of x, so $u = 0$ beyond this point.

Exercises 9.3.6

■ **28** The problem is best solved by using MATLAB.
Explicit

```
n=5;L=0.25;x=[0:1/(n-1):1];z=zeros(1,5);
zz=.25*[0 .25 .5 .25 0]
```

```
zzz=[0,2*zz([2:n-1])-z([2:n-1])+L*(zz([1:n-2])
    -2*zz([2:n-1])+zz([3:n])),0]
      % gives  0  0.1250 0.2188  0.1250  0
z=zz;zz=zzz;
zzz=[0,2*zz([2:n-1])-z([2:n-1])+L*(zz([1:n-2])
    -2*zz([2:n-1])+zz([3:n])),0]
     % gives  0  0.1797  0.2656  0.1797  0
```

Implicit

```
n=5;L=0.25;
a=[-L 2*(1+L)-L];A=eye(n);for i=2:n-1,A(i,i-1:i+1)=a;end
b=[L/2 1-L L/2];C=eye(n);for i=2:n-1,C(i,i-1:i+1)=b;end
u=[0 0 0 0 0]';v=C*u+.25*[0 .25 .5 .25 0]';
B=inv(A);
w=4*B*v-u;w'
   % gives  0  0.1224  0.2245  0.1224  0
u=v;v=w;w=4*B*v-u;w'
   % gives  0  0.1741  0.2815  0.1741  0
```

■ **29** Again MATLAB is a convenient method for the explicit calculation.

```
n=6;L=0.01;delt=0.02;
format long
z=eye(1,6);zz=[sin(delt/2*pi),0 0 0 0 0]
% gives  0.0314107  0  0  0  0  0
zzz=[sin(2*delt/2*pi),2*zz([2:n-1])-z([2:n-1])
    +L*(zz([1:n-2])-2*zz([2:n-1])+zz([3:n])),0]
    % gives  0.062790  0.000314 0  0  0  0
z=zz;zz=zzz;
zzz=[sin(3*delt/2*pi),2*zz([2:n-1])-z([2:n-1])
    +L*(zz([1:n-2])-2*zz([2:n-1])+zz([3:n])),0]
    % gives 0.094108 0.001249  0.000003  0  0  0
```

■ **30** Care must be taken to include the '+2' term but the MATLAB implementation is quite straightforward.

Explicit

```
n=6;L=0.25;delt=0.2;x=[0:0.2:1];z=x.*(1-x)
zz=[0,(1-L)*z([2:n-1])+L*(z([1:n-2])
  +z([3:n]))/2,0]+[0,delt^2*ones(1,4),0]
  % gives 0 0.1900  0.2700  0.2700  0.1900   0
zzz=[0,2*zz([2:n-1])-z([2:n-1])+L*(zz([1:n-2])-2*zz([2:n-1])
   +zz([3:n])),0] +[0,2*delt^2*ones(1,4),0]
% gives 0 0.2725  0.3600  0.3600  0.2725   0
z=zz;zz=zzz;
zzz=[0,2*zz([2:n-1])-z([2:n-1])+L*(zz([l:n-2])-2*zz([2:n-1])
   +zz([3:n])),0]+[0,2*delt^2*ones(1,4),0]
% gives 0  0.3888  0.5081  0.5081  0.3888  0
```

Implicit

```
n=6;L=0.25;delt=0.2;x=[0:0.2:1];
a=[-L2*(1+L)-L];A=eye(n);for i=2:n-1,A(i,i-1:i+1)=a;end
b=[L/2 1-L L/2];C=eye(n);for i=2:n-1,C(i,i-1:i+1)=b;end
u=(x.*(1-x))';v=C*u+[0;delt^2*ones(4,1);0]
% gives      0  0.1900  0.2700   0.2700 0.1900  0
B=inv(A);
w=B*(4*v+[0;2*delt^2*ones(4,1);0])-u
%gives       0   0.2319  0.3191   0.3191 0.2319  0
u=v;v=w;w=B*(4*v+[0;2*delt^2*ones(4,1);0])-u
%gives       0   0.2785  0.3849  0.3849  0.2785  0
```

■ **31** The problem is now more difficult since there is an infinite region. The simplest way to cope with this difficulty for small times is to impose boundaries some distance from the region of interest. Hopefully, the effect of any sensible boundary condition would only affect the solution marginally. For longer times, an alternative strategy must be sought. In the current problem, the region $x = -1$ to 2 is chosen with the solution quoted in the region $x = 0$ to 1.

Explicit

```
n=16;L=0.25;delt=0.2;
x=[-1:0.2:2];z=x.*(1-x);
```

```
zz=[-2,(1-L)*z([2:n-1])+L*(z([l:n-2])
  +z([3:n]))/2,-2]+delt^2*ones(1,16)
% gives 0.0300  0.1900  0.2700  0.2700  0.1900  0.0300
zzz=[-2,2*zz([2:n-1])-z([2:n-1])+L*(zz([1:n-2])-2*zz([2:n-1])
  +zz([3:n])),-2]+2*delt^2*ones(1,16)
% gives 0.1200  0.2800  0.3600  0.3600  0.2800  0.1200  -0.1200
z=zz;zz=zzz;
zzz=[-2,2*zz([2:n-1])-z([2:n-1])+L*(zz([1:n-2])-2*zz([2:n-1])
  +zz([3:n])),-2]+2*delt^2*ones(1,16)
% gives 0.2700  0.4300  0.5100 0.5100  0.4300  0.2700
```

Implicit

```
n=16;L=0.25;delt=0.2;
x=[-1:0.2:2];
a=[-L 2*(1+L)-L];A=eye(n);for i=2:n-1,A(i,i-1:i+1)=a;end
b=[L/2 1-L L/2];C=eye(n);for i=2:n-1,C(i,i-1:i+1)=b;end
u=(x.*(1-x))';v=C*u+delt^2*ones(16,1)
%  gives 0.0300  0.1900  0.2700  0.2700  0.1900  0.0300
B=inv(A);
w=B*(4*v+2*delt^2*ones(16,1))-u
%   gives   0.0800  0.2400  0.3200  0.3200  0.2400  0.0800
u=v;v=w;w=B*(4*v+2*delt^2*ones(16,1))-u
%   gives   0.1495  0.3099  0.3900  0.3900  0.3099  0.1495
```

Exercises 9.4.3

■ **32** From the set of separated solutions in equation (9.40), the only ones that satisfy condition (a) are $u = e^{-\alpha t}\cos\lambda x$ and the second condition (b) implies $\cos\lambda = 0 \Rightarrow \lambda = \left(n + \frac{1}{2}\right)\pi$ where n is an integer.
The third condition (c) can be rewritten as

$$u = \frac{a}{2}\left[\cos\left(\frac{3\pi x}{2}\right) + \cos\frac{\pi x}{2}\right] \text{ for } 0 \le x \le 1 \text{ when } t = 0$$

Thus, the complete solution is

$$u = \frac{a}{2}[\exp(-\kappa\pi^2 t/4)\cos(\pi x/2) + \exp(-9\kappa\pi^2 t/4)\cos(3\pi x/2)]$$

■ **33** If $v = ru$, then differentiating produces

$$v_r = u + ru_r$$

$$v_{rr} = 2u_r + ru_{rr}$$

and hence $u_{rr} + 2\frac{u_r}{r} = \frac{1}{r}v_{rr}$.
Applying these expressions into the spherically symmetric heat conduction equation gives

$$\frac{1}{r}v_{rr} = \frac{1}{\kappa r}v_t$$

Cancelling out the r, it is seen that v satisfies the standard heat conduction equation. If v remains bounded it may be noted that $u \to 0$ as $r \to \infty$.
Since $u = v/r$, the boundary conditions for v are

$$u(a,t) = T_0 \overset{yields}{\longrightarrow} v(a,t) = aT_0 \text{ for } t > 0$$
$$u(b,t) = 0 \overset{yields}{\longrightarrow} v(b,t) = 0 \text{ for } t > 0$$
$$u(r,0) = 0 \overset{yields}{\longrightarrow} v(r,0) = 0 \text{ for } a < r < b$$

The first two of these conditions are satisfied by the given expression

$$v(r,t) = aT_0\left[\frac{b-r}{b-a} - \sum_{N=1}^{\infty} A_N e^{-\kappa\lambda^2 t}\sin\left(\frac{r-a}{b-a}\right)N\pi\right]$$

and the third gives the Fourier problem

$$\frac{b-r}{b-a} = \sum_{N=1}^{\infty} A_N \sin\left(\frac{r-a}{b-a}\right)N\pi$$

The coefficients can be obtained from integration or from standard tables of Fourier series as $A_N = 2/\pi N$.

■ **34** Substituting into the partial differential equation gives the ODE

$$4\eta F'' + (2+\eta)F' - \alpha F = 0$$

and applying $F = \exp(\kappa\eta)$ gives the equation

$$4\eta\kappa^2 + (2+\eta)\kappa - \alpha = 0$$

which is clearly satisfied by $\kappa = -\frac{1}{4}$ and $\alpha = -\frac{1}{2}$ and produces the classic similarity solution.

■ **35** Differentiating

$$\frac{\partial u}{\partial t} = -\beta f(x)\cos(x-\beta t) \text{ and } \frac{\partial u}{\partial x} = f'(x)\sin(x-\beta t) + f(x)\cos(x-\beta t)$$

and

$$\frac{\partial^2 u}{\partial x^2} = f''(x)\sin(x-\beta t) + 2f'(x)\cos(x-\beta t) - f(x)\sin(x-\beta t)$$

Putting these expressions into the heat conduction equation and equating the sine and the cosine terms gives

$$-\beta f = 2f' \text{ and } f'' - f = 0$$

Both equations can be satisfied only if $\beta = 2$ and $f = Ae^{-x}$; the solution is then

$$u = Ae^{-x}\sin(x-2t)$$

Physically, the slab of material is given an initial temperature of $Ae^{-x}\sin x$, the temperature is zero at infinity and at the end $x = 0$ the temperature is periodic taking the form $u(0,t) = -A\sin 2t$ and hence $A = -u_0$.

■ **36** The suggested substitution gives, on differentiation,

$$\theta - \theta_0 = ue^{-ht}$$

$$\theta_t = (u_t - hu)e^{-ht}$$

$$\theta_{xx} = u_{xx}e^{-ht}$$

Putting the expressions into the given equation and cancelling the exponential gives

$$u_t - hu = \kappa u_{xx} - hu \Rightarrow u_t = \kappa u_{xx}$$

and produces the standard equation for u. The term $h(\theta - \theta_0)$ is a heat loss term proportional to the excess temperature over an ambient temperature θ_0; this is the usual Newton cooling through a surface.

■ **37** First it is clear that the final steady solution is $U = 0$. The general separated solution in equation (9.40) is

$$u = e^{-\alpha t}(A \sin \lambda x + B \cos \lambda x) \quad \text{where} \quad \lambda^2 = \alpha/\kappa$$

Condition (a) can only be satisfied at $x = 0$ if $A = 0$. Condition (b) then implies that

$$\cos \lambda l = 0 \text{ so that } \lambda l = \left(n + \frac{1}{2}\right)\pi$$

and hence the solution takes the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n \exp\left[-\kappa\left(n + \frac{1}{2}\right)^2 \frac{\pi^2 t}{l^2}\right] \cos\left[\left(n + \frac{1}{2}\right)\frac{\pi x}{l}\right]$$

The initial condition given in (c) leads to the Fourier problem of evaluating the coefficients in the expression

$$u_0\left(\frac{1}{2} - \frac{x}{l}\right) = \sum_{n=0}^{\infty} a_n \cos\left[\left(n + \frac{1}{2}\right)\frac{\pi x}{l}\right]$$

These can be evaluated by standard integration or using the standard series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos\left[\left(n + \frac{1}{2}\right)\frac{\pi x}{l}\right] = \frac{\pi}{4} \text{ for } -l < x < l$$

and

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\left[\left(n + \frac{1}{2}\right)\frac{\pi x}{l}\right] = \frac{\pi^2}{8}\left(1 - \frac{x}{l}\right) \text{ for } 0 < x < 2l$$

Thus, the coefficients can be calculated as a combination of the last two expressions as

$$a_n = u_0\left[\frac{8}{\pi^2(2n+1)^2} - \frac{2}{\pi}\frac{(-1)^n}{2n+1}\right]$$

■ **38** At any time t, the sine term ensures that the sum is zero at $x = 0$ and L so only the first term survives at the end points, and therefore $v = v_0$ at $x = 0$ and $v = 0$ at $x = L$. From the basic solutions obtained in the text, or by inspection, it is clear that the heat equation is also satisfied. The additional condition at $t = 0$ leads to the Fourier series problem

$$0 = v_0 \left(1 - \frac{x}{L}\right) + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

with the coefficients evaluated from

$$0 = v_0 \int_0^L \left(1 - \frac{x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx + \frac{L}{2} c_n$$

An integration by parts gives $c_n = -\frac{2v_0}{n\pi}$ as required.

■ **39** At the ends of the bar the conditions are
(a) $u = 0$ at $x = 0$ for $t > 0$, (b) $u = 0$ at $x = l$ for $t > 0$ and the initial condition is (c) $u = 10$ for $0 < x < l$ at $t = 0$. From the set of separated solutions in equation (9.40), the only ones that satisfy condition (a) are $u = e^{-\alpha t} \sin \lambda x$. The condition (b) then gives $\sin \lambda l = 0 \Rightarrow \lambda = \frac{nx}{l}$ where n is an integer. The solution is therefore of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(\frac{-\kappa n^2 \pi^2 t}{l^2}\right) \sin\left(\frac{n\pi x}{l}\right)$$

The third condition (c) reduces the problem to a Fourier series, namely

$$10 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

Integrating in the usual way over the interval

$$\int_0^l 10 \sin\left(\frac{n\pi x}{l}\right) dx = \frac{l}{2} a_n$$

gives $a_n = \frac{20}{n\pi}(1 - \cos n\pi)$ and the required result.

■ **40** Taking Laplace transforms of the equation with respect to t leads to

$$s\bar{\phi} - \phi(x,0) = a\bar{\phi}'' + \frac{b}{s}$$

so using condition (b) the required equation is

$$\bar{\phi}'' = \frac{s}{a}\bar{\phi} - \frac{b}{as}$$

This equation has the obvious particular integral $\bar{\phi} = b/s^2$ and it is convenient to write the complementary function in terms of sinh and cosh functions. The solution is thus

$$\bar{\phi} = \frac{b}{s^2} + A\sinh\left(\sqrt{\frac{s}{a}}x\right) + B\cosh\left(\sqrt{\frac{s}{a}}x\right)$$

The boundary conditions in (a) transform to $\bar{\phi}(\pm h, s) = 0$ and hence

$$0 = \frac{b}{s^2} + A\sinh\left(\sqrt{\frac{s}{a}}h\right) + B\cosh\left(\sqrt{\frac{s}{a}}h\right)$$
$$0 = \frac{b}{s^2} - A\sinh\left(\sqrt{\frac{s}{a}}h\right) + B\cosh\left(\sqrt{\frac{s}{a}}h\right)$$

Clearly, $A = 0$ and B is easily calculated to give

$$\bar{\phi} = \frac{b}{s^2}\left[1 - \frac{\cosh\left(\sqrt{\frac{s}{a}}x\right)}{\cosh\left(\sqrt{\frac{s}{a}}h\right)}\right]$$

To transform back to the real plane needs either some tricky integrations or the use of advanced tables of Laplace transform pairs. Tables give the solution,

$$\frac{\phi}{b} = \frac{1}{2a}(h^2 - x^2) + \frac{16h^2}{a\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\exp\left[-\frac{(2n-1)^2\pi^2 at}{4h^2}\right]\cos\left[\frac{(2n-1)\pi x}{2h}\right]$$

Exercises 9.4.5

■ **41** In the explicit formulation equation (9.48), the MATLAB implementation can be written using the 'colon' notation to great effect.

```
n=6;L=0.5;x=0:0.2:1,u=x.^2
```

```
v=[0,L*(u([1:n-2])+u([3:n]))+(1-2*L)*u([2:n-1]),1]
% gives  0  0.0800  0.2000  0.4000  0.6800  1.0000
u=v;v=[0,L*(u([l:n-2])+u([3:n]))+(1-2*L)*u([2:n-1]),1]
% gives 0  0.1000  0.2400  0.4400  0.7000  1.0000
```

Repeating the last line gives successive time steps.

■ **42** Again a MATLAB formulation solves the problem very quickly; lamda (L in the program) is chosen to be 0.4 and time step 0.05.

Explicit

```
n=6;L=0.4;u=[0 0 0 0 0 1];
v=[0,L*(u([1:n-2])+u([3:n]))+(1-2*L)*u([2:n-1]),exp(-0.05)]
% gives  0      0      0      0     0.4000    0.9512
for p=2:20,u=v;v=[0,L*(u([1:n-2])+u([3:n]))+(1-2*L)
          *u([2:n-1]),exp(-p*0.05)];end
v
% gives the values at t = 1
as 0 0.1094  0.2104  0.2939  0.3497  0.3679
```

Repeating the last two lines produces the solution at successive times.

Implicit

There are some slight differences in the solution depending on how the right hand boundary is treated. Equation (9.49) is constructed in MATLAB again using the 'colon' notation

```
L=0.4;M=2*(1+L);N=2*(1-L);
n=6;u=zeros(n,1);u(n)=1;
p=[-L M -L];A=eye(n);for i=2:n-1,A(i,i-1:i+1)=p;end
q=[L N L];B=eye(n);for i=2:n-1,B(i,i-1:i+1)=q;end
DD=inv(A)*B;
v=DD*u;v(n)=exp(-0.05); % for first step
for p=2:20,u=v;v=DD*u;v(n)=exp(-p*0.05);end
% gives for t=1   0 0.1082 0.2096 0.2955 0.3551 0.3679
```

Repeat the last line of code for further time steps.

■ **43** The equations are easily produced in MATLAB. Because of the derivative boundary condition, the region is extended to $x = -0.2$ and $u(-0.2, t)$ is obtained from

$$u(0.2, t) - u(-0.2, t) = 0.4$$

```
L=0.5;M=2*(1+L);N=2*(1-L);n=7;
p=[-L M -L];A=eye(n);for i=2:n-1,A(i,i-1:i+1)=p;end
A(1,3)=1;A(1,1)=-1 % gives LHS matrix
q=[L N L];B=eye(n);for i=2:n-1,B(i,i-1:i+1)=q;end
B(1,1)=0   % gives RHS matrix
rhs=[0.4 0 0 0 0 0 0]';
% gives vector from derivative condition at x=0
AA=inv(A);
x=0:0.2:1,u=[-0.24,x.*(1-x)]' % starting data
v=AA*(B*u+rhs)
% produces next time step
-0.2800 -0.0400 0.1200 0.2002 0.2012 0.1269 0
u=v;v=AA*(B*u+rhs)
% produces next time step
-0.3197 -0.0799 0.0803 0.1613 0.1657 0.1034 0
```

Repeating the last line produces further time steps.

Exercises 9.5.2

■ **44** From equation (9.52), the only separated solution to satisfy $u \to 0$ as $y \to \infty$ is

$$u = (A \sin \mu x + B \cos \mu x)(Ce^{\mu y} + De^{-\mu y}) \text{ with } C = 0$$

Thus

$$u = (a \sin \mu x + b \cos \mu x)e^{-\mu y}$$

To satisfy the boundary conditions,

$$u = 0 \text{ at } x = 0 \Rightarrow b = 0$$

$$u = 0 \text{ at } x = 1 \Rightarrow \sin \mu = 0 \Rightarrow \mu = n\pi \text{ where } n \text{ is an integer.}$$

The condition at $y = 0$ can be satisfied by a sum of terms over n.

On $y = 0, u = \sum_{n=1}^{\infty} a_n \sin n\pi x = \frac{1}{16}(10 \sin \pi x - 5 \sin 3\pi x + \sin 5\pi x)$
and the a_n can be obtained by inspection to give the required solution.

■ **45** The four boundary conditions are satisfied by inspection and the Laplace equation is satisfied by straightforward differentiation.

■ **46** It can easily be checked that the function $x^2 y$ satisfies the given Poisson equation. The boundary conditions on $u(x, y)$ become

$$u(x, 0) = 0 \text{ for } 0 \leq x \leq 1$$
$$u(x, 1) = \sin \pi x \text{ for } 0 \leq x \leq 1$$
$$u(0, y) = 0 \text{ for } 0 \leq y \leq 1$$
$$u(1, y) = 0 \text{ for } 0 \leq y \leq 1$$

The only solution in equation (9.52d) that satisfies these conditions is

$$u = \sin \pi x \frac{\sinh \pi y}{\sinh \pi}$$

and hence the final result

$$\phi = x^2 y + \sin \pi x \frac{\sinh \pi y}{\sinh \pi}$$

■ **47** Differentiating

$$u_r = Bnr^{n-1} \sin n\theta$$
$$u_{rr} = Bn(n-1)r^{n-2} \sin n\theta$$
$$u_{\text{ee}} = -Bn^2 r^n \sin n\theta$$

and substitution gives

$$\text{LHS} = B \sin n\theta r^{n-2}[n(n-1) + n - n^2] = 0 = \text{RHS}$$

and hence the Laplace equation in plane polars is satisfied. To be periodic in θ the constant n must be an integer. A solution of the equation is a sum of the expressions given, so that

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta$$

Putting the condition on the rim, $r = a$, gives the Fourier problem to calculate B_n as $B_1 = 3/4a$ and $B_3 = -1/4a^2$ and otherwise zero. Thus,

$$u(r,\theta) = \frac{3}{4}\left(\frac{r}{a}\right)\sin\theta - \frac{1}{4}\left(\frac{r}{a}\right)^3 \sin 3\theta$$

■ **48** Let $D = x^2 + y^2 + 2x + 1$; then the derivatives can be computed as

$$u_x = \frac{2y(2x+2)}{D^2} \text{ and } u_{xx} = \frac{4y}{D^2} - \frac{4y(2x+2)^2}{D^3}$$

$$u_y = \frac{-2}{D} + \frac{2y2y}{D^2} \text{ and } u_{yy} = \frac{4y}{D^2} + \frac{8y}{D^2} - \frac{4y^2 4y}{D^3}$$

Adding the two second derivatives gives

$$\nabla^2 u = \frac{1}{D^3}\left[16y(x^2+y^2+2x+1) - 16y(x+1)^2 - 16y^3\right]$$

The RHS can easily be checked to be zero and hence the Laplace equation is satisfied. A similar process shows that v also satisfies the Laplace equation.
The u and v come from the complex variable expression

$$u + jv = \frac{j(x-1+jy)}{x+1+jy} = \frac{j(x-1+jy)(x+1-jy)}{(x+1+jy)(x+1-jy)}$$

Multiplying out

$$u + jv = \frac{-2y + j(x^2+y^2-1)}{x^2+y^2+2x+1}$$

gives the expressions quoted in the question.
A check can be made by using MAPLE.

```
u:=2*y/(x^2+y^2+2*x+1);
simplify(diff(u,x,x)+diff(u,y,y));
# gives zero as required -v follows similarly
h:=I*(x+I*y-1)/(x+I*y+1);
simplify(evalc(h));
# gives the u and v of the question
```

For fixed u and v the two expressions can be rearranged as

$$\left(x+\frac{v}{v-1}\right)^2+y^2=\frac{1}{(v-1)^2} \text{ and } (x+1)^2+\left(y+\frac{1}{u}\right)^2=\frac{1}{u^2}$$

which are circles with radii $\frac{1}{v-1}$ and $\frac{1}{u}$, centres $(\frac{-v}{v-1},0)$ and $(-1,\frac{-1}{u})$ respectively. Note that all the circles pass through the point $(-1,0)$.

■ **49** This is an important example that illustrates that sensible solutions can only be obtained if correct boundary conditions are set. First, it is a matter of simple differentiation to verify that the given function satisfies the Laplace equation. Again, since the sinh function is zero at $x=0$ the first condition is satisfied. Differentiating with respect to x,

$$\frac{\partial u}{\partial x}=\frac{1}{n}\cosh nx\sin ny, \text{ so at } x=0, \ \frac{\partial u}{\partial x}=\frac{1}{n}\sin ny$$

The solution therefore satisfies all the conditions of the problem. It is known that the solution is unique.

For any given n, however large, sinh nx can be made as large as required and even when divided by n^2 it is still large; for instance, $n=10, x=5$ and $y=\pi/200$ gives $u=4.1\times10^{18}$.

The 'neighbouring' problem has a boundary condition $u_x=0$ and solution u identically zero. For the values chosen for illustration, the maximum change at the boundary is 0.1; yet the solution changes by 10^{18}. Such behaviour is very unstable; these boundary conditions give a unique solution; yet small changes in the boundary produce huge changes in the solution. Figure 9.59 should be referred for a summary of the 'correctness' of boundary conditions.

■ **50** It is useful in solution by separation to try to modify the problem so that the function is zero on two opposite boundaries. Apply $u=x+f(x,y)$; then f satisfies the Laplace equation and the four boundary conditions become

$$f(0,y)=0 \qquad f(1,y)=0 \qquad \text{for } 0<y<1$$
$$f(x,0)=-x \quad f(x,1)=1-x \text{ for } 0<x<1$$

The solution given in equation (9.52d) is the appropriate one and the cosine can be omitted since it cannot satisfy the first of the four conditions. Thus,

$$f = \sin \mu x(a \cosh \mu y + \sinh \mu y)$$

The second condition now gives $\sin \mu = 0 \Rightarrow \mu = N\pi$ where N is an integer. The solution therefore takes the form

$$f = \sum_{N=1}^{\infty} \sin N\pi x(a_N \cosh N\pi y + b_N \sinh N\pi y)$$

and the coefficients are derived as Fourier series from the other two sides of the boundary as

$$-x = \sum_{N=1}^{\infty} a_N \sin N\pi x$$
$$1 - x = \sum_{N=1}^{\infty} \sin N\pi x(a_N \cosh N\pi + b_N \sinh N\pi)$$

Straightforward integration gives

$$a_N = \frac{2 \cos N\pi}{N\pi} \text{ and } (a_N \cosh N\pi + b_N \sinh N\pi) = \frac{2}{N\pi}$$

The final solution is obtained by substituting back

$$u = x + \frac{2}{\pi} \sum_{N=1}^{\infty} \frac{1}{N} \sin N\pi x \left[\cos N\pi \cosh N\pi y + (1 - \cos N\pi \cosh N\pi) \frac{\sinh N\pi y}{\sinh N\pi} \right]$$

which can be tidied up to the given solution.

■ **51** The boundary conditions on the four sides are

$$u(0, y) = 0 \quad u(a, y) = 0 \quad \text{for} \quad 0 < y < a$$

and $$u(x, 0) = 0 \quad u(x, a) = u_0 \quad \text{for} \quad 0 < x < a$$

Clearly, the only relevant separated solutions involve $\sin \lambda x \sinh \lambda y$ since these are the ones that satisfy the conditions on $x = 0$ and $y = 0$. The condition $u(a, y) = 0$ implies that $\sin \lambda a = 0 \Rightarrow \lambda a = mx$ where m is an integer. Thus, the solution takes the form

$$u = \sum_{m=1}^{\infty} b_m \sin \left(\frac{m\pi x}{a} \right) \sinh \left(\frac{m\pi y}{a} \right)$$

and the final condition gives the Fourier problem of calculating the coefficients from

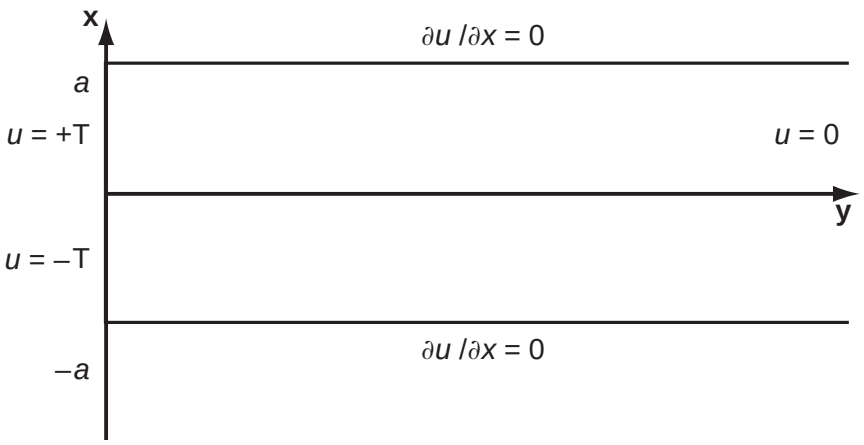
$$u_0 = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{a}\right) \sinh(m\pi)$$

The usual integration gives

$$\frac{a\sinh(m\pi)b_m}{2} = \int_0^a u_0 \sin\left(\frac{m\pi x}{a}\right) dx = u_0 \frac{a}{m\pi}(1 - \cos m\pi)$$

The only coefficients that survive are the odd ones and putting $m = 2n+1$ produces the result quoted.

■ **52** The configuration is illustrated in the figure



From the available separated solutions, the appropriate ones are $u = e^{-\lambda y}\sin\lambda x$ and to satisfy $u_x = 0$ at $x = \pm a$ requires $\lambda a = \left(n + \frac{1}{2}\right)\pi$. Thus, the solution takes the form

$$u = \sum_{n=0}^{\infty} b_n \exp\left[\frac{-\left(n+\frac{1}{2}\right)\pi y}{a}\right] \sin\left[\frac{\left(n+\frac{1}{2}\right)\pi x}{a}\right]$$

so that at $y = 0$ the coefficients are required from the Fourier problem

$$\sum_{n=0}^{\infty} b_n \sin\left[\frac{\left(n+\frac{1}{2}\right)\pi x}{a}\right] = \begin{cases} +T \text{ for } 0 < x < a \\ -T \text{ for } -a < x < 0 \end{cases}$$

The integration gives $b_n = \dfrac{4T}{(2n+1)\pi}$ as required.

■ **53** The separated solutions are put into the equation to give

$$\frac{r^2}{R}(R'' + \frac{1}{r}R') = -\frac{\ddot{\Theta}}{\Theta} = \lambda^2$$

where the separation constant has been chosen to be positive to ensure that periodic solution in Θ is possible. The equation

$$\ddot{\Theta} + \lambda^2\Theta = 0 \text{ has solution } \Theta = A\sin\lambda\theta + B\cos\lambda\theta$$

and is periodic only if $\lambda = N$, which is an integer. Since $T = 0$ on $\theta = 0$ and $\theta = \pi$, the constant B is given by $B = 0$.
The R equation is $r^2R'' + rR' - N^2R = 0$ and trying solutions of the form $R = r^P$ shows that $p = N$ or $-N$. The solution must be finite at the origin; so the negative powers are omitted and the original equation has a solution

$$T = \sum_{N=1}^{\infty} b_N r^N \sin N\theta$$

The condition on the boundary is

$$T_0 = \sum_{N=1}^{\infty} b_N a^N \sin N\theta$$

and the coefficients are obtained by Fourier analysis. The integrations are given by

$$\frac{\pi}{2}b_N a^N = \int_0^{\pi} T_0 \sin N\theta d\theta = \frac{T_0}{N}(1 - \cos N\pi)$$

and the result follows applying $N = 2n + 1$ and starting the summation from $n = 0$.

■ **54** The suggested separated form produces the equations

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{y\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dy}{d\theta}\right) = k(k+1)$$

where the RHS conforms with the given expression. The equation for y is obtained by making the substitution $x = \cos\theta$. Now

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{1}{\sin\theta}\frac{dy}{d\theta}$$

so putting this into the θ equation gives

$$\frac{d}{dx}\left(\sin^2\theta\frac{dy}{dx}\right)+k(k+1)y=0 \text{ or } \frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right)+k(k+1)y=0$$

To solve these equations, first put $R=r^n$ then substitution leads to

$$n(n+1)=k(k+1) \qquad \text{with solution} \qquad n=k \text{ or } -k-1$$

Secondly, the only solutions for y that remain finite at $x=+1$ and -1 are the solutions given in the question (see an advanced book on Legendre polynomials). Thus, the solution that involves $k=1,2,3$ is

$$V=A+\frac{B}{r}+\cos\theta\left(A'r+\frac{B'}{r^2}\right)+\frac{1}{2}(3\cos^2\theta-1)\left(A''r^2+\frac{B''}{r^3}\right)$$

The given boundary conditions involve only even functions of θ, so $A'=B'=0$. The first of the two conditions gives

$$0=A+\frac{B}{a}+\frac{1}{2}(3\cos^2\theta-1)\left(A''a^2-\frac{B''}{a^3}\right) \text{ for all } \theta$$

Hence $B=-aA$ and $B''=-a^5A''$ and the solution can be rewritten

$$V=A\left(1-\frac{a}{r}\right)+\frac{A''}{2}(3\cos^2\theta-1)\left(r^2-\frac{a^5}{r^3}\right)$$

The final condition implies

$$\alpha\sin^2\theta=A\left(1-\frac{a}{b}\right)+\frac{A''}{2}(2-3\sin^2\theta)\left(b^2-\frac{a^5}{b^3}\right)$$

so that identifying the two parts gives

$$A''=-\frac{2\alpha}{3\left(b^2-\frac{a^5}{b^3}\right)} \quad \text{and} \qquad A=\frac{2\alpha}{3\left(1-\frac{a}{b}\right)}$$

and the solution follows.

Exercises 9.5.4

■ **55** For rectangular regions MATLAB has an easy setup procedure

```
G=numgrid('S',5) % gives the node numbering as
```

```
G= 0 0 0 0 0
   0 1 4 7 0
   0 2 5 8 0
   0 3 6 9 0
   0 0 0 0 0
rhs(1)=0.25;rhs(4)=0.5;rhs(7)=0.75+0.9375;rhs(8)=0.75;
rhs(9)=0.4375;
rhs     % sets up the right hand side, one entry for each of the 9 equations
(default = 0), as the transpose of
rhs=  0.2500 0 0 0.5000 0 0 1.6875 0.7500 0.4375
A=delsq(G); % MATLAB sets up the matrix stored in sparse form
full(A) % gives the full matrix
      4  -1   0  -1   0   0   0   0   0
     -1   4  -1   0  -1   0   0   0   0
      0  -1   4   0   0  -1   0   0   0
     -1   0   0   4  -1   0  -1   0   0
      0  -1   0  -1   4  -1   0  -1   0
      0   0  -1   0  -1   4   0   0  -1
      0   0   0  -1   0   0   4  -1   0
      0   0   0   0  -1   0  -1   4  -1
      0   0   0   0   0  -1   0  -1   4
A\rhs' % gives the required solution
0.2026  0.1462 0.0776  0.4141  0.3047 0.1641  0.6490 0.4944 0.2740
```

■ **56** For the small grid

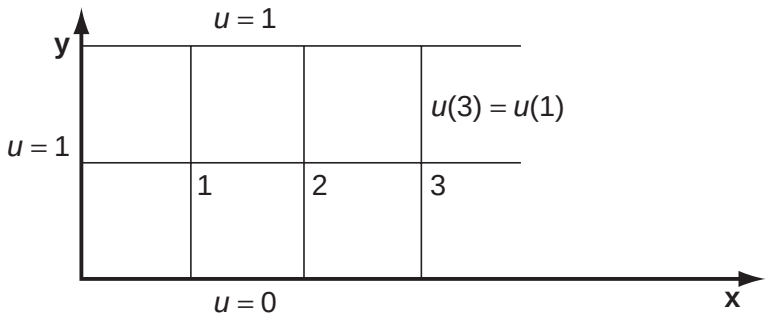
The equations are

$4u_1 = 1 + 1 + 0 + u_2$

$4u_2 = 1 + u_1 + 0 + u_1$

which can be solved as

$u_1 = \frac{9}{14}$ and $u_2 = \frac{4}{7}$

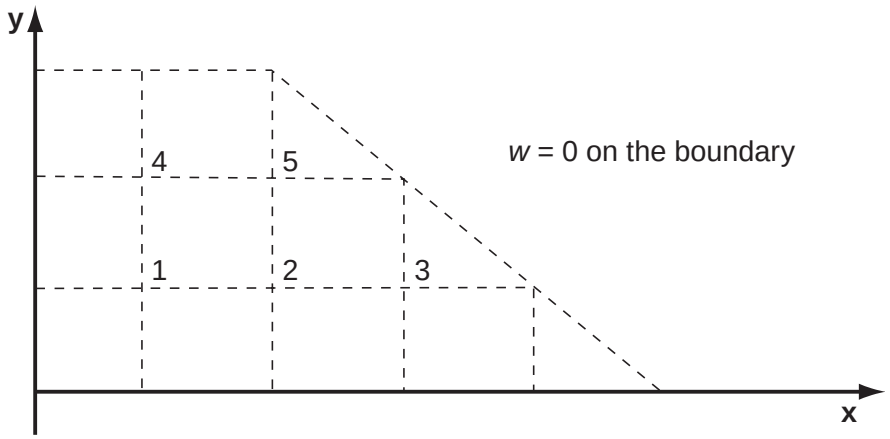


For the larger grid the MATLAB implementation needs more care since the derivative boundary condition modifies the matrix set up by the package.

```
G=numgrid('S',5);
A=delsq(G); % A is modified in the next four lines
B=zeros(9,3);B(7,1)=-1;B(8,2)=-1;B(9,3)=-1;C=[A B];
D=zeros(3,12);D(1,10)=4;D(2,11)=4;D(3,12)=4;
D(1,7)=-2;D(1,11)=-1;
D(2,8)=-2;D(2,10)=-1;D(2,12)=-1;D(3,9)=-2;D(3,11)=-1;
E=[C;D];
full(E) % prints out the modified A
rhs=zeros(12,1);rhs(1)=2;rhs(2)=1;
rhs(3)=1;rhs(4)=1;rhs(7)=1;rhs(10)=1;
E\rhs % gives the solution
```

```
0.9008 0.7684 0.5348 0.8348 0.6379 0.3709 0.8007 0.5774 0.3110
0.7904 0.5602 0.2955
```

■ **57** The mesh for this problem is shown in the figure



Using the notation of the problem, at a typical node 0 with neighbours 1, 2, 3 and 4, the adaptation of equation (9.55) gives

$$\frac{w_1 + w_2 + w_3 + w_4 - 4w_0}{h^2} + 20 = 0$$

which is easily rearranged into the given form. Applying this formula to the problem in hand produces the five equations

$$4w_1 = w_2 + w_4 + 20$$
$$4w_2 = w_1 + w_3 + w_5 + 20$$
$$4w_3 = w_2 + 20$$
$$4w_4 = w_1 + w_5 + 20$$
$$4w_5 = w_4 + w_2 + 20$$

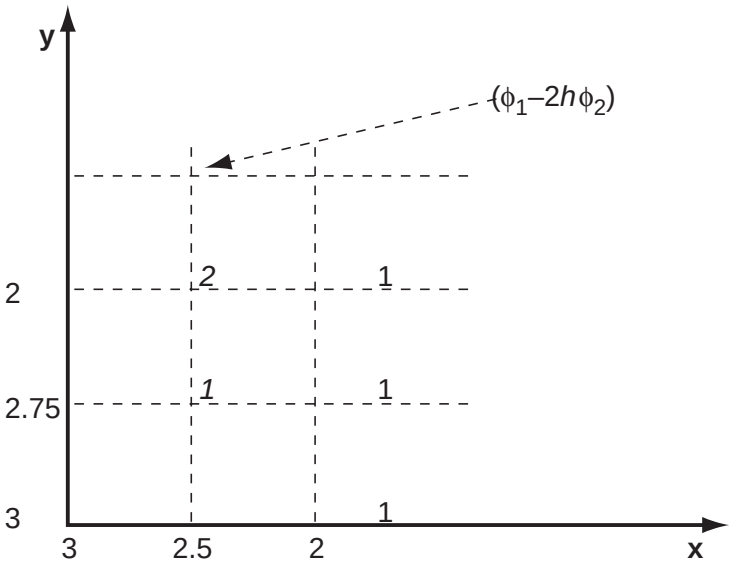
which can be solved by a spreadsheet or MATLAB to give

0	0	0			
0	10.337	10.674	0		
0	10.674	12.360	8.090	0	
0	0	0	0	0	0

■ **58** The problem has more complicated boundary data and so requires a little more effort to set up. The top boundary condition is derived from

$$\frac{\phi_+ - \phi_-}{2h} = -\phi_0$$

For case (a) when $h = 1/2$ the problem only has two unknowns.



The equations are

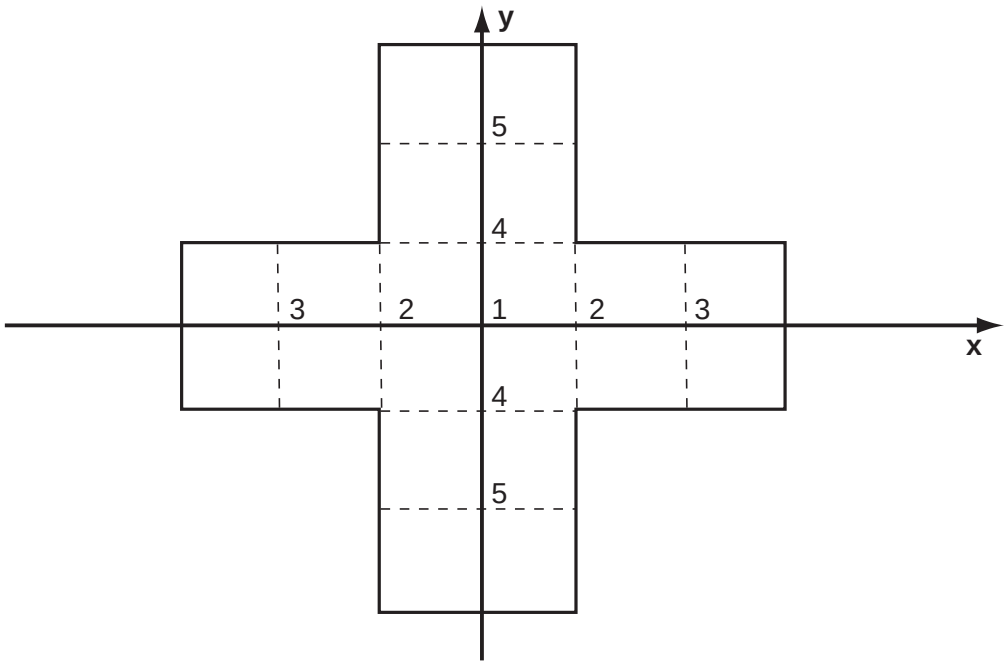
$$2.75 + 2.5 + 1 + \phi_2 - 4\phi_1 = 0.125$$
$$2 + \phi_1 + 1 + (\phi_1 - \phi_2) - 4\phi_2 = 0.125$$

which are easily solved to give the quoted answer.

(b) For the larger mesh, the MATLAB version is as follows; note the powerful matrix building techniques:

```
G=numgrid('S',5);A=delsq(G);
b=eye(3);c=zeros(3);d=[4.5-1 0;-1 4.5-1; 0-1 4.5];
A=[[A;c c-2*b][c;c;-b;d]];full(A) % A is printed out
rhs=[2.75+2.9375;2.5;3.25;2.75;0;1;2.4375;0;1;2;0;1];
z=[1/64;1/32;3/64];z=[z;z;z;z];
rhs=rhs-z      % rhs is printed out
A\rhs          % gives the final answer
         1.6016  1.2868  1.0565
         1.9680  1.5818  1.2572
         2.2666  1.8465  1.4375
         2.5175  2.1314  1.6930
```

■ **59** In this problem, there are essentially five unknowns because of the symmetry as



seen in the figure

The five equations are set up in matrix form as

$$\begin{bmatrix} -4 & 2 & 0 & 2 & 0 \\ 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -17 \\ -2 \\ -2 \end{bmatrix}$$

These equations can be solved on any suitable package to give

1.5909

2.0909

4.7727

1.0909

0.7727

Exercises 9.6.1

The exercises can only be solved sensibly on a computer package and need considerable computer experience, in the present case MATLAB. Some M-files are required by all the problems.

coeff.m

```
function [a0,a1,a2]=coeff(a,b,c)
A=[a 1];B=[b 1];C=[c 1];1x=[1 0 0];1y=[0 1 0];
den=0.5/det([A;B;C]);
L0=[det([1x;B;C]) det([1y;B;C])];
L1=[det([1x;C;A]) det([1y;C;A])];
L2=[det([1x;A;B]) det([1y;A;B])];
a0=L0*L0'*den;a1=L1*L0'*den;a2=L2*L0'*den;
```

stiff.m

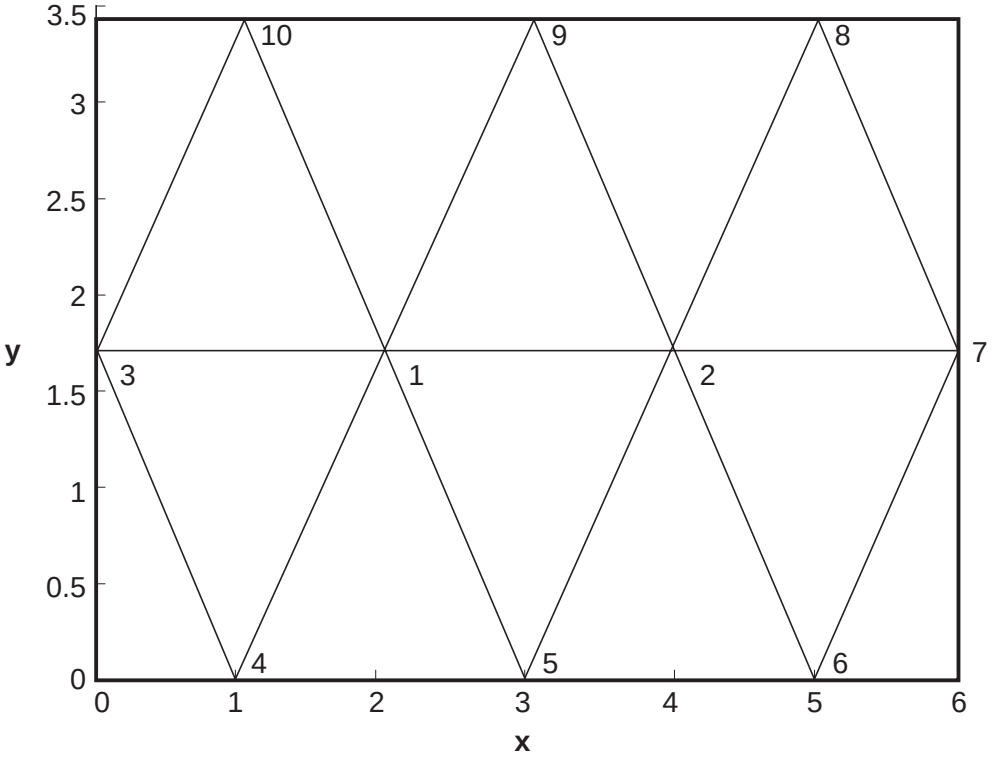
```
function a=stiff(mm,k,ll)
%mm=no of neighbours,k=current point,
ll=row of k's neighbours
global CO
a=zeros(1,mm+1);
```

```
            for p=1:mm-1
                    [l,m,n]=coeff(CO(k,:),CO(ll(p),:),CO(ll(p+1),:));
                    a(1)=a(1)+1;a(p+1)=a(p+1)+m;a(p+2)=a(p+2)+n;
            end
                    [l,m,n]=coeff(CO(k,:),CO(ll(mm),:),CO(ll(1),:));
                    a(1)=a(1)+1;a(mm+1)=a(mm+1)+m;a(2)=a(2)+n;
calcArhs.m % the following code needs to be copied to the main
program
            for k=1:nin
            r=link(k,:);m=nnz(r);
            z=stiff(m,k,r);
            A(k,k)=A(k,k)+z(1);
                   for i=1:m
                           if r(i)<=nin
                           A(k,r(i))=A(k,r(i))+z(i+1);
                           else
                           rhs(k)=rhs(k)-z(i+1)*bdry(r(i)-nin);
                           end
                   end
            end
```

■ **60** Each of the FE problems has its own input, for this problem

60(a)

```
inform60.m    % the node labelling is shown in the Figure
              nin=2;nbdry=8;%number of internal and boundary nodes
              global CO
              r3=sqrt(3);
              CO=[2,r3;4,r3;0,r3;1,0;3,0;5,0;6,
               r3;5,2*r3;3,2*r3;1,2*r3];
              %coords of points internal first then bdry
              link=[3 4 5 2 9 10;1 5 6 7 8 9];
              %links from interior points to neighbours,in CO order
              bdry=[0 0 0 0 1 0 0 0];%boundary values,in CO order
              A=zeros(nin);rhs=zeros(nin,1);
```



The calculation then proceeds using `inform.60` and the code from `calcArhs.m`

```
global C0
inform60
for k=1:nin
r=link(k,:);m=nnz(r);
z=stiff(m,k,r);
A(k,k)=A(k,k)+z(1);
       for i=1:m
               if r(i)<=nin
               A(k,r(i))=A(k,r(i))+z(i+1);
               else
               rhs(k)=rhs(k)-z(i+1)*bdry(r(i)-nin);
               end
        end
 end
```

```
A,rhs  % gives A=3.4641  -0.57774 rhs=  0
                 0.5774   3.4641        0.5774
A \rhs % gives the final result  0.0286
                                 0.1714
```

60(b) The larger mesh is treated similarly. The mesh information is in `inform60b.m`, where an obvious node numbering has been used – print out `CO` and `link` for details.

`inform60b.m`

```
nin=5;nbdry=12;%number of internal and boundary nodes
global CO
r3=sqrt(3);
a0=[3,r3;1.5,2*r3/3;4.5,2*r3/3;4.5,4*r3/3;1.5,4*r3/3];
a1=zeros(5,1);a2=[0;1;2;3;4]*1.5;a3=ones(5,1)*r3*2;
CO=[a0;a2,a1;6,r3;flipud(a2),a3;0,r3];
%coords of points internal first then bdry
link=[2 8 3 4 14 5; 6 7 8 1 5 17; 8 9 10 11 4 1;
        1 3 11 12 13 14; 17 2 1 14 15 16];
%links from interior points to neighbours, in CO order
bdry=[0 0 0 0 1 1 1 0 0 0 0 0];%boundary values, in
      CO order
A=zeros(nin);rhs=zeros(nin,1);
```

The calculation is the same as part (a)

```
global CO
inform60
for k=1:nin
r=link(k,:);m=nnz(r);
z=stiff(m,k,r);
A(k,k)=A(k,k)+z(1);
```

```
        for i=1:m
                if r(i)<=nin
                A(k,r(i))=A(k,r(i))+z(i+1);
                else
                rhs(k)=rhs(k)-z(i+1)*bdry(r(i)-nin);
                end
        end
end
A,rhs %prints out A and rhs
A\rhs %gives the required answer
                0.1024 0.0208 0.2920 0.2920 0.0208
```

■ **61** Because the problem now involves a Poisson equation, some modifications are required. The M-file `coeff.m` is modified to

`coeffr.m`

```
function [a0,a1,a2,a3]=coeffr(a,b,c)
A=[a 1];B=[b 1];C[c 1];1x=[1 0 0];1y=[0 1 0];
den=0.5/det([A;B;C]);
L0=[det([1x;B;C]) det([1y;B;C)];
L1=[det([1x;C;A]) det([1y;C;A)];
L2=[det([1x;A;B]) det([1y;A;B)];
a0=L0*L0'*den;a1=L1*L0'*den;a2=L2*L0'*den;
a3=-20/(12*den);
```

and `stiff.m` to

`stiffr.m`

```
function a=stiffr(mm,k,ll)
%mm=no of neighbours,k=current point,ll=row
of k's neighbours
global CO
a=zeros(1,mm+2);
for p=1:mm-1
        [l,m,n,q]=coeffr(CO(k,:),CO(ll(p),:),
        CO(ll(p+1),:));
```

```
            a(1)=a(1)+l;a(p+1)=a(p+1)+m;a(p+2)=a(p+2)+n;
            a(mm+2)=a(mm+2)+q;
      end
            [l,m,n,q]=coeffr(CO(k,:),CO(ll(mm),:),CO(ll(1),:));
            a(1)=a(1)+l;a(mm+1)=a(mm+1)+m;a(2)+n;
            a(mm+2)=a(mm+2)+q;
```

The mesh information for the problem is contained in

```
inform61
            nin=5;nbdry=12;%number of internal and
            boundary nodes
            global CO
            CO=[1 2;1 1;2 2;2 1;3 1;0 0;1 0;2 0;3 0;
            4 0;4 1;3 2;2 3;1 3;0 3;0 2;0 1];
            %coords of points internal first then bdry
            link=[2 4 3 14 15 16;1 16 17 7 4 3;1 4 5 12 13 14;
            1 2 8 9 5 3;4 9 10 11 12 3];
            %links from interior points to neighbours, in CO order
            bdry=[0 0 0 0 0 0 0 0 0 0 0 0];%boundary values,
            in CO order
            A=zeros(nin);rhs=zeros(nin,1);
```

The calculation now proceeds as

```
inform61
for k=1:nin
r=link(k,:);m=nnz(r);
z=stiffr(m,k,r);
A(k,k)=A(k,k)+z(1);
       for i=1:m
              if r(i)<=nin
              A(k,r(i))=A(k,r(i))+z(i+l);
              else
               rhs(k)=rhs(k)-z(i+l)*bdry(r(i)-nin);
              end
           end
```

```
rhs(k)=rhs(k)-z(m+2);
and
A,rhs % prints out A and rhs
A\rhs % gives the final result
10.3371 10.6742  10.6742 12.3596 8.0899
```

■ **62** Although the problem has a great deal of symmetry, all the 9 internal nodes and 16 boundary nodes are used in this implementation. An interesting exercise is to modify the program using the full symmetry. The mesh data is

```
inform62
            nin=9;nbdry=16;%number of internal and boundary nodes
            global CO
            r=1/3;s=2/3;
            CO=[0 0;0 -r;r 0; 0 r;-r 0; 0 -s;s 0;0 s;-s 0;
            -r -r;-r -1;0 -1;r -1;r -r;1 -r;1 0; 1 r;r r;r 1;
            0 1;-r 1; -r r;-1 r;-1 0;-1 -r];
            %coords of points internal first then bdry
            link=[2 14 3 18 4 22 5 10; 1 10 6 14 0 0 0 0;
            1 14 7 18 0 0 0 0;1 18 8 22 0 0 0 0;
            1 22 9 10 0 0 0 0;2 10 11 12 13 14 0 0;
            3 14 15 16 17 18 0 0;4 18 19 20 21 22 0 0;
            5 22 23 24 25 10 0 0];
            %links from interior points to neighbours, in CO order
            bdry=[1 1 0 1 1 9 9 9 1 1 0 1 1 9 9 9];
            %boundary values, in CO order
            A=zeros(nin);rhs=zeros(nin,1);
```

The equation is Laplace and so no modifications are required to the other M-files.

```
            global CO
            inform53
            for k=1:nin
            r=link(k,:);m=nnz(r);
            z=stiff(m,k,r);
            A(k,k)=A(k,k)+z(1);
```

```
                for i=1:m
                        if r(i)<=nin
                        A(k,r(i))=A(k,r(i))+z(i+1);
                        else
                         rhs(k)=rhs(k)-z(i+1)*bdry(r(i)-nin);
                        end
                end
        end
        A,rhs  % prints out A and rhs
        A\rhs  % gives the final answer for the internal
        nodes - note all the symmetries
1.6818 1.1152 2.2485 1.1152 2.2485 0.7788 5.3121 0.7788 5.3121
```

Exercises 9.7.4

■ **63** The Poisson formula gives

$$T(r,\theta) = \frac{T_0}{2\pi}\int_0^{\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - s)} ds$$

The integral can be evaluated by a package such as MAPLE or by using the substitution $z = \tan\, {}^1\!/_2(\theta - s)$ to give

$$T(r,\theta) = \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{a+r}{a-r}\tan\frac{\theta}{2}\right) + \tan^{-1}\left(\frac{a+r}{a-r}\cot\frac{\theta}{2}\right)\right]$$

It is instructive to show that $T = T_0/2$ when $r = 0$ and to check that $T = T_0$ when $r = a$ and $0 \le \theta \le \pi$.

■ **64** Substitution readily shows that the Laplace equation is satisfied. Also by inspection, it is easily seen that $G = 0$ on $x = 0$ and $y = 0$. No additional singularities have been added in the quarter plane, so the function is the Green's function for the problem. Equation (9.74) thus becomes

$$T(x_0,y_0) = \int_0^{\infty} f(x)\frac{\partial G}{\partial y}(x,0;x_0,y_0)dx + \int_0^{\infty} g(y)\frac{\partial G}{\partial x}(0,y;x_0,y_0)dy$$

Differentiating G with respect to y and putting $y = 0$ gives after careful algebra

$$\frac{y_0}{\pi}\left[\frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2}\right]$$

and similarly differentiating G with respect to x and putting $x = 0$ gives the expression given in the question.

If $f(x) = 1$ and $g(x) = 0$ then the solution is

$$\begin{aligned} T(x_0, y_0) &= \frac{y_0}{\pi}\int_0^\infty \left[\frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2}\right] dx \\ &= \frac{1}{\pi}\left[\tan^{-1}\left(\frac{(x-x_0)}{y_0}\right) - \tan^{-1}\left(\frac{(x+x_0)}{y_0}\right)\right]_0^\infty = \frac{2}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right) \end{aligned}$$

■ **65** We first note that if $x = r\cos\theta, y = r\sin\theta, x_0 = r_0\cos\theta_0, y_0 = r_0\sin\theta_0$ we have

$$(x-x_0)^2 + (y-y_0)^2 = r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)$$

Hence the given expression has the correct singularity. A similar calculation shows that the other term does not have a singularity inside the circle $r \le a$. Applying $r = a$ into the given Green's function it follows almost immediately that $G = 0$ as required. The equation (9.74) now produces the solution as

$$u(r_0, \theta_0) = -\int_0^{2\pi} f(\theta)\frac{\partial G(a, \theta; r_0, \theta_0)}{\partial r} a d\theta$$

Evaluating the derivative of the Green's function gives

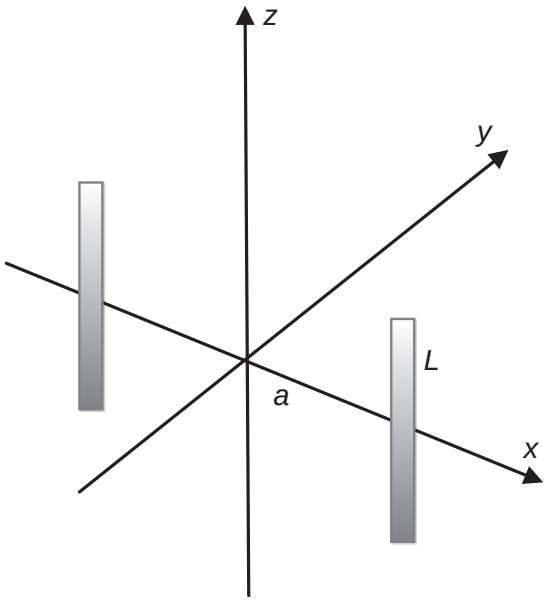
$$\frac{\partial G(a, \theta; r_0, \theta_0)}{\partial r} = \frac{1}{2\pi a}\frac{a^2 - r_0^2}{a^2 + r_0^2 - 2ar_0\cos(\theta-\theta_0)}$$

and hence the Poisson formula

$$u(r_0, \theta_0) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{a^2 - r_0^2}{a^2 + r_0^2 - 2ar_0\cos(\theta-\theta_0)} d\theta$$

■ **66** The figure illustrates the configuration. At the point (x, y, z), the temperature from the line source $x = a, y = 0, -L \le z \le L$ is given in Example 9.44 as

$$\frac{q_L}{4\rho c\kappa\pi}\left[\sinh^{-1}\left(\frac{z+L}{(x-a)^2+y^2}\right) - \sinh^{-1}\left(\frac{z-L}{(x-a)^2+y^2}\right)\right]$$



Note that the basic expression in Example 9.44 has been shifted to make the origin $x = a,\ y = 0, z = 0$. Clearly, this result does not satisfy the condition that the temperature is zero on $x = 0$. A second constant sink of length L must be added symmetrically at $x = -a, y = 0$, $-L \leq z \leq L$. When the source and the sink are added together we obtain

$$T(x,y,z) = \frac{q_L}{4\rho c\kappa\pi}\left[\sinh^{-1}\left(\frac{z+L}{(x-a)^2+y^2}\right) - \sinh^{-1}\left(\frac{z-L}{(x-a)^2+y^2}\right)\right]$$
$$- \frac{q_L}{4\rho c\kappa\pi}\left[\sinh^{-1}\left(\frac{z+L}{(x+a)^2+y^2}\right) - \sinh^{-1}\left(\frac{z-L}{(x+a)^2+y^2}\right)\right]$$

It can be seen that when $x = 0$ then $T = 0$, and hence we have the solution to the problem posed.

■ **67** From equation (9.76), the temperature generated from the instantaneous source $ad\theta$ is

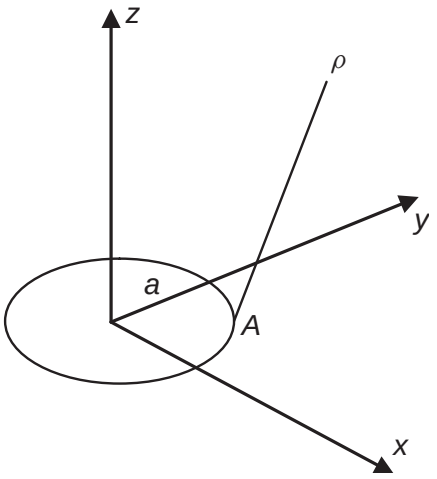
$$T(R,z,\theta,t) = \frac{qa}{8\rho c(\pi\kappa t)^{3/2}} \exp\left(-\frac{r^2}{4\kappa t}\right) d\theta$$

where $r = AP$. From the diagram, A is the point $\mathbf{a} = (a\cos\theta, a\sin\theta, 0)$ and P is the point $\mathbf{p} = (R\cos\phi, R\sin\phi, z)$. Thus, we can calculate $AP = |\boldsymbol{a} - \boldsymbol{p}|$ as

$$r^2 = R^2 + a^2 + z^2 - 2aR\cos(\theta - \phi)$$

The temperature can now be calculated by substituting and integrating the above expression.

$$T(R,z) = \frac{qa}{8\rho c(\pi\kappa t)^{3/2}} \exp\left(-\frac{R^2+a^2+z^2}{4\kappa t}\right) \int_0^{2\pi} \exp\left(\frac{aR\cos(\theta-\phi)}{2\kappa t}\right) d\theta$$



The integral is then identified with the definition given for the modified Bessel function.

Exercises 9.8.3

■ **68(a)** Now '$AC - B^2$' $= 0$ in this case so the equation is parabolic. Applying $r = x - y$ and $s = x + y$ the derivatives are

$$\begin{aligned} u_x &= u_r + u_s \\ u_y &= -u_r + u_s \end{aligned} \quad \text{and} \quad \begin{aligned} u_{xx} &= u_{rr} + 2u_{rs} + u_{ss} \\ u_{yy} &= u_{rr} - 2u_{rs} + u_{ss} \\ u_{xy} &= -u_{rr} + u_{ss} \end{aligned}$$

Substituting into the given equation gives the parabolic equation $u_{ss} = 0$.

68(b) Now '$AC - B^2$' $= 4$ in this case so the equation is elliptic. From the theory in Section 9.8.1 substitute $r = -3x + y$ and $s = x + y$; the derivatives are

$$\begin{aligned} u_x &= -3u_r + u_s \\ u_y &= u_r + u_s \end{aligned} \quad \text{and} \quad \begin{aligned} u_{xx} &= 9u_{rr} - 6u_{rs} + u_{ss} \\ u_{yy} &= u_{rr} + 2u_{rs} + u_{ss} \\ u_{xy} &= -3u_{rr} - 2u_{rs} + u_{ss} \end{aligned}$$

Substituting into the given equation gives the elliptic equation

$$u_{rr} + u_{ss} - \frac{9}{8}u_r + \frac{3}{8}u_s + \frac{1}{8}u = 0$$

68(c) Now '$AC - B^2$' $= -\frac{25}{4} - 3 \times 2 < 0$ in this case and so the equation is hyperbolic. From the theory in Section 9.8.1 substitute $r = 9x + y$ and $s = x + y$; the derivatives are

$$\begin{aligned} u_x &= 9u_r + u_s \\ u_y &= u_r + u_s \end{aligned} \quad \text{and} \quad \begin{aligned} u_{xx} &= 81u_{rr} + 18u_{rs} + u_{ss} \\ u_{yy} &= u_{rr} + 2u_{rs} + u_{ss} \\ u_{xy} &= 9u_{rr} + 10u_{rs} + u_{ss} \end{aligned}$$

Substituting into the given equation gives the hyperbolic equation

$$49u_{rr} = u_{ss}$$

■ **69** The characteristic directions from Exercise 68 are $r + 7s = 16x + 8y$ and $r - 7s = 2x - 6y$. This suggests that the solution is of the form

$$u = f(2x + y) + g(x - 3y)$$

where f and g are arbitrary functions. Substituting, the given equation

$$\text{LHS} = 3(4f'' + \ddot{g}) - 5(2f'' - 3\ddot{g}) - 2(f'' + 9\ddot{g}) = 0 = \text{RHS}$$

is satisfied, as expected.

■ **70** The chain rule gives

$$\begin{aligned} f_x &= f_u u_x + f_v u_x \\ &= f_u + f_v \end{aligned} \quad \text{and} \quad \begin{aligned} f_y &= f_u u_y + f_v u_y \\ &= f_u - f_v \end{aligned}$$

and the second derivatives are

$$f_{xx} = f_{uu} + 2f_{uv} + f_w, \; f_{yy} = f_{uu} - 2f_{uv} + f_w$$

$$f_{xy} = f_{uu} - f_w$$

Substituting into the equation (9.84) reduces to $f_w = 0$.
Integrating with respect to v

$$f_v = F(u) \qquad \text{where } F \text{ is an arbitrary function of } u.$$

Integrating again with respect to v

$$f = vF(u) + G(u) \qquad \text{where } G \text{ is another arbitrary function of } u.$$

Hence,

$$f = (x-y)F(x+y) + G(x+y)$$

■ **71** Now '$AC - B^2$' $= y$ and there are three cases (a) $y > 0$ where the equation is elliptic. (b) $y = 0$ where the equation is parabolic and (c) $y < 0$ where the equation is hyperbolic.

From equation (9.83), the characteristics are obtained from the solution of the equation

$$\frac{dy}{dx} = \pm\frac{\sqrt{-}y}{y}$$

The equation only makes sense in the hyperbolic region where $y < 0$. Substitute $z = -y$ and the differential equation becomes

$$\frac{dz}{dx} = \pm\frac{\sqrt{z}}{z} = \pm\frac{1}{\sqrt{z}}$$

which is easily integrated to

$$z^{\frac{3}{2}} = \pm\frac{3}{2}x + K \Rightarrow (-y)^{\frac{3}{2}} \pm \frac{3}{2}x = K$$

as the characteristics.

■ **72** Differentiating

$$f_x = \left(3Ax^2 - \frac{2B}{x^3}\right) y(1-y^2) \text{ and } f_y = \left(Ax^3 + \frac{B}{x^2}\right)(1-3y^2)$$

$$f_{xx} = 6\left(Ax + \frac{B}{x^4}\right) y(1-y^2) \text{ and } f_{yy} = \left(Ax^3 + \frac{B}{x^2}\right)(-6y)$$

The given equation can be checked by substitution.

Now '$AC - B^2$' $= x^2(1-y^2)$ so

$$\begin{cases} \text{elliptic if } |y| < 1 \\ \text{parabolic if } x = 0 \text{ or } y = \pm 1 \\ \text{hyperbolic if } |y| > 1 \end{cases}$$

■ **73** Calculating '$AC - B^2$' $= -4p^2+4q^2$, it can be deduced that the equations are

(a) $p > q$ or $p < -q$ then the equations are hyperbolic

(b) $p = q$ then the equations are parabolic

(c) $-q < p < q$ then the equations are elliptic

Using the substitution $-p^2+q^2 = -\frac{1}{4}(x^4-2x^2y^2+y^4-x^4-2x^2y^2-y^4) = x^2y^2 > 0$, hence leads to the elliptic region when it is expected that the equation will look like the Laplace equation.

$$\begin{aligned} v_x &= xv_p + xv_q \\ v_y &= -yv_p + yv_q \end{aligned} \quad \text{and} \quad \begin{aligned} v_{xx} &= v_p + v_q + x^2(v_{pp} + 2v_{pq} + v_{qq}) \\ v_{yy} &= -v_p + v_q + y^2(v_{pp} - 2v_{pq} + v_{qq}) \\ v_{xy} &= xy(-v_{pp} + v_{qq}) \end{aligned}$$

It may be noted that

$$v_{xx} + v_{yy} = 2v_q + (x^2 + y^2)v_{pp} + 2(x^2 - y^2)v_{pq} + (x^2 + y^2)v_{qq}$$

from which the required transformation to the Laplace follows immediately.

■ **74** In this case, '$AC - B^2$' $= -(xy)^2$ so the equation is hyperbolic away from the axes. The characteristics are computed from equation (9.83) as

$$\frac{dy}{dx} = \pm\frac{\sqrt{x^2y^2}}{x^2} = \pm\frac{y}{x}$$

and the solution is obtained by integration as

$$\ln\ y = \pm \ln\ x + K \Rightarrow y = ax \text{ and } yx = b$$

so one set of characteristics are straight lines through the origin and the other are rectangular hyperbolas. The domain of dependence and the range of influence can now be sketched.

Review exercises 9.11

■ **1** The boundary and initial conditions on $y(x,t)$ are

$$y = 0 \qquad \text{at} \qquad x = 0 \quad \text{and } a$$

String initially at rest $\Rightarrow \frac{\partial y}{\partial t} = 0 \quad \text{at} \quad t = 0$

Initial displacement $y = f(x) = \begin{cases} \frac{\varepsilon x}{b} & \text{for } 0 \le x \le b \\ \frac{\varepsilon(a-x)}{a-b} & \text{for } b \le x \le a \end{cases}$ at $t = 0$

Out of the possible separated solutions (9.25) that satisfy the wave equation, the only one that satisfies the conditions at $x = 0$ and $y = 0$ is $y = \sin\lambda x \cos\lambda ct$. The condition at $x = a$ gives $\lambda a = n\pi$ where n is an integer. Thus, the solution is a sum of such terms

$$y = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi ct}{a}\right)$$

and the final condition produces the Fourier series

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)$$

The coefficients are evaluated from the integral

$$\frac{1}{2}aA_n = \int_0^b \frac{\varepsilon x}{b} \sin\left(\frac{n\pi x}{a}\right) dx + \int_b^a \frac{\varepsilon(a-x)}{a-b} \sin\left(\frac{n\pi x}{a}\right) dx$$

which, after some careful integration by parts, gives the required coefficient

$$A_n = \frac{2\varepsilon a^2}{n^2\pi^2 b(a-b)} \sin\left(\frac{n\pi b}{a}\right)$$

It is interesting to look at the solution for various values of b since the solution gives strengths of the harmonics for different musical instruments. It is these values that give the characteristic sound of the instrument. For example, a violin has $b = a/7$; it is seen that $A_7 = 0$ and sevenths do not occur for this instrument.

■ **2** Taking the Laplace transform of the equation and the boundary conditions gives

$$\bar{\phi}'' = s^2\bar{\phi} - sx^2$$

and $$\bar{\phi}(0, s) = 0,\ \bar{\phi}'(l, s) = \frac{2l}{s}$$

The most convenient form for the complementary function is

$$\bar{\phi} = A\cosh s(x-l) + B\sinh s(x-l)$$

The particular integral is a quadratic in x which when substituted into the equation gives

$$\text{particular integral} = \frac{x^2}{s} + \frac{2}{s^3}$$

The complete solution is therefore

$$\bar{\phi} = A\cosh s(x-l) + B\sinh s(x-l) + \frac{x^2}{s} + \frac{2}{s^3}$$

Putting the two conditions on $\bar{\phi}$ into the solution

$$\bar{\phi}'(0,s) = 0 \Rightarrow 0 = A\cosh(-sl) + B\sinh(-sl) + \frac{2}{s^3}$$

$$\bar{\phi}'(l,s) = \frac{2l}{s} \Rightarrow \frac{2l}{s} = sB + \frac{2l}{s} \Rightarrow B = 0$$

The Laplace transform of the solution is

$$\bar{\phi} = \frac{x^2}{s} + \frac{2}{s^3} - \frac{2}{s^3}\frac{\cosh s(x-l)}{\cosh sl}$$

The solution in real space can be obtained from advanced tables of transform pairs.

■ **3** Take the separated solutions that are quoted and first note that the conditions $y(0) = y(l) = 0$ are satisfied. Secondly, substitute into equation (9.92)

$$\frac{1}{c^2}\left(T_n'' + \frac{1}{\tau}T_n'\right) = -T_n\left(\frac{n\pi}{l}\right)^2$$

This equation can then be solved in the standard way by looking for solutions of the form $T_n = e^{\alpha t}$ which produces the quadratic equation

$$\alpha^2 + \frac{1}{\tau}\alpha + \left(\frac{cn\pi}{l}\right)^2 = 0$$

The equation has roots

$$\alpha = \frac{1}{2}\left(-\frac{1}{\tau} \pm \sqrt{\left(\frac{1}{\tau}\right)^2 - 4\left(\frac{cn\pi}{l}\right)^2}\right) = -\frac{1}{2\tau} \pm j\frac{cn\pi}{l}\sqrt{1 - \left(\frac{l}{2\tau cn\pi}\right)^2}$$

and hence the solution is

$$T_n(t) = \exp\left(-\frac{t}{2\tau}\right)(a_n\cos\omega_n t + b_n\sin\omega_n t)$$

and general solution

$$y = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{l}\right)$$

The condition $\frac{\partial y}{\partial t}(x, 0) = 0$ implies, on differentiation and substituting $t = 0$,

$$0 = -\frac{1}{2\tau} a_n + b_n \omega_n$$

From the other condition it is seen that the only term to survive is when $n = 3$. Thus,

$$4 \sin\left(\frac{3\pi x}{l}\right) = a_n \exp\left(-\frac{0}{2\tau}\right)\left(\cos \omega_3 0 + \frac{1}{2\tau\omega_3} \sin \omega_3 0\right) \sin\left(\frac{3\pi x}{l}\right)$$

and hence the solution satisfying all the conditions is

$$y(x, t) = 4\exp\left(-\frac{t}{2\tau}\right)\left(\cos \omega_3 t + \frac{1}{2\tau\omega_3} \sin \omega_3 t\right) \sin\left(\frac{3\pi x}{l}\right)$$

and ω_3 given in the question.

■ **4** This problem is the extension of the wave equation to beams. Simply substituting the given form into the beam equation gives the equation (9.93) for V and the end conditions follow immediately. Again simply substituting sin, sinh, cos and cosh into the equation shows they are solutions of (9.93). A linear combination of the functions is also a solution and since it contains four arbitrary constants, it is the general solution.

To satisfy the end conditions

$$V(0) = 0 \Rightarrow A + B = 0$$

$$V'(0) = 0 \Rightarrow C + D = 0$$

$$V(l) = 0 \quad \Rightarrow \quad A\cosh \alpha l + B\cos \alpha l + C\sinh \alpha l + D\sin \alpha l = 0$$

$$V'(l) = 0 \quad \Rightarrow \quad A\sinh \alpha l - B\sin \alpha l + C\cosh \alpha l + D\cos \alpha l = 0$$

Thus,

$$A(\cosh \alpha l - \cos \alpha l) = D(\sinh \alpha l - \sin \alpha l)$$

$$A(\sinh \alpha l + \sin \alpha l) = D(-\cos \alpha l + \cosh \alpha l)$$

Dividing these two equations, and using the trigonometric and hyperbolic identities, produces the equation

$$\cos \alpha l \cosh \alpha l = 1$$

from which the natural frequencies of vibration of the beam can be calculated.

■ **5** The separated solutions in equation (9.40) that satisfy the condition at $x = 0$ are $\theta = e^{-\alpha t} \cos \lambda x$. To satisfy the condition at $x = l$ requires that

$$\lambda l = \left(n + \frac{1}{2}\right) \pi \text{ where } n \text{ is an integer}$$

Thus, the solution takes the form

$$\theta(x,t) = \sum_{n=0}^{\infty} A_{2n+1} \cos\left[\frac{(2n+1)\pi x}{2l}\right] \exp\left[-\left(\frac{(2n+1)\pi}{2al}\right)^2 t\right]$$

and the initial condition now produces the usual Fourier series problem with

$$\frac{1}{2} A_{2n+1} = \int_0^l f(x) \cos\left[\frac{(2n+1)\pi x}{2l}\right] dx$$

Integration by parts is required to obtain the coefficient for the given function $f(x) = \theta_0(l - x)$ as

$$A_{2n+1} = \frac{8\theta_0 l}{\pi^2 (2n+1)^2}$$

and the temperature can be obtained from the series solution.

■ **6** Evaluating the partial derivatives

$$\frac{\partial \phi}{\partial x} = \frac{1}{\sqrt{t}} f', \ \frac{\partial \phi}{\partial t} = -\frac{1}{2} \frac{x}{t^{3/2}} f' \text{ and } \frac{\partial^2 \phi}{\partial x^2} = \frac{1}{t} f''$$

Substituting the derivatives back into the equation

$$\kappa \frac{1}{t} f'' = -\frac{1}{2} \frac{x}{t^{3/2}} f' \Rightarrow f'' = -\frac{1}{2\kappa} z f'$$

the problem has been reduced to an ordinary differential equation. Rearranging the equation

$$\frac{df'}{f'} = -\frac{1}{2\kappa} z dz \text{ which integrates to } \ln f' = -\frac{z^2}{4\kappa} + C$$

or $f' = A \exp\left(-\frac{z^2}{4\kappa}\right)$ which on further integration gives

$$f = A \int_0^z \exp\left(-\frac{v^2}{4\kappa}\right) dv + B$$

Substitute $u = \frac{v}{2\sqrt{\kappa}}$ then the equation reduces to

$f = a \int_0^{\frac{z}{2\sqrt{\kappa}}} e^{-u2} du + b$ where a and b are new arbitrary constants.
The solution is as required

$$f = a \text{ erf}\left(\frac{z}{2\sqrt{\kappa}}\right) + b$$

For the particular problem,

$$\begin{array}{lll} \text{at } t = 0 & \phi(x, 0) = 0 & \text{for all } x > 0 \\ \text{at } x = 0 & \phi(0, t) = \phi_0 & \text{for } t > 0 \end{array}$$

In the expression containing the error function, these conditions give respectively

$$0 = a \text{ erf}(\infty) + b \text{ and } \phi_0 = a \text{ erf}(0) + b$$

Since $\text{erf}(0) = 0$ and $\text{erf}(\infty) = 1$, the solution can be constructed as

$$T(x, t) = T_0 + \phi_0 \left[1 - \text{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]$$

■ **7** The problem is standard except for the treatment of the derivative boundary conditions. Note the way that they are handled in the MATLAB implementation.

Explicit

```
u=[1 1 1 1 1 1];
u=0.1*[u([2:6]),u(5)-0.4*u(6)]+0.8*u([1:6])
+.1*[u(2)-0.4*u(1),u([1:5])]
% gives 0.9600 1.0000 1.0000 1.0000 1.0000 0.9600
u=0.1*[u([2:6]),u(5)-0.4*u(6)]
+0.8*u([1:6])+.1*[u(2)-0.4*u(1),u([1:5])]
% gives 0.9296 0.9960 1.0000 1.0000 0.9960 0.9296
u=0.1*[u([2:6]),u(5)-0.4*u(6)]+
0.8*u([1:6])+.1*[u(2)-0.4*u(1),u([1:5])]
% gives 0.9057 0.9898 0.9996 0.9996 0.9898 0.9057
```

Repeating the last line of code produces subsequent time steps.
Implicit

```
A=[-2.4 2 0 0 0 0;1 -2 1 0 0 0;0 1 -2 1 0 0;0 0 1 -2 1 0;
0 0 0 1 -2 1;0 0 0 0 2 -2.4]
u=[1;1;1;1;1;1];
B=2*eye(6)-0.1*A
C=2*eye(6)+0.1*A
E=inv(B)*C
u=E*u
% gives 0.9641 0.9984 0.9999 0.9999 0.9984 0.9641
u=E*u
% gives 0.9354 0.9941 0.9996 0.9996 0.9941 0.9354
u=E*u
% gives 0.9120 0.9881 0.9988 0.9988 0.9881 0.9120
```

Repeating the last line of code produces subsequent time steps.

■ **8** From the possible solutions in equation (9.52), one that can be chosen to satisfy the conditions on $x = 0$ and $y = a$ is

$$u = \sin \mu x \sinh \mu(a - y)$$

On $x = a,\ u = 0 \Rightarrow \sin \mu a = 0 \Rightarrow \mu a = n\pi$ where n is an integer. The final condition on $y = 0$ can be satisfied by taking a sum of such solutions

$$x(a - x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \sinh n\pi$$

The problem is the usual evaluation of the Fourier coefficients

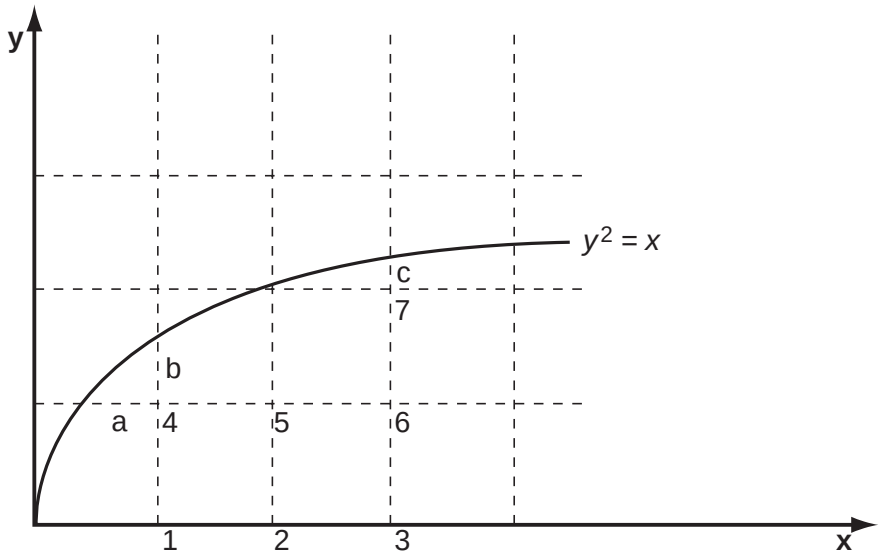
$$\frac{a}{2} b_n \sinh n\pi = \int_0^a (a - x) \sin\left(\frac{n\pi x}{a}\right) dx$$

and after two integrations by parts gives

$$\frac{a}{2} b_n = 2\left(\frac{a}{n\pi}\right)^3 (1 - \cos n\pi)$$

All the even terms go to zero and putting $n = 2r + 1$, the expression quoted is recovered.

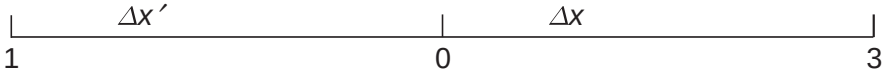
■ **9** This exercise is a harder one since the regular mesh points do not lie on the boundary.



At a point such as node 4 the lengths of the mesh are not uniform, so the second derivative needs to be approximated as

$$f'' = \left[\left(\frac{f_3 - f_0}{\Delta x}\right) - \left(\frac{f_0 - f_1}{\Delta x'}\right)\right] \frac{2}{\Delta x + \Delta x'}$$

for a typical configuration



The curve does not pass through the mesh points and the lengths are calculated as $a = 0.25, b = \sqrt{0.5} - 0.5$ and $c = \sqrt{1.5} - 1$. Working through the equations one at a time, using this formula where appropriate

$$1 + u_2 + 2u_4 - 4u_1 = 0$$

$$u_1 + u_3 + 2u_5 - 4u_2 = 0$$

$$u_2 + 1 + 2u_6 - 4u_3 = 0$$

$$\frac{2}{0.5 + a}\left(\frac{u_5 - u_4}{0.5} + \frac{1 - u_4}{a}\right) + \frac{2}{0.5 + b}\left(\frac{1 - u_4}{b} + \frac{u_1 - u_4}{0.5}\right) = 0.5(0.5)^2$$

$$u_4 + u_2 + 1 + u_6 - 4u_5 = 0.0625$$

$$u_5 + u_3 + u_7 + 1 - 4u_6 = 0.09375$$

$$\frac{1 + 1 - 2u_7}{0.5^2} + \frac{2}{c + 0.5}\left(\frac{1 - u_7}{c} + \frac{u_6 - u_7}{0.5}\right) = 1.5$$

The equations can be transformed to matrix form as

$$\mathbf{a} = \begin{bmatrix} -4 & 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 2 & 0 \\ 5.6569 & 0 & 0 & -35.3137 & 5.3333 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5.5192 & -25.7980 \end{bmatrix}$$

$$\mathbf{b}^T = [\,-1 \quad 0 \quad -1 \quad -24.1985 \quad -0.9375 \quad -0.9063 \quad -18.7788\,]$$

and the solution of $\mathbf{Ax} = \mathbf{b}$ can be obtained from any package, for example MATLAB, as

$$\mathbf{x}^T = [\,0.9850 \quad 0.9648 \quad 0.9602 \quad 0.9876 \quad 0.9570 \quad 0.9380 \quad 0.9286\,]$$

■ **10** Clearly, at $z = 0$ the velocity $u = U\cos\omega t$ so the given solution has the correct velocity on the wall. It also satisfies the equation of motion since substituting the derivatives

$$u_t = -U\omega e^{-\alpha z}\sin(\omega t - \alpha z)$$

$$u_z = -\alpha U e^{-\alpha z}\cos(\omega t - \alpha z) - \alpha U e^{-\alpha z}\sin(\omega t - \alpha z)$$

$$u_{zz} = \alpha^2 U e^{-\alpha z}\cos(\omega t - \alpha z) - 2\alpha^2 U e^{-\alpha z}\sin(\omega t - \alpha z)$$

$$-\alpha^2 U e^{-\alpha z}\cos(\omega t - \alpha z)$$

into the equation gives $-U\omega = -\nu 2\alpha^2 U$ which agrees precisely with the definition of α.

■ **11** Differentiating

$$U_t = kt^{k-1}\exp\left(-\frac{r^2}{4t}\right) + t^k\left(\frac{-r^2}{4}\right)\left(\frac{-1}{t^2}\right)\exp\left(-\frac{r^2}{4t}\right)$$

$$U_r = t^k\left(\frac{-2r}{4t}\right)\exp\left(\frac{r^2}{4t}\right)$$

$$\frac{\partial}{\partial r}(r^2U_r) = \frac{-t^{k-1}}{2}\left(3r^2 + r^3\left(\frac{-2r}{4t}\right)\right)\exp\left(\frac{r^2}{4t}\right)$$

and applying into the spherically symmetric heat equation

$$-\frac{t^{k-1}}{2}\left(3 - \frac{r^2}{2t}\right)\exp\left(-\frac{r^2}{4t}\right) = t^{k-1}\left(k + \frac{r^2}{4t}\right)\exp\left(-\frac{r^2}{4t}\right)$$

gives the relation $k = -3/2$.

■ **12** The equation is the same as Example 9.7 and can be dealt with in the same way, see also equation (9.14). However, the MAPLE solution is very straightforward.

```
with (PDEtools):
rev 12:=diff(z(x,y),x) + diff(z(x,y),y);
sol:=pdesolve (rev12,z(x,y));
# gives the solution          sol:=z(x,y)=_F1(y-x)
```

The boundary conditions $x = s, y = -s, z = 2s$ for $s > 0$ give the solution

$$z(x, y) = x - y \quad \text{for} \quad x > y$$

and not defined otherwise.

■ **13** The separated solutions in equation (9.52) that tend to zero for large y must be of the form

$$\phi = (A \sin \mu x + B \cos \mu x) e^{-\mu y}$$

The condition $\phi(0, y) = 0 \Rightarrow B = 0$ and $\phi(\pi, y) = 0 \Rightarrow \sin \mu\pi = 0$; thus, the required solution is

$$\phi = \sum_{n=1}^{\infty} c_n e^{-ny} \sin nx$$

Yet again, the final condition requires the evaluation of the coefficients by Fourier analysis. Integration by parts gives

$$\frac{\pi}{2} c_n = \int_0^{\pi} x(\pi - x) \sin nx dx = \frac{2}{n^3}(1 - \cos n\pi)$$

When n is even, the coefficient is zero and the odd values give the result in the exercise.

■ **14** First observe that the function $\chi = a^2 - x^2$ satisfies $\nabla^2 \chi = -2$ and second that a separated solution of the Laplace equation derived in equation (9.52) is $\cosh \mu y \cos \mu x$ and the overall solution is a combination of terms of these types. The conditions that $\chi = 0$ on $x = \pm a \Rightarrow \cos \mu a = 0 \Rightarrow \mu a = \left(n + \frac{1}{2}\right) \pi$ where n is an integer. Thus, an appropriate solution is

$$\chi(x, y) = a^2 - x^2 + \sum_{n=0}^{\infty} A_{2n+1} \cosh \left[\left(n + \frac{1}{2} \right) \pi \frac{y}{a} \right] \cos \left[\left(n + \frac{1}{2} \right) \pi \frac{x}{a} \right]$$

The function is even, so only the condition at $y = b$ needs to be considered since the other boundary is satisfied by symmetry. Therefore,

$$0 = a^2 - x^2 + \sum_{n=0}^{\infty} A_{2n+1} \cosh \left[\left(n + \frac{1}{2} \right) \pi \frac{b}{a} \right] \cos \left[\left(n + \frac{1}{2} \right) \pi \frac{x}{a} \right]$$

and the coefficients are derived by Fourier analysis. From tables of Fourier series it may be noted that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos\left[(2n+1)\frac{\pi x}{2a}\right] = \frac{\pi^3}{32a^2}(a^2 - x^2) \text{ for } -a < x < a$$

and thus the coefficient can be identified as

$$A_{2n+1} = \frac{32a^2}{\pi^3} \frac{(-1)^{n+1}}{(2n+1)^3 \cosh\left[\frac{(2n+1)\pi b}{2a}\right]}$$

■ **15** The possible separated solutions of the wave equation are given in equation (9.25) and to satisfy conditions (a) and (b), the solution $\sin \lambda x \cos \lambda ct$ must be chosen. The condition at $x = 1$ implies that $\sin \lambda = 0 \Rightarrow \lambda = n\pi$ where n is an integer. Thus, the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \cos n\pi t$$

and condition (c) is substituted to give the Fourier series

$$1 - x = \sum_{n=1}^{\infty} a_n \sin n\pi x$$

The coefficients are obtained from

$$\frac{1}{2}a_n = \int_0^1 (1-x) \sin n\pi x dx$$

which can be integrated by parts to give $a_n = \frac{2}{nx}$ and agrees with the quoted answer.

■ **16** Complete drainage at top and bottom implies $u = 0$ at $z = 0$ and $z = h$. Since there are no sources the pressure tends to zero as time becomes infinite. Thus the solution that is relevant to this problem is $u = e^{-c\alpha^2 t} \sin \alpha z$. It is readily checked that this function is a solution of the consolidation equation. The only

boundary condition not satisfied is at $z = h$; to do this choose $\alpha h = m\pi$ where m is an integer. Therefore,

$$u(z,t) = \sum_{m=1}^{\infty} a_n \sin\left(\frac{m\pi z}{h}\right) \exp\left(-\frac{cm^2\pi^2 t}{h^2}\right)$$

and the initial uniform pressure leads to the Fourier series problem of evaluating the coefficients in

$$A = \sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi z}{h}\right)$$

so

$$\frac{h}{2}a_m = \int_0^h A \sin\left(\frac{m\pi z}{h}\right) dz = \frac{Ah}{m\pi}(1 - \cos m\pi)$$

For m, even the coefficient is zero and substituting $m = 2n+1$ provides the solution quoted in the exercise.

■ **17** Substituting $\phi = X(x)T(t)$ into the equation

$$\frac{X''}{X} = \frac{1}{c^2}\frac{\ddot{T} + K\dot{T}}{T} = -\lambda^2$$

where the separation constant has been chosen to be negative, to ensure periodic solutions for X. The variable X satisfies

$$X'' + \lambda^2 X = 0 \text{ with solution } \quad X = P\cos\lambda x + Q\sin\lambda x$$

From condition (a) the sine term must be zero so $Q = 0$. For the T equation,

$$\ddot{T} + K\dot{T} + c^2\lambda^2 T = 0$$

so trying a solution of the form $T = \exp(at)$ gives the quadratic

$$a^2 + Ka + (c\lambda)^2 = 0 \text{ with solution } a = \frac{1}{2}\left[-K \pm \sqrt{K^2 - 4(c\lambda)^2}\right]$$

The constant λ can be identified as p, so the condition that $(cp)^2 > \frac{1}{4}K^2$ gives an overall solution of the type

$$\phi = \cos px \exp\left(-\frac{Kt}{2}\right)(M\sin bt + N\cos bt)$$

where $b^2 = (cp)^2 - \frac{1}{4}K^2$.

Condition (a) requires that $N = A$

Condition (b) requires the derivative

$$\frac{\partial \phi}{\partial t} = -\frac{K}{2}\cos px \exp\left(-\frac{Kt}{2}\right)(M\sin bt + N\cos bt) + b\cos px \exp\left(-\frac{Kt}{2}\right)(M\cos bt - N\sin bt)$$

At $x = t = 0$

$$-\frac{1}{2}AK = -\frac{1}{2}NK + bM$$

and since $N = A$ then $M = 0$. The required solution is therefore

$$\phi(x,t) = A\cos px \exp\left(-\frac{Kt}{2}\right)\cos bt \text{ where } b^2 = (cp)^2 - \frac{1}{4}K^2$$

Results can be checked easily in MAPLE.

```
f:=cos(p*x)*exp(-k*t/2)*cos(b*t);
simplify(diff(f,x,x)-(diff(f,t,t)+k*diff(f,t))/c^2);
```

gives

$$\frac{-4p^2c^2 + k^2 + 4b^2}{4c^2}\cos(px)\exp\left(-\frac{1}{2}kt\right)\cos(bt)$$

■ **18** Substituting the expressions for v_r and v_θ into the continuity equation, it is satisfied for any stream function

$$\text{LHS} = \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(-\frac{\partial \psi}{\partial r}\right) = 0 = \text{RHS}$$

since the cross partial derivatives are equal for all differentiable functions. For the given stream function,

$$v_r = \frac{1}{r}U\cos\theta\left(r - \frac{a^2}{r}\right) = U\cos\theta\left(1 - \frac{a^2}{r^2}\right)$$

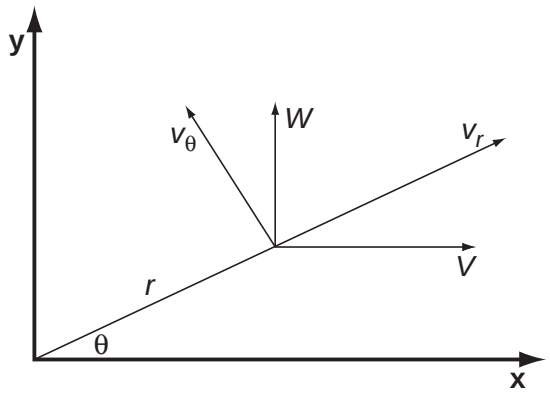
$$v_\theta = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right)$$

On the circle $r = a$, the radial velocity $v_r = 0$ and there is no flow into the circle.
As r gets very large $v_r \to U\cos\theta$ and $v_\theta \to -U\sin\theta$
The velocities parallel and perpendicular to the axes are

$$V = v_r\cos\theta - v_\theta\sin\theta$$
$$\to U$$

and

$$W = v_r\sin\theta + v_\theta\cos\theta$$
$$\to 0$$



and hence the flow far downstream is a steady flow in the x direction with velocity U. Overall the flow represents inviscid, irrotational flow past a circular obstacle in uniform flow.

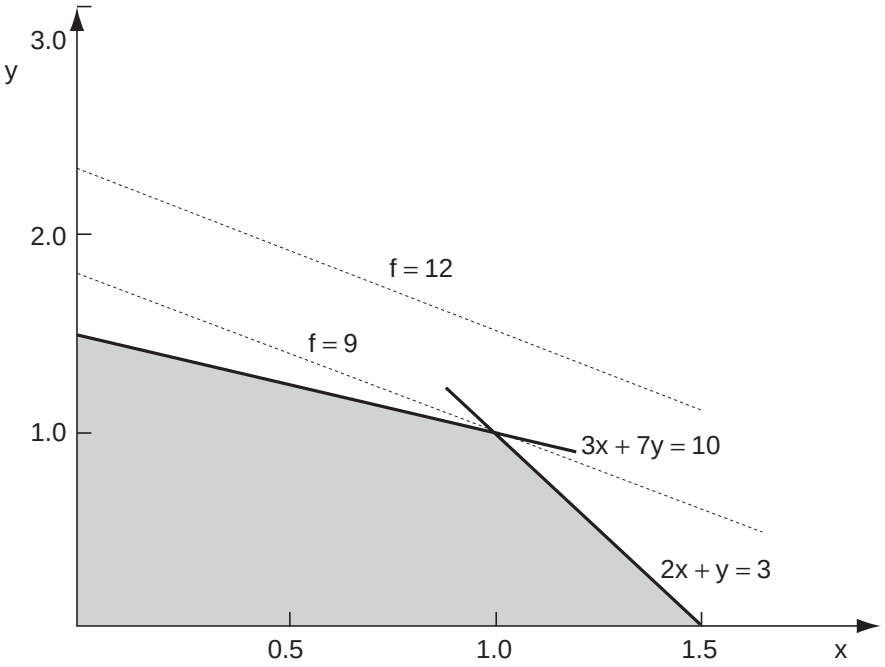
19, 20, and **21** are intended to be open exercises extending the work of the chapter to more investigative work. Consequently, no advice is offered for these problems. For Exercises 19 and 20 the text quotes sources for the work and for Exercise 21 many books on heat transfer will have a version of this problem.

10

Optimization

Exercises 10.2.4

■ **1** The required region is shaded on the graph and the lines of constant cost are all parallel to the lines labelled $f = 9$ and $f = 12$. The point where f is a maximum in the region is at the point $x = 1$ and $y = 1$ giving a maximum cost of $f = 9$.

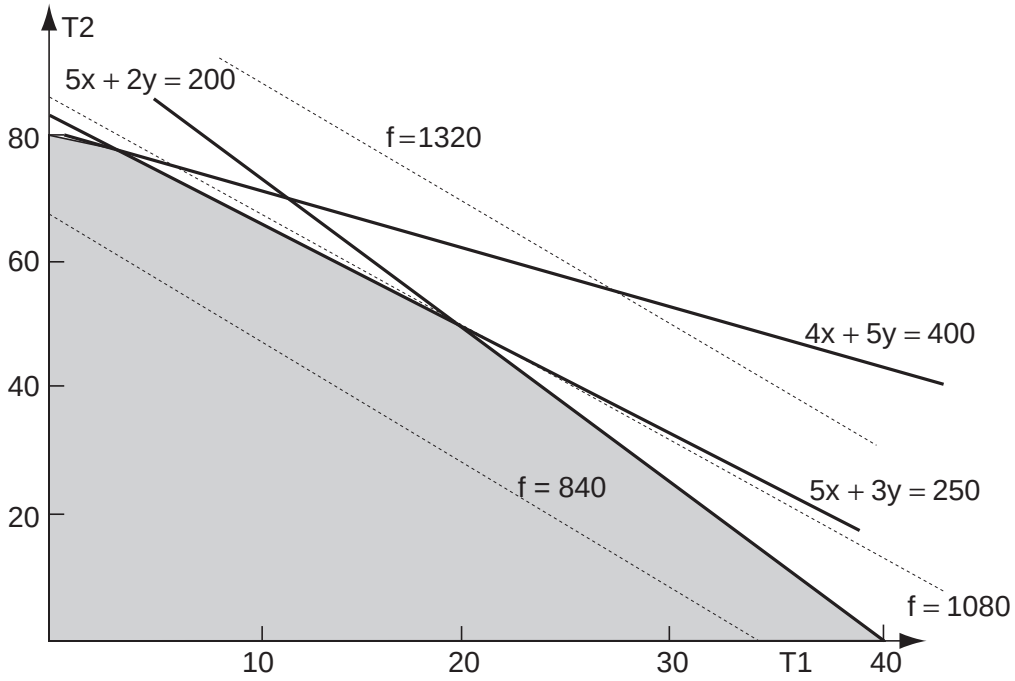


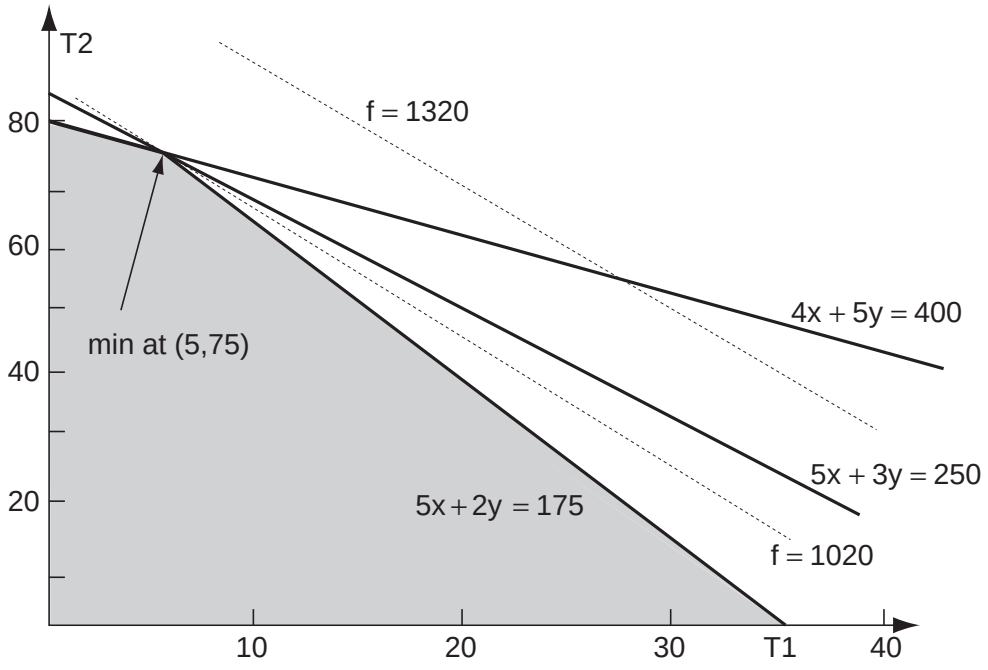
■ **2** A graphical or a tabular solution is possible but the MAPLE solution is given here.

```
with(simplex):
con2:={2*x-y<=6,x+2*y<=8,3*x+2*y<=18,y<=3};
obj2:=x+y;
maximize(obj2,con2,NONNEGATIVE);
# gives the solution {y=2, x=4}
```

■ **3** Let x be number of type 1 and y the number of type 2. The profit from these numbers is

$$f = 24x + 12y$$





and the constraints are, in the appropriate units,

chipboard	$4x + 5y \leq 400$
veneer	$5x + 2y \leq 200$
labour	$5x + 3y \leq 250$

and the obvious constraints $x \geq 0$ and $y \geq 0$. The first figure shows the feasible region bounded by the axes and the three lines $4x + 5y = 400, 5x + 2y = 200$ and $5x + 3y = 250$. These lines intersect at $x = 20, y = 50$ and give the optimum profit of £1080.

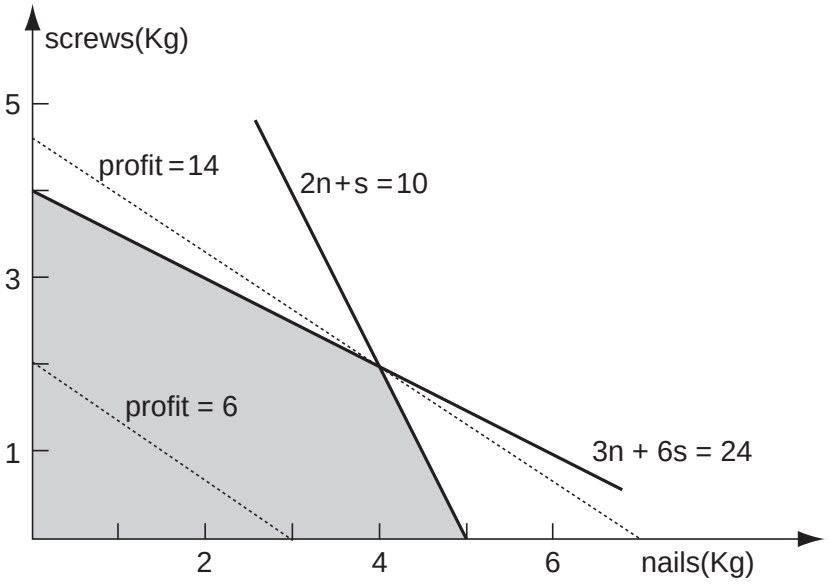
Reducing the available amount of oak veneer to 175 m, the diagram changes as in the second figure. It can be seen that the same two constraints are active. They give the solution $x = 5, y = 75$ and a reduced optimum profit of £1020.

■ **4** Let n and s be the number of kg of nails and screws respectively. The profit made is therefore $z = 2n + 3s$

The constraints are

labour	$3n + 6s \leq 24$
material	$2n + s \leq 10$

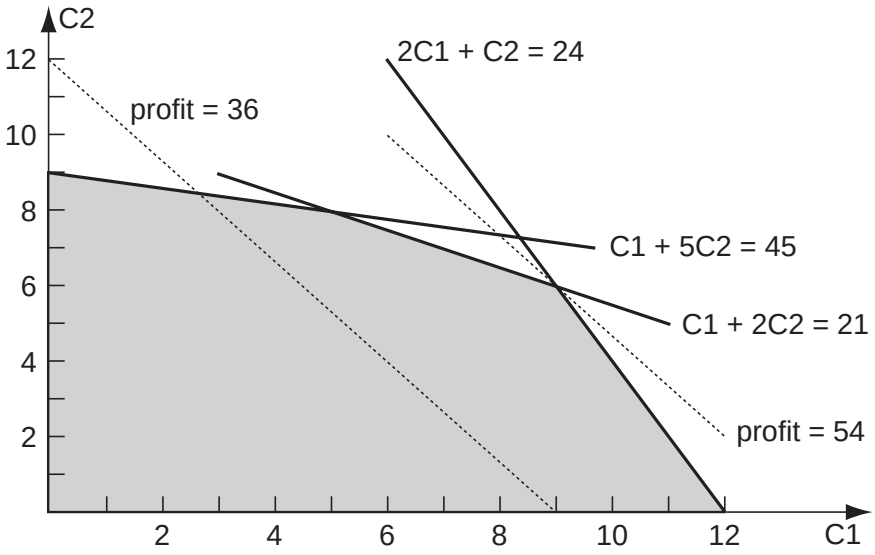
The figure shows the feasible region. The point of intersection of the two constraints gives the maximum profit of 14p making 4 kg of nails and 2 kg of screws.



■ **5** Let $C1$ and $C2$ be the number of cylinders CYL1 and CYL2 produced. The profit is $4C1+3C2$ and the constraints given by the availability of the materials are:

$$\begin{aligned} \text{M1} \quad & C1 + 5C2 \leq 45 \\ \text{M}_2 \quad & C1 + 2C2 \leq 21 \\ \text{M}_3 \quad & 2C1 + C2 \leq 24 \end{aligned}$$

It is clear from the figure that the optimum is $z = 54$ with $C1 = 9$ and $C2 = 6$. The constraints M2 and M3 are active so all these materials are used up. The constraint M1 has some slack, it may be checked that 6 units remain unused.



■ **6** Let y be the number of Yorks and w the number of Wetherbys, then the profit made is

$$z = 25y + 30w$$

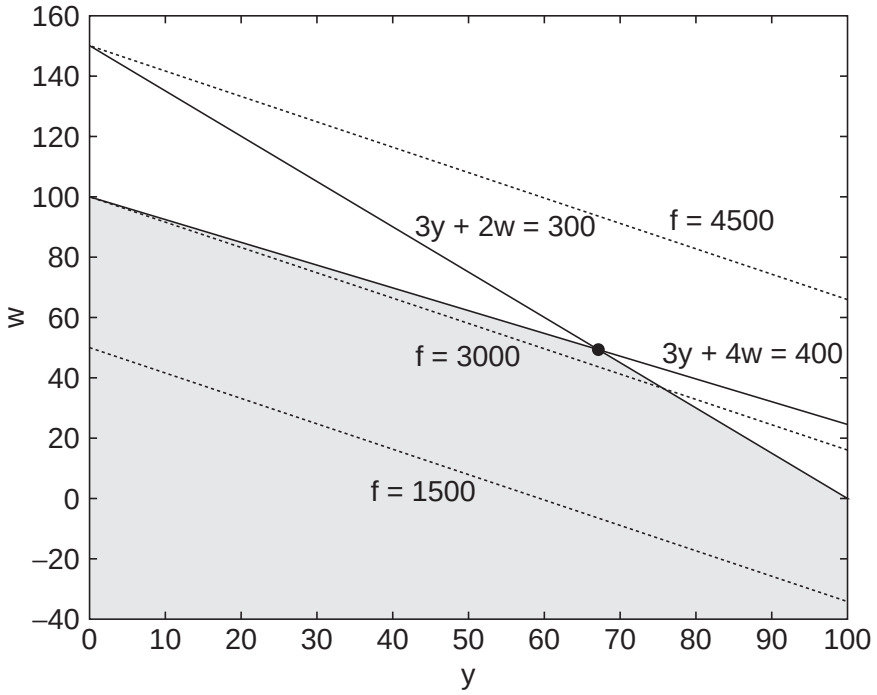
The constraints are

$$\begin{aligned} \text{cloth} \quad & 3y + 4w \leq 400 \\ \text{labour} \quad & 3y + 2w \leq 300 \end{aligned}$$

and the problem is a straightforward LP problem that can be solved graphically. It can be seen from the figure that the optimum is at $y = 66.67$ and $w = 50$ with a profit of £3166.67. Note that the solution must be integral to make sense,

two-thirds of a jacket is not much good to anyone. However, the solution is an approximate one and, to proceed correctly, it is necessary to undertake the problem as an Integer Programming problem. This is much harder and can be found in any advanced book on Mathematical Programming.

If the amount of cloth is increased then the line $3y + 4w = C$ is moved from its present $C = 400$ parallel and upwards. The intersection point moves upwards and the profit is increased. This situation continues until the solution is $y = 0$ and $w = 150$ and all the labour is used on the Wetherbys. The amount of cloth required would be 600 m and the profit in this case is £4500. This is the maximum possible profit even if unlimited cloth is available because of the labour constraint.



■ **7** The initial tableau is

	x_1	x_2	x_3	x_4	Soln
z	$-k$	-20	0	0	0
x_3	1	2	1	0	20
x_4	3	1	0	1	25

Eliminating the x_2 column first

	x_1	x_2	x_3	x_4	Soln
z	$10-k$	0	10	0	200
x_2	0.5	1	0.5	0	10
x_4	2.5	0	-0.5	1	15

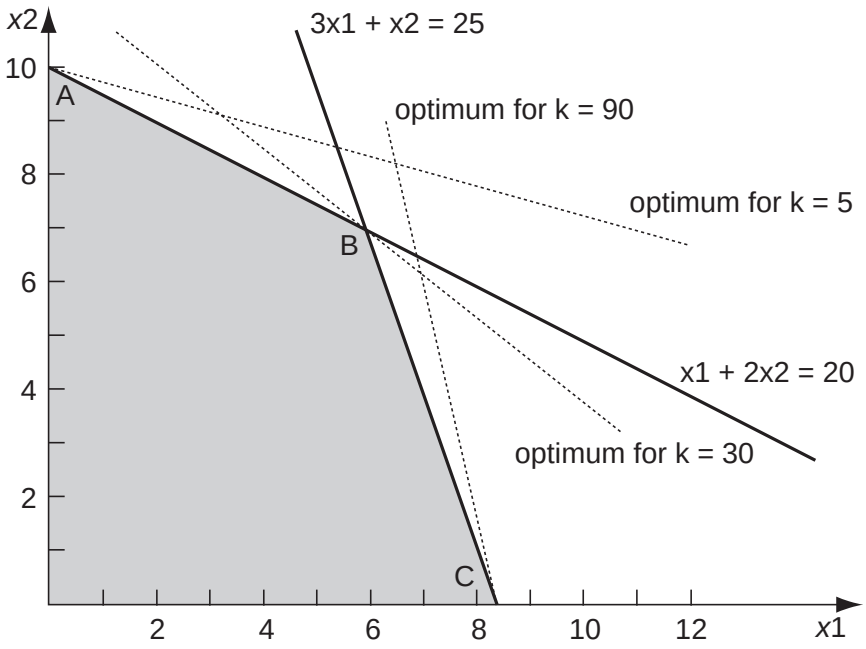
If $k < 10$ then the tableau is optimal and the solution is $x_1 = 0, x_2 = 10$ and the value of z is 200. If, however, $k > 10$ then the method continues with the x_1 column cleared.

	x_1	x_2	x_3	x_4	Soln
z	0	0	$(60-k)/5$	$2(k-10)/5$	$140+6k$
x_2	0	1	0.6	0.2	7
x_1	1	0	-0.2	0.4	6

If $10 < k < 60$ then the solution is optimal and the solution is $x_1 = 6, x_2 = 7$ and z takes the value $140 + 6k$. If, however, $k > 60$ then the solution is not optimal and a further tableau needs to be formed using the x_3 column.

	x_1	x_2	x_3	x_4	Soln
z	0	$-(60-k)/5$	0	$(7k-120)/15$	$25k/3$
x_3	0	5/3	1	1/3	35/3
x_1	1	1/3	0	7/15	25/3

For $k > 60$, the z row is positive and therefore optimal with $x_1 = 25/3, x_2 = 0$ and $z = 25k/3$.



The situation for this problem is illustrated in the figure. The feasible region is shown together with three different cost functions corresponding to the three

different cases. Note that for small k the optimum is at the corner A; as the cost steepens, the corner B is optimum; and finally as the cost steepens further with the highest values of k, the point C is the optimum.

■ **8** This problem has four variables and so it has to be solved using the simplex algorithm. There are no difficulties with this problem and the tableaux are presented, to two decimal places, without comment.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	-2	-1	-4	-1	0	0	0	0
x_5	2	0	1	0	1	0	0	3
x_6	1	0	3	1	0	1	0	4
x_7	0	4	1	1	0	0	1	3

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	-0.67	-1	0	0.33	0	1.33	0	5.33
x_5	1.67	0	0	-0.33	1	-0.33	0	1.67
x_3	0.33	0	1	0.33	0	0.33	0	1.33
x_7	-0.33	4	0	0.67	0	-0.33	1	1.67

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	-0.75	0	0	0.5	0	1.25	0.25	5.75
x_5	1.67	0	0	-0.33	1	-0.33	0	1.67
x_3	0.33	0	1	0.33	0	0.33	0	1.33
x_2	-0.08	1	0	0.17	0	-0.08	0.25	0.42

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	0	0	0	0.35	0.45	1.1	0.25	6.5
x_1	1	0	0	-0.2	0.6	-0.2	0	1
x_3	0	0	1	0.4	-0.2	0.4	0	1
x_2	0	1	0	0.15	0.05	-0.1	0.25	0.5

The solution is read off as $x_1 = 1, x_2 = 0.5, x_3 = 1$ and $x_4 = 0$; the maximum is at $z = 6.5$.

The MAPLE instructions

```
con8:= {2*x1+x3<=3,x1+3*x3+x4<=4,4*x2+x3+x4<=3};
obj8:=2*x1+x2+4*x3+x4;
maximize(obj8,con8,NONNEGATIVE);
```

and the MATLAB instructions

```
f=[-2;-1;-4;-1];A=[2,0,1,0;1,0,3,1;0,4,1,1];b=[3;4;3];
options=optimset('LargeScale','off','Simplex','on');
[x,fval]=linprog(f,A,b,[],[],zeros(4,1),[],[],options)
```

also produce the solution {`x4=0, x1=1, x3=1, x2=1/2`}.

■ **9** If $b1$,$b2$,$b3$ are the respective number $(\times 1000)$ of books printed then the profit will be

$$z = 900b1 + 800b2 + 300b3$$

and the constraints are

sales restriction	$b1 + b2 \leq 15$
paper	$3b1 + 2b2 + b3 \leq 60$

The first tableau is easily set up and the other tableaux follow by the usual rules.

	$b1$	$b2$	$b3$	$b4$	$b5$	Soln
z	-900	-800	-300	0	0	0
$b4$	1	1	0	1	0	15
$b5$	3	2	1	0	1	60

	$b1$	$b2$	$b3$	$b4$	$b5$	Soln
z	0	100	-300	900	0	13,500
$b1$	1	1	0	1	0	15
$b3$	0	-1	1	-3	1	15

	$b1$	$b2$	$b3$	$b4$	$b5$	Soln
z	0	-200	0	0	300	18,000
$b1$	1	1	0	1	0	15
$b3$	0	-1	1	-3	1	15

	$b1$	$b2$	$b3$	$b4$	$b5$	Soln
z	200	0	0	200	300	21,000
$b2$	1	1	0	1	0	15
$b3$	1	0	1	-2	1	30

Thus the optimal profit is made if none of book 1, 15,000 of book 2 and 30,000 of book 3 are printed giving a profit of £21,000.

■ **10** Let L, M, S be the respective number of long, medium and short range aircraft purchased. The profit is

$$z = 0.4L + 0.3M + 0.15S$$

and the constraints are

money available	$4L + 2M + S \leq 60$
pilots	$L + M + S \leq 25$
maintenance	$2L + 1.5M + S \leq 30$

The initial tableau is

	L	M	S	P	Q	R	Soln
z	-0.4	-0.3	-0.15	0	0	0	0
P	4	2	1	1	0	0	60
Q	1	1	1	0	1	0	25
R	2	1.5	1	0	0	1	30

The first column is chosen, since it has the most negative entry, and when the ratios are calculated, the P and R rows produce the same value of 15. From the two, the P row is chosen arbitrarily.

	L	M	S	P	Q	R	Soln
z	0	-0.1	-0.05	0.1	0	0	6
L	1	0.5	0.25	0.25	0	0	15
Q	0	0.5	0.75	-0.25	1	0	10
R	0	0.5	0.5	-0.5	0	1	0

In this tableau the M, R element is the pivot and performing the elimination gives

	L	M	S	P	Q	R	Soln
z	0	0	0.05	0	0	0.2	6
L	1	0	-0.25	0.75	0	-1	15
Q	0	0	0.25	0.25	1	-1	10
M	0	1	1	-1	0	2	0

Note that the final column has not changed between the last two tableaux because of the zero in the very last entry. The optimal solution is to purchase 15 long range aircraft and no medium or short range aircraft; the estimated profit is £6 m.

In the problem, it would seem that some operational constraints have been omitted. In many optimisation problems it takes several steps to achieve a sensible cost and set of constraints.

■ **11** The first tableau is set up from the data. Choosing the pivot in the usual way, x_1 and x_5 are interchanged and the elimination follows the basic rules. Again, the pivot is found and x_3 and x_6 are interchanged; the elimination produces a top row that is all positive so the solution is optimal. The solution is read from the tableau as $x_1 = 1.5,\ x_2 = 0,\ x_3 = 2.5,\ x_4 = 0$ and $f = 14$.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln	Ratio
z	−6	−1	−2	−4	0	0	0	0	
x_5	2	1	0	1	1	0	0	3	1.5
x_6	1	0	1	1	0	1	0	4	4
x_7	1	1	3	2	0	0	1	10	10

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln	Ratio
z	0	2	−2	−1	3	0	0	9	
x_1	1	0.5	0	0.5	0.5	0	0	1.5	
x_6	0	−0.5	1	0.5	−0.5	1	0	2.5	2.5
x_7	0	0.5	3	1.5	−0.5	0	1	8.5	2.83

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln	Ratio
z	0	1	0	0	2	2	0	14	
x_1	1	0.5	0	0.5	0.5	0	0	1.5	
x_3	0	−0.5	1	0.5	−0.5	1	0	2.5	
x_7	0	2	0	0	1	−3	1	1	

The MAPLE implementation just gives the 'answer'. Some of the detail can be extracted from MAPLE and the various tableau can be identified.

```
con11:={2*x1+x2+x4<=3,x1+x3+x4<=4,x1+x2+3*x3+2*x4<=10};
obj11:=6*x1+x2+2*x3+4*x4;
```

```
maximize(obj11,con11,NONNEGATIVE);
# gives the same solution {x4=0, x2=0, x1=3/2, x3=5/2}
# for the detail
z:=setup(con11);
          z:={_SL1=3-2 x1-x2-x4,_SL2=4-x1-x3-x4,
                    _SL3=10-x1-x2-3 x3-2 x4}
piv:=pivoteqn(z,x1);
          piv:=[_SL1=3-2 x1-x2-x4]
z:=pivot(z,x1,piv);
          z:={x1=3/2-1/2_SL1-1/2 x2-1/2 x4,
               _SL2=5/2+1/2_SL1+1/2 x2-1/2 x4-x3,
                   _SL3=17/2+1/2 _SL1-1/2 x2-3/2 x4-3 x3}


obj11:=eval(obj11,z);
          obj11:= 9-3_SL1 - 2x2 + x4 + 2 x3
# compare with second tableau
piv:= pivoteqn(z,x3);
          piv:=[_SL2=5/2 + 1/2_SL1 + 1/2x2 - 1/2x4-x3]
z:=pivot(z,x3,piv);
          z:= {_SL3=-_SL1 + 1 + 3_SL2 - 2 x2,
                  x1 = 3/2 - 1/2_SL1 - 1/2 x2 - 1/2 x4,
                         x3=1/2_SL1 + 5/2 -_SL2 + 1/2 x2
                              - 1/2 x4}
obj11:=eval(obj11,z);
          obj11:=-2_SL2 + 14 - 2_SL1-x2
# compare with the final tableau
```

Note that the entry in position (z, x_4) is zero so we expect many solutions – this is easily identified from the tableau but not from the MAPLE results. Continuing with MAPLE code

```
piv:=pivoteqn(z,x4);
       piv:=[x1=3/2 - 1/2_SL1 - 1/2 x2 - 1/2 x4]
z:=pivot(z,x4,piv);
```

```
z:={_SL3 = -_SL1 + 1 + 3_SL2 - 2 x2,
          x4 = -2 x1+3 -_SL1 - x2,
             x3 =_SL1 + 1 -_SL2 + x2 + x1}
obj11:=eval(obj11,z);
       obj11:=-2_SL2+14-2_SL1-x2
```

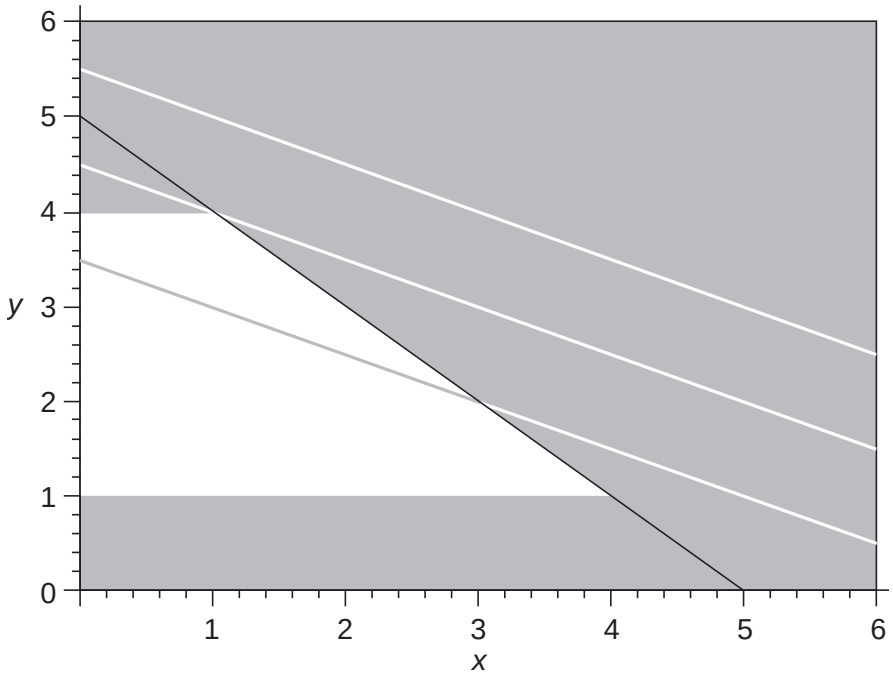
The new solutions are $x_1 = x_2 = 0, x_3 = 1, x_4 = 3$ and $f = 14$, giving, as expected, the same function value.

The MATLAB instructions give the same result but no detail

```
f=[-6;-1;-2;-4];A=[2,1,0,1;1,0,1,1;1,1,3,2];b=[3;4;10];
options=optimset('LargeScale','off','Simplex','on');
[x,fval]=linprog(f,A,b,[],[],zeros(4,1),[],[],options)
```

Exercises 10.2.6

■ **12** The problem can be easily plotted and it is clear that $x = 1, y = 4$ gives the solution $f = x + 2y = 9$.



The MAPLE check is as follows:

```
with (simplex):
con12:={y>=1,y<=4,x+y<=5};
obj12:=x+2*y;
maximize(obj12,con12,NONNEGATIVE);
# gives      {y=4,x=1}
```

The graph can be plotted using the instructions

```
with(plots):
F:=inequal(con12,x=0..6,y=0..6):
G:=plot([(-x+7)/2,(-x+9)/2,(-x+11)/2],x=0..6,
thickness=3,labels=['x','y'],color=yellow):
display(F,G);
```

■ **13** The problem has 'greater than' constraints, so the two-phase method is required. Note that a surplus variable x_4 and an artificial variable x_7 need to be introduced. At the end of phase 1 the artificial variable x_7 will be eliminated from the tableau. The cost function in phase 1 is $-x_7$, but recall that this cost has to be modified to ensure that the tableau is in standard form.

Phase 1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	−2	−1	0	1	0	0	0	−12
x_3	4	1	1	0	0	0	0	32
x_7	2	1	0	−1	0	0	1	12
x_5	2	−1	0	0	1	0	0	4
x_6	−2	1	0	0	0	1	0	8

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	0	−2	0	1	1	0	0	−8
x_3	0	3	1	0	−2	0	0	24
x_7	0	2	0	−1	−1	0	1	8
x_1	1	−0.5	0	0	0.5	0	0	2
x_6	0	0	0	0	1	1	0	12

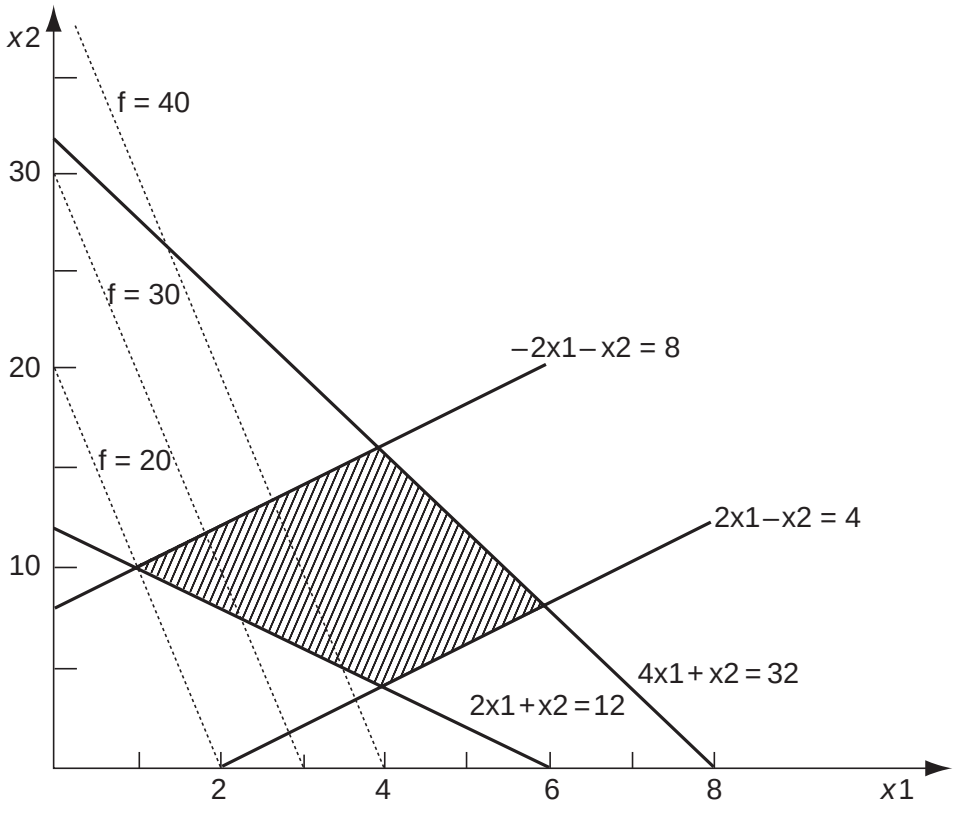
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	0	0	0	0	0	0	1	0
x_3	0	0	1	1.5	−0.5	0	−1.5	12
x_2	0	1	0	−0.5	−0.5	0	0.5	4
x_1	1	0	0	−0.25	0.25	0	0.25	4
x_6	0	0	0	0	1	1	0	12

The artificial cost has been driven to zero so the artificial variable x_7 can be eiiminated and the original cost function reinstated.

Phase 2

Note that the new cost is negative because the problem is a minimisation problem.

	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	0	0	0	3	−2	0	−44
x_3	0	0	1	1.5	−0.5	0	12
x_2	0	1	0	−0.5	−0.5	0	4
x_1	1	0	0	−0.25	0.25	0	4
x_6	0	0	0	0	1	1	12



	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	0	0	0	3	0	2	-20
x_3	0	0	1	1.5	0	0.5	18
x_2	0	1	0	-0.5	0	0.5	10
x_1	1	0	0	-0.25	0	-0.25	1
x_5	0	0	0	0	1	1	12

The solution is read off as $x_1 = 1, x_2 = 10$ with a minimal cost of 20.

■ **14** Let S be the number of shoes produced and B the number of boots, then the problem is to maximize the profit

$$z = 8B + 5S$$

production	$2B + S \leq 250$
sales	$B + S \leq 200$
customer	$B \geq 25$

The tableaux are constructed in the usual way.

Phase 1

	B	S	P	Q	R	T	Soln
z	-1	0	0	0	1	0	-25
P	2	1	1	0	0	0	250
Q	1	1	0	1	0	0	200
T	1	0	0	0	-1	1	25

	B	S	P	Q	R	T	Soln
z	0	0	0	0	0	1	0
P	0	1	1	0	2	-2	200
Q	0	1	0	1	1	-1	175
B	1	0	0	0	-1	1	25

Phase 2

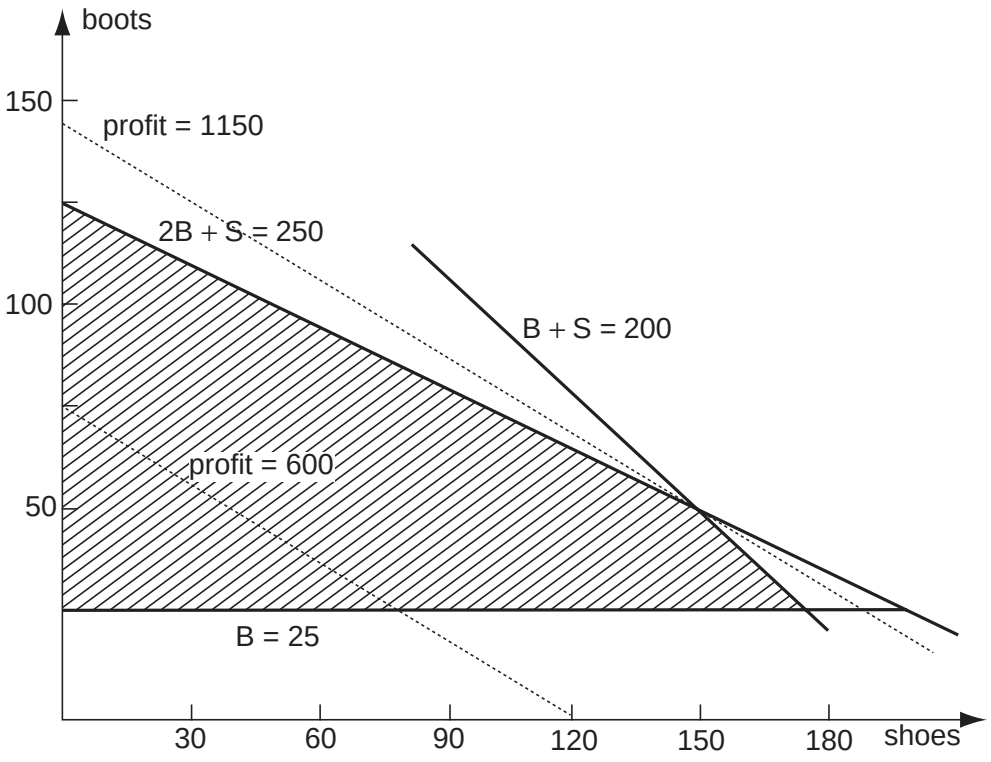
	B	S	P	Q	R	Soln
z	0	-5	0	0	-8	200
P	0	1	1	0	2	200
Q	0	1	0	1	1	175
B	1	0	0	0	-1	25

	B	S	P	Q	R	Soln
z	0	-1	4	0	0	1000
R	0	0.5	0.5	0	1	100
Q	0	0.5	-0.5	1	0	75
B	1	0.5	0.5	0	0	125

	B	S	P	Q	R	Soln
z	0	0	3	2	0	1150
R	0	0	1	-1	1	25
S	0	1	-1	2	0	150
B	1	0	1	-1	0	50

Thus, the manufacturer should make 50 pairs of boots and 150 pairs of shoes and the maximum profit is £1150.

A graphical solution is possible since the problem has only two variables. It is of interest to follow the progress of the simplex solution on the graph.



■ **15** There are techniques for adding a constraint to an existing solution and thence determining the solution. However, here the whole problem will be recomputed. From Exercise 9 the initial tableau can be constructed, but the additional constraint is a 'greater than' constraint so the procedure for entering phase 1 must be followed.

	$b1$	$b2$	$b3$	$b4$	$b5$	$b6$	$b7$	Soln
z	-1	-1	-1	0	0	1	0	-50
$b4$	1	1	0	1	0	0	0	15
$b5$	3	2	1	0	1	0	0	60
$b7$	1	1	1	0	0	-1	1	50

Note that the last row gives the new constraint and that the z row involves the artificial variable $b7$, which has been eliminated to bring the tableau to standard form.

Phase I

	$b1$	$b2$	$b3$	$b4$	$b5$	$b6$	$b7$	Soln
z	0	0	-1	1	0	1	0	-35
$b1$	1	1	0	1	0	0	0	15
$b5$	0	-1	1	-3	1	0	0	15
$b7$	0	0	1	-1	0	-1	1	35

	$b1$	$b2$	$b3$	$b4$	$b5$	$b6$	$b7$	Soln
z	0	-1	0	-2	1	1	0	-20
$b1$	1	1	0	1	0	0	0	15
$b3$	0	-1	1	-3	1	0	0	15
$b7$	0	1	0	2	-1	-1	1	20

	$b1$	$b2$	$b3$	$b4$	$b5$	$b6$	$b7$	Soln
z	0	0	0	0	0	0	1	0
$b1$	1	0.5	0	0	0.5	0.5	-0.5	5
$b3$	0	0.5	1	0	-0.5	-1.5	1.5	45
$b4$	0	0.5	0	1	-0.5	-0.5	0.5	10

Phase 2

	$b1$	$b2$	$b3$	$b4$	$b5$	$b6$	Soln
z	0	-200	0	0	300	0	18,000
$b1$	1	0.5	0	0	0.5	0.5	5
$b3$	0	0.5	1	0	-0.5	-1.5	45
$b4$	0	0.5	0	1	-0.5	-0.5	10

	$b1$	$b2$	$b3$	$b4$	$b5$	$b6$	Soln
z	400	0	0	0	500	200	20,000
$b1$	2	1	0	0	1	1	10
$b3$	-1	0	1	0	-1	-2	40
$b4$	1	0	0	1	-1	-1	5

The solution is optimal with the production schedule as none for book 1, 10,000 of book 2 and 40,000 of book 3. The profit is down to £20,000 because of the additional constraint.

■ **16** The MAPLE solution is presented.

```
with (simplex):
con16:={x>=1,x+2*y<=3,y+3*z<=4};
obj16:=x+y+z;
maximize(obj16,con16,NONNEGATIVE);
# gives the solution {y=0,z=4/3,x=3}, f=x+y+z=13/3
```

Note that there is no detail of the two-phase method and just the 'answer' is given. It is not easy to rewrite phase 1 into MAPLE so that the detail can be extracted. The same is true of MATLAB, the following instructions produce the correct result

```
f=[-1;-1;-1];A=[1,2,0;0,1,3;-1,0,0];b=[3;4;-1];
options=optimset('LargeScale','off,'Simplex','on');
[x,fval]=linprog(f,A,b,[],[],zeros(3,1),[],[],options)
```

■ **17** This is a standard two-phase problem with surplus variables x_5, x_6 and artificial variables x_7, x_8. With the cost constructed from the artificial variables x_7, x_8, and the tableau reduced to standard form, the sequence of tableaux is presented.

Phase 1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Soln
z	-1	-1	0	1	1	1	0	0	-2
x_7	1	0	-1	-1	-1	0	1	0	0
x_9	0	1	1	0	0	-1	0	1	2

There is an arbitrary choice of the columns since two columns have the value -1; the second one is chosen.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Soln
z	-1	0	1	1	1	0	0	1	0
x_7	1	0	-1	-1	-1	0	1	0	0
x_2	0	1	1	0	0	-1	0	1	2

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Soln
z	0	0	0	0	0	0	1	1	0
x_1	1	0	-1	-1	-1	0	1	0	0
x_2	0	1	1	0	0	-1	0	1	2

The solution is optimal so phase 1 ends and the tableau is reconstituted for phase 2.
Phase 2

	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	0	0	-1	7	2	7	-14
x_1	1	0	-1	-1	-1	0	0
x_2	0	1	1	0	0	-1	2

	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	0	1	0	7	2	6	-12
x_1	1	1	0	-1	-1	-1	2
x_3	0	1	1	0	0	-1	2

The solution is now optimal with $x_1 = 2, x_2 = 0, x_3 = 2, x_4 = 0$ and the cost function being 12.

■ **18** Let the company buy *a,b,c* litres of the products A,B,C, then the cost of the materials will be

Cost	$1.8a + 0.9b + 1.5c$
For a total of 100 litres	$a + b + c = 100$
Glycol	$0.65a + 0.25b + 0.8c \geq 50$
Additive	$0.1a + 0.03b \geq 5$

There are two 'greater than' constraints and an equality constraint. Recall that an equality constraint is dealt with via an artificial variable. Thus, phase 1 is entered

with two surplus variables and three artificial variables; the cost row is manipulated until the tableau has the standard form.

Phase 1

	a	b	c	p	q	r	s	t	Soln
z	−1.75	−1.28	−1.8	1	1	0	0	0	−155
r	1	1	1	0	0	1	0	0	100
s	0.65	0.25	0.8	−1	0	0	1	0	50
t	0.1	0.03	0	0	−1	0	0	1	5

	a	b	c	p	q	r	s	t	Soln
z	−0.29	−0.72	0	−1.25	1	0	2.25	0	−42.5
r	0.19	0.69	0	1.25	0	1	−1.25	0	37.5
c	0.81	0.31	1	−1.25	0	0	1.25	0	62.5
t	0.1	0.03	0	0	−1	0	0	1	5

	a	b	c	p	q	r	s	t	Soln
z	−0.1	−0.03	0	0	1	1	1	0	−5
p	0.15	0.55	0	1	0	0.8	−1	0	30
c	1	1	1	0	0	1	0	0	100
t	0.1	0.03	0	0	−1	0	0	1	5

	a	b	c	p	q	r	s	t	Soln
z	0	0	0	0	0	1	1	1	0
p	0	0.51	0	1	1.5	0.8	−1	−1.5	22.5
c	0	0.7	1	0	10	1	0	−10	50
a	1	0.3	0	0	−10	0	0	10	50

The artificial cost has been driven to zero so phase 1 is complete and the three artificial columns are deleted and the actual cost introduced. Recall that this is a minimisation problem so the phase 2 tableau can now be extracted from the tableau above.

Phase 2

	a	b	c	p	q	Soln
z	0	−0.69	0	0	3	−165
p	0	0.51	0	1	1.5	22.5
c	0	0.7	1	0	10	50
a	1	0.3	0	0	−10	50

	a	b	c	p	q	Soln
z	0	0	0	1.37	5.05	−134.26
b	0	1	0	1.98	2.97	44.55
c	0	0	1	−1.39	7.92	18.81
a	1	0	0	−0.59	−10.89	36.63

The tableau is optimal so the solution can be read off as $a = 36.63\%$, $b = 44.55\%, c = 18.81\%$ and a minimum cost of £134.26.
MATLAB solves the problem as follows

```
f=[1.8;0.9;1.5];A=[-0.65,-0.25,-0.8;-0.1,-0.03,0];b=[-50;-5];
options=optimset('LargeScale','off,'Simplex','on');
[x,fval]=linprog(f,A,b,[1,1,1],[100],zeros(3,1),[],[],options)
```

% Note how MATLAB deals with the equality constraint

■ **19** Let s_1, s_2, s_3 be the number of houses of the three styles the builder decides to construct. His profit ($\times$ £100) is

$$10s_1 + 15s_2 + 25s_3$$

and the constraints are

plots	$s_1 + 2s_2 + 2s_3 \leq 40$
facing stone	$s_1 + 2s_2 + 5s_3 \leq 58$
weather boarding	$3s_1 + 2s_2 + s_3 \leq 72$
local authority	$-s_1 + s_2 \geq 5$

The solution requires the two-phase method. The tableaux are listed, to two decimal places, without comment.
Phase 1

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	Soln
z	1	−1	0	0	0	0	1	0	−5
s_4	1	2	2	1	0	0	0	0	40
s_5	1	2	5	0	1	0	0	0	58
s_6	3	2	1	0	0	1	0	0	72
s_8	−1	1	0	0	0	0	−1	1	5

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	Soln
z	0	0	0	0	0	0	0	1	0
s_4	3	0	2	1	0	0	2	−2	30
s_5	3	0	5	0	1	0	2	−2	48
s_6	5	0	1	0	0	1	2	−2	62
s_2	−1	1	0	0	0	0	−1	1	5

Phase 2

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	Soln
z	−25	0	−25	0	0	0	−15	75
s_4	3	0	2	1	0	0	2	30
s_5	3	0	5	0	1	0	2	48
s_6	5	0	1	0	0	1	2	62
s_2	−1	1	0	0	0	0	−1	5

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	Soln
z	0	0	−8.33	8.33	0	0	1.67	325
s_1	1	0	0.67	0.33	0	0	0.67	10
s_5	0	0	3	−1	1	0	0	18
s_6	0	0	−2.33	−1.67	0	1	−1.33	12
s_2	0	1	0.67	0.33	0	0	−0.33	15

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	Soln
z	0	0	0	5.56	2.78	0	1.67	375
s_1	1	0	0	0.56	−0.22	0	0.67	6
s_3	0	0	1	−0.33	0.33	0	0	6
s_6	0	0	0	−2.44	0.78	1	−1.33	26
s_2	0	1	0	0.56	−0.72	0	−0.33	11

The tableau is optimal, so the solution should build 6,11 and 6 houses of styles 1,2 and 3 respectively and make a profit of £37,500.

---

20 Let $x_1 x_2, x_3$ be the amounts ($\times 1000$ m^2) of carpet of type C1,C2,C3 produced, then the maximum of

$$2x_1 + 3x_2$$

is required subject to the constraints

M1	$x_1 + x_2 + x_3 \leq 5$
M2	$x_1 + x_2 \leq 4$
policy1	$x_1 \geq 1$
policy2	$x_1 - x_2 + x_3 \geq 2$

Phase 1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	Soln
z	2	1	−1	0	0	1	1	0	0	−3
x_4	1	1	1	1	0	0	0	0	0	5
x_5	1	1	0	0	1	0	0	0	0	4
x_6	1	0	0	0	0	−1	0	1	0	1
x_9	1	−1	1	0	0	0	−1	0	1	2

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	Soln
z	0	1	1	0	0	−1	1	2	0	−1
x_4	0	1	1	1	0	1	0	−1	0	4
x_5	0	1	0	0	1	1	0	−1	0	3
x_1	1	0	0	0	0	−1	0	1	0	1
x_9	0	−1	1	0	0	1	−1	−1	1	1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	Soln
z	0	0	0	0	0	0	0	1	1	0
x_4	0	2	0	1	0	0	1	0	−1	3
x_5	0	1	0	0	1	1	0	−1	0	3
x_1	1	0	0	0	0	−1	0	1	0	1
x_3	0	−1	1	0	0	1	−1	−1	1	1

Phase 2

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	0	3	0	0	0	−2	0	2
x_4	0	8	0	1	0	0	1	9
x_5	0	1	0	0	1	1	0	3
x_1	1	0	0	0	0	−1	0	1
x_3	0	1	1	0	0	1	−1	1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	0	0	0	1.5	0	−2	1.5	6.5
x_2	0	1	0	0.5	0	0	0.5	1.5
x_4	0	0	0	0.5	1	1	0.5	1.5
x_1	1	0	0	0	0	0	0	1.5
x_3	0	0	1	0.5	0	0	−0.5	2.5

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	0	0	0	0.5	2	0	0.5	9.5
x_2	0	1	0	0.5	0	0	0.5	1.5
x_6	0	0	0	-0.5	1	1	-0.5	1.5
x_1	1	0	0	-0.5	1	0	-0.5	2.5
x_3	0	0	1	1	-1	0	0	1

Hence, the optimum is $2500\,\mathrm{m}^2$ of C1, $1500\,\mathrm{m}^2$ of C2 and $1000\,\mathrm{m}^2$ of C3 giving a profit of £9500.

Exercises 10.3.3

■ **21** Introducing a Lagrange multiplier, the minimum is obtained from

$$2x + y + \lambda = 0$$

$$x + 2y + \lambda = 0$$

$$x+y = 1$$

with solution $x = y = 1/2$. To prove that it is a minimum, put

$$x = \tfrac{1}{2} + \varepsilon \text{ and } y = \tfrac{1}{2} - \varepsilon$$

and substitute into f to give

$$f = \tfrac{3}{4} + \varepsilon^2$$

and hence a minimum.

■ **22** The answer to this exercise is geometrically obvious since it is just the length of the minor axis which in this case, is a. It is required to find the minimum of

$$D^2 = x^2 + y^2 \text{ subject to } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The Lagrange equations are

$$0 = 2x + \frac{2\lambda x}{a^2} \text{ and } 0 = 2y + \frac{2\lambda y}{b^2}$$

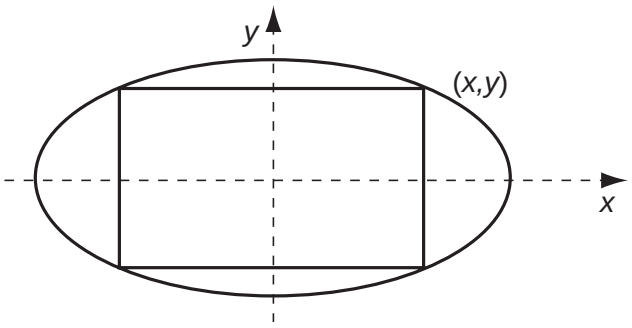
Since λ cannot equal $-a^2$ and $-b^2$ at the same time then either $x = 0$ or $y = 0$. If $a < b$ the it is clear that $y = 0$ and $x = \pm a$ gives the minimum

---

■ **23** The area of the rectangle is

$$A = 4xy$$

and since the point must lie on the ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and hence a Lagrange multiplier problem is produced. The two equations are

$$0 = 4y + \frac{2\lambda x}{a^2} \text{ and } 0 = 4x + \frac{2\lambda y}{b^2}$$

It is soon checked that the required solution is $x = a/\sqrt{2},\ y = b/\sqrt{2},\ A = 2ab$.

■ **24** The necessary conditions for an optimum are

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0 = y^2 z + \lambda \tag{1}$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0 = 2xyz + 2\lambda \tag{2}$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z} = 0 = xy^2 + 3\lambda \tag{3}$$

together with the given constraint

$$x + 2y + 3z = 6 \tag{4}$$

Dividing equation (2) by two and then subtracting the first two equations gives

$$yz(x - y) = 0$$

Thus, there are three cases

$\underline{y = 0}$ Note that all the equations except the constraint are satisfied since $\lambda = 0$. Thus, a possible solution is $(6 - 3\alpha, 0, \alpha)$ for any α.

$\underline{z = 0}$ Again, this implies that $\lambda = 0$ so from equation (3) $x = 0$ and hence from (4) $y = 3$. The case when $y = 0$ in equation (3) has been covered in the first case.

$\underline{x = y}$ Equations (2) and (3) give $\frac{1}{3}x^3 = x^2 z$ so there are two cases

either $x = 0$ and hence $y = 0$ and $z = 2$

$$\text{or } x = 3z \text{ and hence from (4) } z = 1/2 \text{ and } x = 3/2, y = 3/2$$

Equations (1) and (2) have been used to eliminate λ; so a similar exercise must be undertaken with (2) and (3) to check whether any new solutions arise. They give the equation

$$xy(y - 3z) = 0$$

and again three cases must be studied.

$\underline{x = 0}$ In this case $\lambda = 0$ so it reduces to one of the above.

$\underline{y = 0}$ Again, it reduces to the first case above.

$\underline{y = 3z}$ From equations (1) and (2) it can be seen that the same cases arise and correspond to the solutions already obtained.

Note that in the solutions quoted, (6,0,0) is included in the solution $(6 - 3\alpha, 0, \alpha)$ with $\alpha = 0$.

■ **25** Using λ and μ as the Lagrange multipliers, the equations that give the optimum are as quoted

$$0 = 2x + \lambda + (2z - 2y)\mu$$
$$0 = 2y + \lambda + (z - 2x)\mu$$
$$0 = 2z - \lambda + (y + 2x)\mu$$

together with the two given constraints.

Adding the last two equations

$$2(y + z) + (y + z)\mu = 0 \Rightarrow \text{ either } y = -z \text{ or } \mu = -2$$

Adding the first and last equations

$$2(x + z) + (2z + 2x - y)\mu = 0$$

So, λ has been eliminated.

$\underline{y + z = 0}$ In the first constraint $x = -y + z = 2z$. Thus, $(2t, -t, t)$ gives, in the second constraint, $7t^2 = 1$ so $t = \pm 1/\sqrt{7}$ and two of the quoted solutions are obtained.

$\underline{\mu = -2}$ From above $2(x + z) + (2z + 2x - y)(-2) = 0$ and so this expression and the second constraint give a pair of equations

$$0 = -x + y - z$$
$$0 = x + y - z$$

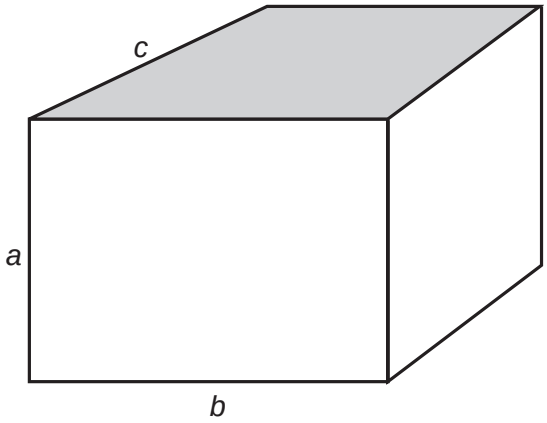
and hence, $x = 0$ and $y = z$. Putting these values into the second constraint gives $y = \pm 1$ and possible optimum points at $(0,\ 1,\ 1)$ and $(0, -1, -1)$.

■ **26** The volume of the figure is

$$V = abc$$

and the surface area is

$$A = 2ab + 2ac + bc$$



The problem is a Lagrange multiplier problem and the three equations are

$$0 = bc + \lambda(2b + 2c)$$
$$0 = ac + \lambda(2a + c)$$
$$0 = ab + \lambda(2a + b)$$

The last two equations give

$$a(c - b) + \lambda(c - b) = 0 \Rightarrow \text{either } c = b \text{ or } \lambda = -a$$

$\underline{c = b}$ Putting back into the first equation gives

$$b^2 + 4b\lambda = 0 \text{ so either } b = 0 \text{ or } \lambda = -b/4$$

The case $b = 0$ can be dismissed as geometrically uninteresting. In the third equation this value of λ gives $0 = ab - \frac{1}{4}b(2a + b) \Rightarrow b = 2a$ since again the case $b = 0$ is uninteresting.
Calculating a gives the possible maximum as

$$c = b = 2a \text{ with } a = \sqrt{\frac{A}{12}} \text{ and } V = 4a^3$$

$\underline{\lambda = -a}$ Putting this value back into the third of the Lagrange equations gives $a = 0$ so gives zero volume and is geometrically uninteresting again.

■ **27** This is a very tough problem but more typical of realistic problems in engineering rather than most of the illustrative exercises in the book.

The exercise will be solved using MAPLE. Note the efficient way it performs the algebra and integrations for the simplest approximation

```
z:=A*cos(Pi*x/2)^2*cos(Pi*y/2)^2;
I1:int(int(z^2,x=-1..1),y=-1..1);
# gives
                          I1:=9/16 A^2
f:=diff(z,x,x)+diff(z,y,y);
I2:=int(int(f^2,x=-1..1),y=-1..1);
# gives
                        I2:=1/2 A^2 Pi^4
freq:=I2/I1;
# gives
                        freq:=8/9 Pi^4 = 86.58 as required.
```

The first approximation could be done by hand but the next approximation needs a package like MAPLE.

```
zz:=cos(Pi*x/2)^2*cos(Pi*y/2)^2
    *(A+B*cos(Pi*x/2)*cos(Pi*y/2));
I3:=evalf(int(int(zz^2,x=-1..1),y=-1..1));
# gives
I3:=.5624999999 A^2 + .9222479291 A B + .3906249999 B^2
ff:=diff(zz,x,x)+diff(zz,y,y);
I4:=evalf(int(int(ff^2,x=-1..1),y=-1..1));
# gives
    I4:=58.21715209 B^2 + 48.70454554 A^2 + 92.64268668 A B
dLbydA:=diff(I4+lam*I3,A);
dLbydB:=diff(I4+lam*I3,B);
fsolve({dLbydA,dLbydB,I3=1}, {A,B,lam});
# gives the solution
    {lam=-81.38454660, A=-1.829151961, B=.6086242001}
```

The frequency is given by $-lam$ which should be compared with the first approximation.

■ **28** The problem asks for a minimum, which is equivalent to the maximum of

$$-2x_1^2 - x_2^2 - 2x_1x_2$$

The Kuhn–Tucker conditions for this problem are

$$\begin{gathered} -4x_1 - 2x_2 - \mu = 0 \\ -2x_1 - 2x_2 + \mu = 0 \\ x_1 - x_2 - \alpha \leq 0 \\ \mu(x_1 - x_2 - \alpha) = 0 \\ \mu \geq 0 \end{gathered}$$

There are two cases

$\underline{\mu = 0}$ The first two equations give $x_1 = x_2 = 0$ and these can only give the optimum if the inequality is satisfied, namely $\alpha \geq 0$.

$\underline{x_1 - x_2 = \alpha}$ Adding the first two equations gives the solution in terms of the simultaneous linear equations

$$\begin{gathered} 3x_1 + 2x_2 = 0 \\ x_1 - x_2 = \alpha \end{gathered}$$

which clearly have the solution $x_1 = 2\alpha/5$ and $x_2 = -3\alpha/5$. Calculation from the first or second equation gives $\mu = -2\alpha/5$ which is only optimal if $\alpha \leq 0$.

Exercises 10.4.2

■ **29(a)** The MATLAB version of this problem (note it deals with the maximum) is as follows:

```
a=0.1;h=0.2;nmax=10;n=0;
zold=q29(a);a=a+h;z=q29(a);c=[z];
while(z(2)>zold(2))&(n<nmax)
n=n+1;zoldold=zold;zold=z;h=2*h;a=a+h;z=q29(a);c=[c,z];
end
% gives x        0.3000  0.7000  1.5000  3.1000
        f       -11.4111 -2.7408 -1.9444 -3.2041
        f'       73.0741  4.8309 -0.4074 -0.9329
```

where the function `q29` is in the M-file

```
function a=q29(x)
a=[x;-x-1/x^2;-1+2/x^3];
```

The maximum is in the region 0.7 to 3.1 only if the function values are known but in the region 0.7 to 1.5 if the derivative is used.

29(b) Using the bracket and the mid point to start the maximization

```
p=[q29(0.7),q29(1.9),q29(3.1)]; u=p(1,:),v=p(2,:)
% gives   u= 0.7000  1.9000  3.1000
          v=-2.7408 -2.1770 -3.2041
[u,v]=qapp(u,v)
% gives   u= 0.7000  1.7253  1.9000
          v=-2.7408 -2.0612 -2.1770
[u,v]=qapp(u,v)
% gives   u= 0.7000  1.5127  1.7253
          v=-2.7408 -1.9497 -2.0612
```

where the M-file `qapp.m` contains the quadratic algorithm

```
function [x,f]=qapp(a,b) %note written for max problem
% a=[a1,a2,a3] is the input vector of three points from
  bracketing
% b=[f(a1),f(a2),f(a3)] is the vector of function values
x=a;f=b;p=polyfit(a,b,2);
xstar=-0.5*p(2)/p(1);z=q29(xstar);fstar=z(2);% for other
  problems change q29
if fstar>b(2)
        if xstar<a(2),  x(3)=a(2);f(3)=b(2);
        else x(1)=a(2);f(1)=b(2);  end
        x(2)=xstar;f(2)=fstar;
```

```
else
        if xstar<a(2), x(1)=xstar;f(1)=fstar;
        else  x(3)=xstar;f(3)=fstar; end
end
% x contains the three points of the new bracket and f
  the function values
```

29(c) For comparison the same bracket is used

```
p=q29(0.7);q=q29(3.1);p′,q′
% gives   x=0.7000   f=-2.7408   f′=4.8309
% and     x=3.1000   f=-3.2041   f′=-0.9329
[p,q]=cufit(p,q);p′,q′
% gives   x=0.7000   f=-2.7408   f′=4.8309
% and     x=1.5129   f=-1.9498   f′=-0.4224
[p,q]=cufit(p,q);p′,q′
% gives   x=1.1684   f=-1.9009   f′=0.2538
% and     x=1.5129   f=-1.9498   f′=-0.4224
```

where the cubic algorithm is contained in the M-file `cufit.m`

```
function [an,bn]=cufit(a,b)
% a=[x1 f(x1) fdash(x1)] and b=[x2 f(x2) fdash(x2)] are
  the input vectors
f=[a(2);b(2);a(3);b(3)];
A=[a(1)^3 a(1)^2 a(1) 1;b(1)^3 b(1)^2 b(1) 1;
  3*a(1)^2 2*a(1) 1 0;3*b(1)^2 2*b(1) 1 0];
p=A\f;xstar=(-p(2)-sqrt(p(2)^2-3*p(1)*p(3)))/(3*p(1));
  c=q29(xstar);
% for other problems change the function q29
if c(3)>0 an=c;bn=b; else bn=c;an=a;end
% an and bn contain the new bracket values
```

■ **30** If the derivatives are not used, then the bracket is $0 < x < 3$, whilst using the derivatives gives the much narrower range of $0 < x < 1$ but at the expense of twice the number of function evaluations.

30(a) The calculation is identical to Exercise 29 with q29 replaced by q30 in qapp

```
p=[q30(0),q30(1),q30(3)]; u=p(1,:),v=p(2,:)
% gives   u= 0  1      3
           v= 0  0.4207 0.0141
[u,v]=qapp(u,v)
% gives   u= 0  1.0000 1.5113
           v= 0  0.4207 0.3040
[u,v]=qapp(u,v)
% gives   u= 0  0.9898 1.0000
           v= 0  0.4222 0.4207
```

where the function q30 is in the M-file

```
function a=q30(x)
a=[x;sin(x)/(1+x^2);cos(x)/(1+x^2)-2^*x^*sin(x)/(1+x^2)^2];
```

30(b) Again, as in Exercise 29, with q29 replaced by q30 in cufit

```
p=q30(0);q=q30(1);p′,q′
% gives       x=0           f=0           f′= 1
% and         x=1.0000      f=0.4207      f′=-0.1506
[p,q]=cufit(p,q);p′,q′
% gives       x=0           f=0           f′= 1
% and         x=0.8667      f=0.4352      f′=-0.0612
[p,q]=cufit(p,q);p′,q′
% gives       x=0           f=0           f′= 1
% and         x=0.8242      f=0.4371      f′=-0.0247
```

The built in MATLAB procedure *fminbnd* gives $x = 0.7980, f = 0.4374$ in 9 iterations

```
f=@(x)-sin(x)/(1+x^2);
options=optimset('display','iter');
x=fminbnd(f,0,1,options)
```

■ **31** The problem is a standard exercise to illustrate the use of the two basic algorithms for a single variable search.

31(a) An adaptation of Exercise 29 with the appropriate function used gives

```
p=[q31(1),q31(5/3),q31(3)];u=p(1,:),v=p(2,:)
% gives   u=1.0000  1.6667  3.0000
          v=0.2325  0.2553  0.1419
[u,v]=qapp(u,v)
% gives   u=1.000   1.6200  1.6667
          v=0.2325  0.2571  0.2553
[u,v]=qapp(u,v)
% gives   u=1.0000  1.4784  1.6200
          v=0.2325  0.2602  0.2571
```

where `q31` is the M-file

```
function a=q31(x)
a=[x;x*(exp(-x)-exp(-2*x));(exp(-x)-exp(-2*x))
  -x*(exp(-x)-2*exp(-2*x))];
```

31(b) Use the same bracket and adapt the algorithm in Exercise 29.

```
p=q31(1);q=q31(3);p',q'
% gives     x=1.0000       f=0.2325       f'=  0.1353
% and       x=3.0000       f=0.1419       f'= -0.0872
[p,q]=cufit(p,q);p',q'
% gives     x=1.0000       f=0.2325       f'=  0.1353
% and       x=1.5077       f=0.2599       f'= -0.0136
[p,q]=cufit(p,q);p',q'
```

```
% gives     x=1.0000       f=0.2325       f'=  0.1353
% and       x=1.4462       f=0.2603       f'= -0.0001
```

The next iteration gives the three-figure accuracy required in (c).
x = 1.4456 f = 0.2603 f′ = 0.0000

■ **32** The difficulty in this problem is that the eigenvalues must be computed at each function evaluation. With a package such as MATLAB this is less of a problem since the instruction `eig(A)` will give them almost instantly.
The M-file `q32` performs this computation. Note the `-max(eig(A))` since `qapp` looks for a maximum

```
function a=q32(x)
A=[x -1 0;-1 0 -1;0 -1 x^2];
a=[x;-max(eig(A))];
```

The calculations are performed as in Exercise 29

```
p=[q32(-1),q32(0),q32(1)];u=p(1,:),v=p(2,:)
% gives   u=-1          0          1
           v=-1.7321   -1.4142   -2.0000
[u,v]=qapp(u,v)
% gives   u=-1.0000   -0.1483    0
           v=-1.7321   -1.3854   -1.4142
[u,v]=qapp(u,v)
% gives   u=-1.0000   -0.2356   -0.1483
           v=-1.7321   -1.3769   -1.3854
```

See the text for the MATLAB *fminbnd* code for this problem.

■ **33** To establish the formula is a matter of simple substitution into (10.10).
To find when $f'(x) = 0$, from the Newton method, requires the iteration of the formula

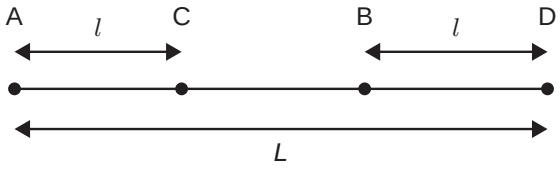
$$xnew = x - \frac{f'(x)}{f''(x)}$$

If the function is known at $x - h, x, x + h$ with values f_1, f_2, f_3 respectively, then the derivatives can be replaced by their approximations

$$f'(x) \approx \frac{f_3 - f_1}{2h} \quad \text{and} \quad f''(x) \approx \frac{f_1 - 2f_2 + f_3}{h^2}$$

and the formula follows immediately.

■ **34(a)** To obtain the Golden Section ratio, the figure gives



$$\frac{AC}{AD} = \frac{AD}{AB} \overset{yields}{\longrightarrow} \frac{l}{L-l} = \frac{L-l}{L}$$

Solve for $a = l/L$ to obtain $a = \frac{1}{2}(3 - \sqrt{5})$.

The Golden Section algorithm can be written in MATLAB as

```
alpha = (3-sqrt(5))/2;a=[0;1;1;2.5];
f=@(x)x*sin(x);
a=[0;(1-alpha)*a(1)+alpha*a(4);alpha*a(1)+(1-alpha)*a(4);2.5];
F=[f(a(1));f(a(2));f(a(3));f(a(4))];aa=[a];FF=[F];
for i=1:5
    if F(2)>F(3)a(4)=a(3);a(3)=a(2);F(4)=F(3);F(3)=F(2);
         a(2)=(1-alpha)*a(1)+alpha*a(4);F(2)=f(a(2));
    else a(1)=a(2);a(2)=a(3);F(1)=F(2);F(2)=F(3);
         a(3)=alpha*a(1)+(1-alpha)*a(4);F(3)=f(a(3));
    end, aa=[aa,a];FF=[FF,F];
end
```

The instructions *aa, FF* prints out x and f at five successive iterations

```
aa =    0       0.9549    1.5451    1.9098    1.9098    1.9098
      0.9549    1.5451    1.9098    2.1353    2.0492    1.9959
      1.5451    1.9098    2.1353    2.2746    2.1353    2.0492
      2.5000    2.5000    2.5000    2.5000    2.2746    2.1353
```

```
FF =    0       0.7795    1.5446    1.8011    1.8011    1.8011
      0.7795    1.5446    1.8011    1.8040    1.8191    1.8183
      1.5446    1.8011    1.8040    1.7341    1.8040    1.8191
      1.4962    1.4962    1.4962    1.4962    1.7341    1.8040
```

with the best value at $x = 2.0492$ and $f = 1.8191$.

- **34(b)** The algorithm is the same as part (a) except the function is now

```
f=@(x)(log(x)+2*(1-x)/(1+x))/(1-x)^2;
```

The five iterations give

```
aa =
   1.5000   1.8820   1.8820   2.0279   2.1180   2.1180
   1.8820   2.1180   2.0279   2.1180   2.1738   2.1525
   2.1180   2.2639   2.1180   2.1738   2.2082   2.1738
   2.5000   2.5000   2.2639   2.2639   2.2639   2.2082
FF =
   0.0218604   0.0260436   0.0260436   0.0265463   0.0266783   0.0266783
   0.0260436   0.0266783   0.0265463   0.0266783   0.0267059   0.0266998
   0.0266783   0.0266780   0.0266783   0.0267059   0.0267051   0.0267059
   0.0262879   0.0262879   0.0266780   0.0266780   0.0266780   0.0267051
```

Note that the function is so flat that it has been quoted to six figures; the best result is $x = 2.1738, f = 0.0267059$.

Exercises 10.4.4

- **35** The gradient is given by

$$\mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so, at the first step

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ f = 1.5, \ \mathbf{G} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The first search is for $\min_{\mu}[f(1-\mu, 1-\mu)]$ and a simple calculation gives $\mu = 2$. The second iteration can be started as

$$\mathbf{a} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \ f = -0.5, \ \mathbf{G} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Note that the two gradients are perpendicular. The search $\min_{\mu}[f(-1-\mu, -1+\mu)]$ is easily performed (exactly in this problem) to give $\mu = 0.4$ and the problem is ready for the next iteration.

$$\mathbf{a} = \begin{bmatrix} -7/5 \\ -3/5 \end{bmatrix}, \ f = -0.9, \ \mathbf{G} = \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix}$$

■ **36** For this function the gradient is

$$\mathbf{G} = \begin{bmatrix} 6x + 2y + 3 \\ 2x + 6y + 2 \end{bmatrix}$$

and it easily checked that $x = -7/16, y = -3/16$ gives zero gradient and hence the minimum at $f = -0.84375$.

The steepest descent method is easily written in MATLAB. For more complicated functions two M-files are needed, one to set up the function and its gradient and the second to set up the line search routine. Anonymous functions can be used for more .straightforward functions. The following lines of code solve the problem.

```
f=@(x)2*(x(1)+x(2))^2+(x(1)-x(2))^2+3*x(1)+2*x(2);
g=@(x)[4*(x(1)+x(2))+2*(x(1)-x(2))+3;4*(x(1)+x(2))-2*(x(1)-x(2))+2];
fm=@(t,xx,G)f(xx-t*G);
% The function, its derivative and the line search are now set
xx=[0,0];ff=f(xx);G=g(xx);XX=[xx];FF=[ff];GG=[G];% Start values
[t,fval]=fminbnd@(t)fm(t,xx,G),0,2);% Note how the minimisation is done
xx=xx-t*G;G=g(xx);ff=f(xx);XX=[XX,xx];GG=[GG,G];FF=[FF,ff];
%Next point
%Repeat the last two lines to iterate
```

The first five values are produced in XX,FF and GG as

```
XX=    0   -0.3824   -0.4296   -0.4365   -0.4374
       0   -0.2549   -0.1841   -0.1887   -0.1874
FF=    0   -0.8284   -0.8435   -0.8437   -0.8437
GG=    3.0000    0.1961   0.0545    0.0036   0.0010
       2.0000   -0.2941   0.0363   -0.0053   0.0007
```

Because the function is a quadratic, the Newton method, which is based on the assumption that the function is approximated by a quadratic, must converge in one iteration.

■ **37** The MATLAB implementation is similar to Exercise 36.
The program is identical except for the instructions that generate the functions, which are now

```
f=@(x)(x(1)-x(2)+x(3))^2+(2*x(1)+x(3)-2)^2+(x(3)^2-1)^2;
g=@(x)[2*(x(1)-x(2)+x(3))+4*(2*x(1)+x(3)-2);2*(x(1)-x(2)+x(3));...
2*(x(1)-x(2)+x(3))+2*(2*x(1)+x(3)-2)+4*x(3)*(x(3)^2-1)];
```

The first few iterations are

```
XX =    2.0000    1.1518    0.4985    0.6175    0.4941    0.5178
        2.0000    2.1696    1.8171    1.7505    1.6616    1.6358
        2.0000    0.4733    0.7984    0.9653    1.0178    1.0300
FF =   29.0000    1.5023    0.4440    0.0729    0.0237    0.0158
GG =   20.0000    2.0184   -1.8592    0.4657   -0.2758    0.0859
       -4.0000    1.0891    1.0405    0.3354    0.2995    0.1762
       36.0000   -1.0044   -2.6078   -0.1980   -0.1414    0.2054
```

To use the built in routine *fminunc*, it is easier to use an M file.

```
function [f,g] = q37(x) %Question 37
f=(x(1)-x(2)+x(3))^2+(2*x(1)+x(3)-2)^2+(x(3)^2-1)^2;
g=[2*(x(1)-x(2)+x(3))+4*(2*x(1)+x(3)-2);2*(x(1)-x(2)+x(3));...
2*(x(1)-x(2)+x(3))+2*(2*x(1)+x(3)-2)+4*x(3)*(x(3)^2-1)];
end
```

The main progam is

```
x=[2;2;2];
options=optimset('GradObj','on','display','iter','HessUpdate',
        'steepdesc','LargeScale','off');
[x,fval]=fminunc(@q37,x,options)
```

The first three and last three iterations are

```
Iteration    Func-count    f(x)
    0            1           29         with starting point x=(2,2,2)
    1            2         3.67901
    2            4         2.35628
   . . . . . . . . . . . . . .
   59          119        8.6485e-010
   60          121        6.17926e-010
   61          123        4.41502e-010  with final point x=(0.5,1.5,1.0)
```

■ **38** The function, gradient and Hessian matrix are computed in the M-file newton38

```
function [a,agrad,ajac]=newton38(z)
t1=z(1)-z(2)+z(3);t2=2*z(1)+z(3)-2;
a=t1^2+t2^2+(z(3)^2-1)^2;
agrad(1)=2*t1+4*t2;
agrad(2)=-2*t1;
agrad(3)=2*t1+2*t2+4*z(3)^3-4*z(3);
ajac(1,1)=10;ajac(1,2)=-2;ajac(1,3)=6;
ajac(2,1)=-2;ajac(2,2)=2;ajac(2,3)=-2;
ajac(3,1)=6;ajac(3,2)=-2;ajac(3,3)=12*z(3)^2;
```

and the Newton iteration proceeds as

```
a=[2;2;2];E=[0;0;0;0];
for i=1:5
        [f,G,J]=newton38(a);E=[E,[f;a]];
```

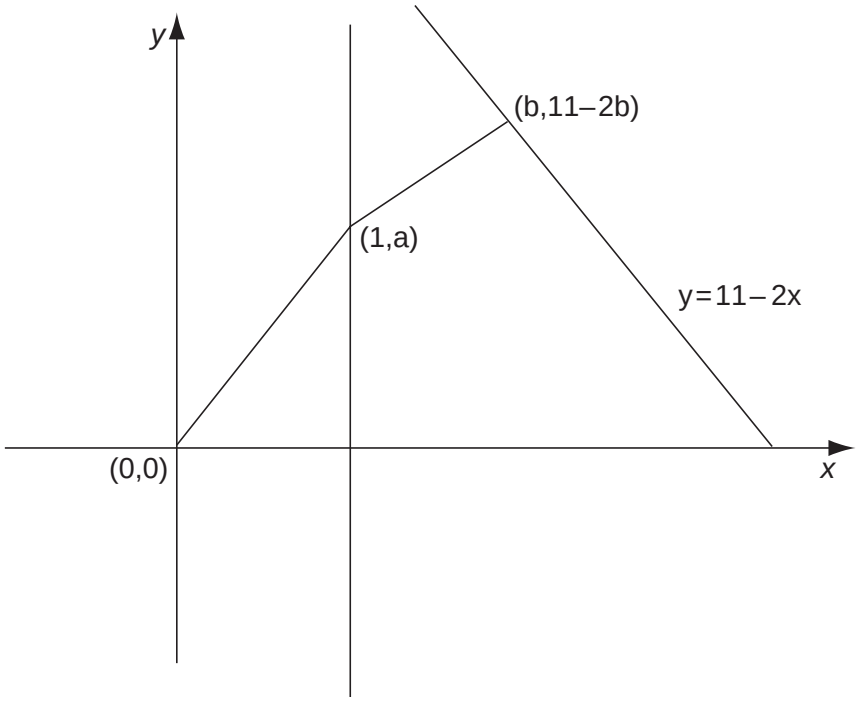
```
a=a-J\G';
end
% gives
```

f	29.0000	1.2448	0.1056	0.0026	0.0000
x	2.0000	0.2727	0.4245	0.4873	0.4995
y	2.0000	1.7273	1.5755	1.5127	1.5005
z	2.0000	1.4545	1.1510	1.0253	1.0009

■ **39** The figure illustrates the cost of the road from (0,0) to $(1,a)$ to $(b,11\text{-}2b)$ is

$$C = 2(1+a^2)^{1/2} + \left[(b-1)^2 + (11-2b-a)^2\right]^{1/2}$$

and any of the minimization methods give $a = 0.2294$, $b = 4.5083$ and $c = 5.9743$, giving equations of lines as $y = 0.2294x, y = 0.5x - 0.2706$ and cost $= 5.974$.



Exercises 10.4.7

■ **40(a)** The gradient can be calculated as $\mathbf{G} = \begin{bmatrix} 10x - 2y - 8 \\ -2x + 2y \end{bmatrix}$ and the initial choice of $\mathbf{H}$ is the unit matrix.

Iteration 1

The computation commences

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} f_1 = 4 \; \mathbf{G}_1 = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \; \mathbf{H}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the minimization takes place in the direction $\mathbf{a} = \begin{bmatrix} 2 - 8\lambda \\ 2 \end{bmatrix}$. The minimum in this direction may be obtained as $\lambda = 0.1$. The new values are

$$\mathbf{a}_2 = \begin{bmatrix} 1.2 \\ 2 \end{bmatrix} f_2 = 0.8 \; \mathbf{G}_2 = \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

and the values of $\mathbf{h}_1$ and $\mathbf{y}_1$ are calculated as

$$\mathbf{h}_1 = \mathbf{a}_2 - \mathbf{a}_1 = \begin{bmatrix} -0.8 \\ 0 \end{bmatrix} \text{ and } \mathbf{y}_1 = \mathbf{G}_2 - \mathbf{G}_1 = \begin{bmatrix} -8 \\ 1.6 \end{bmatrix}$$

Finally, the $\mathbf{H}$ is updated as

$$\begin{aligned} \mathbf{H}_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{66.56} \begin{bmatrix} -8 \\ 1.6 \end{bmatrix} [-8 \; 1.6] + \frac{1}{6.4} \begin{bmatrix} -0.8 \\ 0 \end{bmatrix} [-0.8 \; 0] \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.9615 & -0.1923 \\ -0.1923 & 0.0385 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.1385 & 0.1923 \\ 0.1923 & 0.9615 \end{bmatrix} \end{aligned}$$

Iteration 2

The iteration starts with the variables computed from iteration 1

$$\mathbf{a}_2 = \begin{bmatrix} 1.2 \\ 2 \end{bmatrix} f_2 = 0.8 \; \mathbf{G}_2 = \begin{bmatrix} 0 \\ 1.6 \end{bmatrix} \; \mathbf{H}_2 = \begin{bmatrix} 0.1385 & 0.1923 \\ 0.1923 & 0.9615 \end{bmatrix}$$

The method follows the same pattern as iteration 1 and so the computations are not written down in the same detail.

$$\mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} f_3 = 0 \; \mathbf{G}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \; \mathbf{H}_3 = \begin{bmatrix} 0.1250 & 0.1250 \\ 0.1250 & 0.6250 \end{bmatrix}$$

The minimum has been achieved; this is expected since the function is quadratic and it is known that the method converges in n steps for an n-dimensional quadratic, provided the minimizations are performed exactly.

- **40(b)** The problem has three variables and is not quadratic. The one-dimensional minimizations need a numerical procedure; so it is better to use a package such as MATLAB. The M-file required for the function is

```
function [f,G]=q40b(z)
t1=z(1)-z(2)+z(3);t2=2*z(1)+z(3)-2;
f=t1^2+t2^2+z(3)^4;
G(1)=2*t1+4*t2;
G(2)=-2*t1;
G(3)=2*t1+2*t2+4*z(3)^3;
```

The main DFP segment is

```
1.  a=[0;0;0];H=eye(3);[f,G]=q40b(a);
2.  fm=@(t,xx,GG)q40b(xx-t*GG);
3.  aa=[a];ff=[f];gg=[G];HH=[H];
4.  for n=1:4
5.          D=H*G;[t,fval]=fminbnd(@(t)fm(t,a,D),0,2);
6.          aold=a;Gold=G;a=aold-t*D;[f,G]=q40b(a);
7.          h=a-aold;y=G-Gold;
8.          H=H-H*y*y'*H/(y'*H*y)+h*h'/(h'*y);
9.          aa=[aa,a];ff=[ff,f];gg=[gg,G];HH=[HH,H];
10. end
```

The instructions give the results

```
aa=      0     0.5853     1.0190     1.0186     1.0184
         0          0     0.9813     0.9814     0.9815
         0     0.2926    -0.0372    -0.0372    -0.0369
ff=   4.0000   1.0662     0.0000     0.0000     0.0000
gg=  -8.0000  -0.3916     0.0047    -0.0000    -0.0000
         0    -1.7558    -0.0012    -0.0001    -0.0000
     -4.0000   0.7823     0.0027    -0.0002    -0.0002
```

The convergence looks good but slows because H tends to a singular matrix.

■ **41** Evaluating $\mathbf{H}_{i+1}\mathbf{y}_i$ it is readily shown that equation (10.18) is satisfied. The update is one that is quite effective but suffers from the problem that the denominator can become zero when line searches are not exact. A great deal of remedial action must be taken to ensure that the problem is overcome.

■ **41(a)** The function is quadratic so the solution is obtained in two steps. The program in Exercise 39, suitably adapted, was used to compute the solution.
Iteration 1

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} f = 9 \ \mathbf{G} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \ \mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Iteration 2

$$\mathbf{a} = \begin{bmatrix} 0.4848 \\ -0.0606 \end{bmatrix} f = 0.2424 \ \mathbf{G} = \begin{bmatrix} 0.9697 \\ -0.2424 \end{bmatrix} \mathbf{H} = \begin{bmatrix} 0.9948 & -0.0619 \\ -0.0619 & 0.2577 \end{bmatrix}$$

41(b) The exercise is similar to Exercise 40(b) but with a different function and updating methods.

```
function [f,G]=q41b(z)
t1=z(1);t2=z(1)-z(2)+1;
f=t1^2+t2^2+z(2)^2*z(3)^2;
G(1)=2*t1+2*t2;
G(2)=-2*t2+2*z(2)*z(3)^2;
G(3)=2*z(3)*z(2)^2;
```

In the lines 1,2 and 6 of Exercise 40(b) the M file q41b replaces q40b. In line 8, the Rank 1 and the BFGS updates replace the one in the program. Also, a different start point is used. The results for Rank 1 update (i) are

```
aa=     0.5000    -0.0732    -0.1628    -0.00592   -0.0593
        0.5000     0.8344     0.7747     0.9235     0.9234
        0.5000     0.4522     0.0525    -0.0640    -0.0640
ff=     1.3125     0.1563     0.0321     0.0073     0.0073
gg=     3.0000     0.0386    -0.2006    -0.0839    -0.0839
       -1.7500     0.1564    -0.1207    -0.0270    -0.0270
        0.2500     0.6296     0.0630    -0.1091    -0.1092
```

The results for BFGS update (ii) are

```
aa=     0.5000    -0.0732     0.0969    -0.0726    -0.0348
        0.5000     0.8344     0.7384     0.9408     0.9344
        0.5000     0.4522     0.0659     0.0141     0.0071
ff=     1.3125     0.1563     0.0389     0.0056     0.0022
gg=     3.0000     0.0386     0.1355    -0.1718    -0.0080
       -1.7500     0.1564    -0.3229     0.0271    -0.0614
        0.2500     0.6296     0.0718     0.0250     0.0124
```

Note that BFGS does better than Rank 1 and is the choice of method for the built-in function fminunc.

■ **42** It is easy to check that equation (10.18) is satisfied but note that the notation has changed and $\mathbf{y}$,$\mathbf{h}$ have been replaced by $\mathbf{u}$,$\mathbf{v}$ respectively.

$$\begin{aligned}\mathbf{H}'\mathbf{u} &= \mathbf{H}\mathbf{u} + \mathbf{v}\mathbf{p}^T\mathbf{u} - \mathbf{H}\mathbf{u}\mathbf{q}^T\mathbf{u} \\ &= \mathbf{H}\mathbf{u} + \mathbf{v} - \mathbf{H}\mathbf{u} = \mathbf{v} \text{ since } \mathbf{p}^T\mathbf{u} = \mathbf{q}^T\mathbf{u} = 1\end{aligned}$$

To match with the Davidon formula (10.19) first choose $\beta = \alpha' = 0$ and hence

$$\mathbf{p}^T\mathbf{u} = \alpha\mathbf{v}^T\mathbf{u} = 1 \text{ and } \mathbf{q}^T\mathbf{u} = \beta'\mathbf{u}^T\mathbf{H}\mathbf{u} = 1$$

Substituting gives

$$\mathbf{H}' = \mathbf{H} - \frac{\mathbf{H}\mathbf{u}\mathbf{u}^T\mathbf{H}}{\mathbf{u}^T\mathbf{H}\mathbf{u}} + \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{u}}$$

as required.

The formula in Exercise 41(i) is obtained by putting $\beta = \beta' = -\alpha = -\alpha'$. Thus

$$\mathbf{p} = \mathbf{q} = \alpha(\mathbf{V} - \mathbf{H}\mathbf{u}) \text{ and hence } \mathbf{p}^T\mathbf{u} = \alpha(\mathbf{v} - \mathbf{H}\mathbf{u})^T\mathbf{u} = 1$$

Substituting gives the formula

$$\mathbf{H}' = \mathbf{H} + \frac{(\mathbf{v} - \mathbf{H}\mathbf{u})(\mathbf{v} - \mathbf{H}\mathbf{u})^T}{(\mathbf{v} - \mathbf{H}\mathbf{u})^T\mathbf{u}}$$

This whole class of solutions was devised by Huang. Many general results can be proved for this class and many of the commonly used formulae are included in it.

■ **43 (a)** The Fletcher-Reeves method is easily written as a MATLAB segment

```
global a b p
a=1;b=1;g=[3*a;b];p=-g;
lam=fminbnd(@ffr,0,2)
% gives   lam = 0.3571
a=a+lam*p(1),b=b+lam*p(2),gold=g;g=[3*a;b]
% gives   a= -0.0714   b= 0.6429   g= -0.2143   0.6429
p=-g+p*(g'*g)/(gold'*gold)
% gives   p= 0.0765   -0.6888
lam=fminbnd(@ffr,0,2)
% gives   lam = 0.9333
a=a+lam*p(1),b=b+lam*p(2),gold=g;g=[3*a;b]
% gives   a=-4.0246e-016   b=-1.1102e-016
```

which is the minimum point. The M-file used is

```
function v=ffr(x)
global a b p
v=3*(a+x*p(1))^2+(b+x*p(2))^2;
```

■ **43(b)** The three-variable problem is handled in a similar manner.

```
global a b c p
a=0.5;b=0.5;c=0.5;
f=(a-b+1)^2+a^2*b^2+(c-1)^2;
g=[2*(a-b+1)+4*a*b^2;-2*(a-b+1)+2*a^2*b;2*(c-1)];
p=-g;W=[f;a;b;c];
for i=1:5
gold =g;lam=fminbnd(@ffr2,-2,2);
a=a+lam*p(1);b=b+lam*p(2);c=c+lam*p(3);
f=(a-b+1)^2+a^2*b^2+(c-1)^2;W=[W[f;a;b;c]];
g=[2*(a-b+1)+4*a*b^2;-2*(a-b+1)+2*a^2*b;2*(c-1)];
p=-g+p*(g'*g)/(gold'*gold);
end
```

```
W
% gives the sequence of iterations
   f  1.3125   0.0764   0.0072   0.0007   0.0004   0.0000
   a  0.5000  -0.0950   0.0057  -0.0251  -0.0079  -0.0032
   b  0.5000   0.9165   0.9276   0.9633   0.9742   0.9978
   c  0.5000   0.7380   0.9674   1.0009   1.0044   1.0014
```

Review Exercises 10.7

■ **1** The successive tableaux are as follows:

	x_1	x_2	x_3	x_4	x_5	Soln
z	-12	-8	0	0	0	0
x_3	1	1	1	0	0	350
x_4	2	1	0	1	0	600
x_5	1	3	0	0	1	900

	x_1	x_2	x_3	x_4	x_5	Soln
z	0	-2	0	6	0	3600
x_3	0	0.5	1	-0.5	0	50
x_1	1	0.5	0	0.5	0	300
x_5	0	2.5	0	-0.5	1	600

	x_1	x_2	x_3	x_4	x_5	Soln
z	0	0	4	4	0	3800
x_2	0	1	2	-1	0	100
x_1	1	0	-1	1	0	250
x_5	0	0	-5	2	1	350

Hence the solution is read from the table as $x_1 = 250, x_2 = 100$ and $\mathrm{F} = 3800$.

■ **2** Let x_1, x_2, x_3 be the numbers of sailboard constructed of types 1,2,3. The profit is

$$10x_1 + 15x_2 + 25x_3$$

and the constraints are

$$5x_1 + 10x_2 + 25x_3 \leq 290$$

$$3x_1 + 2x_2 + x_3 \leq 72$$

$$10x_1 + 20x_2 + 30x_3 \leq 400$$

The system only has 'less than' inequalities so the initial and subsequent tableaux can be written down immediately.

	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	-10	-15	-25	0	0	0	0
x_4	5	10	25	1	0	0	290
x_5	3	2	1	0	1	0	72
x_6	10	20	30	0	0	1	400

	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	-5	-5	0	1	0	0	290
x_3	0.2	0.4	1	0.04	0	0	11.6
x_5	2.8	1.6	0	-0.04	1	0	60.4
x_6	4	8	0	-1.2	0	1	52

	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	0	5	0	-0.5	0	1.25	355
x_3	0	0	1	0.1	0	-0.05	9
x_5	0	-4	0	0.8	1	-0.7	24
x_1	1	2	0	-0.3	0	0.25	13

	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	0	2.5	0	0	0.62	0.81	370
x_3	0	0.5	1	0	-0.12	0.04	6
x_4	0	-5	0	1	1.25	-0.88	30
x_1	1	0.5	0	0	0.37	-0.01	22

The solution is $x_1 = 22, x_2 = 0, x_3 = 6$ and maximum profit is £ 370.

■ **3** Let x_1, x_2, x_3 be the number of standard, super and deluxe cars respectively, then the profit function is

$$100x_1 + 300x_2 + 400x_3$$

and the constraints are

$$\begin{gathered} 10x_1 + 20x_2 + 30x_3 \leq 1600 \\ 10x_1 + 15x_2 + 20x_3 \leq 1500 \\ x_2 + x_3 \leq 50 \\ x_1 + x_2 + x_3 \geq 70 \end{gathered}$$

The problem includes a 'greater than' inequality, so the two-phase approach is needed. An artificial variable is introduced and the artificial cost is introduced in phase 1. The z row of the tableau is manipulated to bring it to standard form.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Soln
z	−1	−1	−1	0	0	0	1	0	−70
x_4	10	20	30	1	0	0	0	0	1600
x_5	10	15	20	0	1	0	0	0	1500
x_6	0	1	1	0	0	1	0	0	50
x_8	1	1	1	0	0	0	−1	1	70

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Soln
z	0	0	0	0	0	0	0	1	0
x_4	0	10	20	1	0	0	10	−10	900
x_5	0	5	10	0	1	0	10	−10	800
x_6	0	1	1	0	0	1	0	0	50
x_1	1	1	1	0	0	0	−1	1	70

A feasible solution has been obtained so the method moves to phase 2. The z row is first re-calculated

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	0	−200	−300	0	0	0	−100	7000
x_4	0	10	20	1	0	0	10	900
x_5	0	5	10	0	1	0	10	800
x_6	0	1	1	0	0	1	0	50
x_1	1	1	1	0	0	0	−1	70

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	0	−50	0	15	0	0	50	20500
x_3	0	0.5	1	0.05	0	0	0.5	45
x_5	0	0	0	−0.5	1	0	5	350
x_6	0	0.5	0	−0.05	0	1	−0.5	5
x_1	1	0.5	0	−0.05	0	0	−1.5	25

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Soln
z	0	0	0	10	0	100	0	21,000
x_3	0	0	1	0.1	0	−1	1	40
x_5	0	0	0	−0.5	1	0	5	350
x_2	0	1	0	−0.01	0	2	−1	10
x_1	1	0	0	0	0	−1	−1	20

The solution is read off the table as $x_1 = 20, x_2 = 10, x_3 = 40$ and the maximum profit is £21,000.

Note that the (z, x_7) entry is zero; so a non-unique solution is expected. Interchanging the x_3 and x_7 entries and completing one further tableau gives the alternative solution $x_1 = 60, x_2 = 50, x_3 = 0$ and profit is still £21,000.

The MAPLE solution

```
with(simplex):
cor3:={10*x+20*y+30*z<=1600,10*x+15*y+20*z<=1500,
  y+z<=50,x+y+z>=70};
obr3:=100*x+300*y+400*z;
maximize(obr3,cor3,NONNEGATIVE);
# gives the solution {x=20, z=40,y=10}
```

MAPLE provides the same solution but does not identify the alternative solution. Similarly, for the corresponding MATLAB code, which is

```
f=[-100,-300,-400];A=[10,20,30;10,15,20;0,1,1;-1,-1,-1];b=[1600;1500;50;-70];
options=optimset('LargeScale','off','Simplex','on');
[x,fval]=linprog(f,A,b,[],[],zeros(3,1),[],[],options)
```

■ **4** Let the student buy x_1 kg of bread and x_2 kg of cheese, then the cost to be minimized is

$$60x_1 + 180x_2$$

and the two constraints are

$$1000x_1 + 2000x_2 \geq 3000$$

$$25x_1 + 100x_2 \geq 100$$

There are two surplus and two artificial variables in the tableau and the artificial cost function which has been processed to standard form.

Phase 1

	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	−1025	−2100	1	1	0	0	−3100
x_5	1000	2000	−1	0	1	0	3000
x_6	25	100	0	−1	0	1	100

	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	-500	0	1	-20	0	21	-1000
x_5	500	0	-1	20	1	-20	1000
x_2	0.25	1	0	-0.01	0	0.01	1

	x_1	x_2	x_3	x_4	x_5	x_6	Soln
z	0	0	0	0	1	1	0
x_1	1	0	-0.002	0.04	0.002	-0.04	2
x_2	0	1	0.0002	-0.02	-0.0002	0.02	0.5

Phase 1 is completed so the tableau is reconstituted as

	x_1	x_2	x_3	x_4	Soln
z	0	0	0.03	1.2	-210
x_1	1	0	-0.002	-0.04	2
x_2	0	1	0.0002	0.02	0.5

It may be noted that the z row entries are positive; so the tableau is optimal and there is no need to enter phase 2. Thus the minimum cost of the diet is 210p and is made up of 2 kg of bread and 0.5 kg of cheese.

■ **5** The square of the distance from the origin to the point (x, y) is

$$f = x^2 + y^2$$

so the problem is to optimize this function subject to the condition that it lies on the curve

$$g = x^2 - xy + y^2 - 1 = 0$$

so

$$0 = \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 2x + \lambda(2x - y)$$
$$0 = \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 2y + \lambda(-x + 2y)$$

Subtracting these two equations gives

$$0 = 2(x - y) + \lambda(3x - 3y) \quad \Rightarrow \quad x = y \quad or \qquad \lambda = -\frac{2}{3}$$

Adding the two equations gives

$$0 = 2(x+y) + \lambda(x+y) \quad \Rightarrow \quad x = -y \quad or \qquad \lambda = -2$$

The first possibility, $x = y$, gives the points, (1,1) and $(-1,-1)$ with $f = 2$ in each case and the second possibility, $x = -y$, gives

$$\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \quad \text{and} \quad \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \quad \text{with} \quad f = \frac{2}{3}$$

In this problem the cases $\lambda = -\dfrac{2}{3}$ and $\lambda = -2$ reproduce the identical solutions. Although it is not proved, the first solution is the maximum and the second the minimum.

■ **6** The volume of the solid is

$$V = x^3 + y^3$$

where the sides of the two cubes are x and y. The surface area has essentially the area of two faces of the smaller cube removed so

$$S = 7 = 6x^2 + 4y^2$$

The Lagrange multiplier equations are

$$3x^2 + 12x\lambda = 0$$
$$3y^2 + 8y\lambda = 0$$

The solutions when $x = y = 0$ can be dismissed since the volume is zero. There are three other cases

$$\text{Case 1} \qquad x = 0, \qquad y = \frac{-8}{3}\lambda = \sqrt{\left(\frac{7}{4}\right)}, V = \left(\frac{7}{4}\right)^{\frac{3}{2}}$$

$$\text{Case 2} \qquad y = 0, \qquad x = -4\lambda = \sqrt{\left(\frac{7}{6}\right)}, V = \left(\frac{7}{6}\right)^{\frac{3}{2}}$$

$$\text{Case 3} \qquad x = -4\lambda, \quad y = \frac{-8}{3}\lambda, \quad \Rightarrow \qquad 7 = 96\lambda^2 + \frac{256}{9}\lambda^2$$

and hence $\lambda = \dfrac{\pm 3}{(4\sqrt{10})}$ so the sides have lengths $\dfrac{3}{\sqrt{10}}$ and $\dfrac{2}{\sqrt{10}}$ and volume $\dfrac{35}{(10)^{\frac{3}{2}}}$
The cases when either $x = 0$ or $y = 0$ imply that the problem has collapsed to a single cube so these solutions are omitted as geometrically uninteresting.

■ **7** The problem is to maximize the distance

$$(x-1)^2 + y^2 + z^2$$

subject to

$$2x + y^2 + z = 8$$

The Lagrange equations are

$$2(x-1) + 2\lambda = 0 \Rightarrow x = 1 - \lambda$$
$$2y + 2y\lambda = 0 \Rightarrow y = 0 \text{ or } \lambda = -1$$
$$2z + \lambda = 0 \Rightarrow z = \frac{-1}{2}\lambda$$

There are two cases

$\underline{y = 0}$

$$\text{gives } \lambda = \frac{-12}{5} \quad \text{so} \quad x = \frac{17}{5},\ y = 0,\ z = \frac{6}{5}$$

$\underline{\lambda = -1}$

$$\text{gives } x = 2,\ z = \frac{1}{2} \quad \text{and} \quad y = \pm\sqrt{\frac{7}{2}}$$

The first of these possibilities gives the maximum distance.

■ **8** The Lagrange equations for this problem are

$$1 + 2\lambda x = 0$$
$$2 + 2\lambda y = 0$$
$$3 + 2\lambda z = 0$$

and putting back into the constraint gives $2\lambda = \pm 1$. The local extrema are therefore at (1,2,3) and $(-1,-2,-3)$ with corresponding $F = 14$ and $F = -14$.

To obtain the global extremum all the points on each of the boundaries must be examined. However, in this case, the geometry is sufficiently simple to establish the result. The region is the inside of the sphere of radius $\sqrt{14}$ in the region where all the variables are positive. The cost function comprises a series of parallel planes, so the global maximum in the region is the local maximum at (1,2,3) and the global minimum in the region is when all the variables are zero, namely (0,0,0) and $F = 0$.

■ **9** (i) We are given that $2s = a + b + c$, so maximizing A^2 with respect to b and c, together with this constraint, gives

$$\frac{\partial(A^2)}{\partial b} = s(s-a)(s-c) + \lambda = 0, \ \frac{\partial(A^2)}{\partial c} = s(s-a)(s-b) + \lambda = 0$$

Clearly $b = c$ and the triangle is isosceles.

(ii) Now, as a so to the above equations we add

$$\frac{\partial(A^2)}{\partial a} = s(s-b)(s-c) + \lambda = 0$$

and we see that we must have $a = b = c$ and the triangle is equilateral.

■ **10** We need to consider the problem of maximizing

$$V = \pi r^2 h \text{ subject to the constraint } \left(\frac{a}{r}\right)^2 + \left(\frac{\pi}{h}\right)^2 = b$$

Using the Lagrange multiplier approach we have the two equations

$$\frac{\partial V}{\partial r} = 2\pi r h - \frac{2a^2}{r^3}\lambda = 0$$
$$\frac{\partial V}{\partial h} = \pi r^2 - \frac{2\pi^2}{h^3}\lambda = 0$$

together with the constraint itself. Eliminating λ and solving gives

$$h^2 = \frac{3\pi^2}{b} \text{ and } r^2 = \frac{3a^2}{2b}$$

Note that it has not been proved that these values give the maximum but this can be inferred from physical or geometric reasoning.

■ **11** First notice that as $k \to \infty$ then $F \to 0$ and a careful Taylor expansion shows that $\lim_{k\to 1} F = 0$. Take any $k > 1$, then F takes a positive value so we know there must be a maximum for $k > 1$. We start the bracket procedure with $k = 1.1$ and initial increment of 0.1.

k	1.1	1.2	1.4	1.8	2.6	4.2
F/A	0.00721	0.01258	0.01962	0.02556	0.02602	0.01995

The maximum has been bracketed by $1.8 \le k \le 4.2$ and the quadratic algorithm is given in terms of anonymous functions as

```
qr11=@(x)[x;(log(x)-2*(x-1)/(x+1))/(x-1)^2]; %x and the function value
zz=[qr11(1.8),qr11(2.6),qr11(4.2)] % start values
zz =  1.8000    2.6000   4.2000
      0.0256    0.0260   0.0200
         % repeat from here
p=polyfit(zz(1,:),zz(2,:),2);
xstar=-0.5*p(2)/p(1);zstar=qr11(xstar);
if zstar(2)>zz(2,2)
     if zstar(1)<zz(1,2),zz(:,3)=zz(:,2);else zz(:,1)=zz(:,2);end
     zz(:,2)=zstar;
  else
     if zstar(1)<zz(1,2),zz(:,1)=zstar;else zz(:,3)=zstar;end
  end % to here
```

Successive iterations of the segment produces the results

```
   % iteration 1 gives   u = 1.8000  2.3594  2.6000
                         v = 0.0256  0.0266  0.0260
   % iteration 2 gives   u = 1.8000  2.2573  2.3594
                         v = 0.0256  0.0267  0.0266
   % iteration 3 gives   u = 1.8000  2.2203  2.2573
                         v = 0.0256  0.0267  0.0267
   % iteration 4 gives   u = 1.8000  2.2024  2.2203
                         v = 0.0256  0.0267  0.0267
```

Clearly we are near the solution and a value of $k = 2.2$ would be adequate for practical use in the bearing. Note Exercise 34(b).

■ **12** The function is now sufficiently complicated that hand computations become extremely tedious. At least one set of calculations should be done by hand but a computer implementation should then be encouraged. For a bracket, the calculations give

R	2	2.5	3.5	5.5	9.5
Cost	83089	14899	1124	704	1418

and hence $3.5 < R < 9.5$. Successive computations using the quadratic algorithm described in Review Exercise 11 produce the output

```
   R =   3.5000   5.5000   9.5000
cost =   -1123.5  -704.4   -1417.5
   R =   3.5000   5.5000   6.1210
cost =   -1123.5  -704.4   -801.9
   R =   3.5000   5.2493   5.5000
cost =   -1123.5  -666.3   -704.4
   R =   3.5000   5.0068   5.2493
cost =   -1123.5  -631.6   -666.3
   R =   3.5000   4.8615   5.0068
cost =   -1123.5  -612.8   -631.6
```

The function can be put in an M-file or in the program as the anonymous function

```
qr12=@(x)[x;-2*(1000/x+pi*x^2)*(1+(1-1000/(4*pi*x^3))^2)];
```

■ **13** The problem is now beyond hand computation, except for the first step. The bracket given is not useful since whatever internal point is chosen, the minimum is always estimated at the mid point. It shows that these techniques are not foolproof and a lot of checks must be inserted into any program. The following output was produced from MATLAB.

```
best f  -0.7729  -0.7584  -0.7524  -0.7508  -0.7503  -0.7501  -0.7500
   x1    0.3147   0.5000   0.5629   0.6051   0.6243   0.6364   0.6427
   x2    0.5000   0.5629   0.6051   0.6243   0.6364   0.6427   0.6464
   x3    1.0000   1.0000   1.0000   1.0000   1.0000   1.0000   1.0000
```

where the code in Review Exercise 11 and the function is given in the M-file

```
function z=frev13(x)
t=(-x+sqrt(x^2+4*(1-x^2)))/2;
z=[x;-(1-t+t^2)];
```

■ **14** Note that $x = a\cos\theta + \sqrt{L^2 - a^2\sin^2\theta}$ and, by differentiating, the formula for the velocity follows.
For the minimum, the bracketing

```
a=0;h=0.2;nmax=10;n=0;
zold=qr14(a);a=a+h;z=qr14(a);c=[z];
while((z(2)>zold(2))&(n<nmax))
n=n+1;zoldold=zold;zold=z;a=a+h;z=qr14(a);h=2*h;c=[c,z];
end
```

yields

```
a  0.2000  0.4000  0.8000  1.6000  3.2000
V  0.2637  0.5100  0.8889  0.9893 -0.0389
```

so the bracket is 0.8 to 3.2.
The quadratic algorithm follows the code of Review Exercise 11

```
% repeated application gives
V2  0.9893  1.0452  1.0545  1.0546  1.0546  1.0546  1.0546
a1  0.8000  0.8000  0.8000  0.8000  0.8000  0.8000  0.8000
a2  1.6000  1.3959  1.2935  1.2815  1.2777  1.2773  1.2772
a3  3.2000  1.6000  1.3959  1.2935  1.2815  1.2777  1.2773
```

where the function `qr14` is obtained from the M-file

```
function a=qr14(x)
a=[x;sin(x)*(1+cos(x)/sqrt(9-sin(x)^2))]; % note the sign
```

A similar calculation for the maximum gives a=5.006.

■ **15** The time taken is $t = \frac{D}{\nu}$ so the total cost is

$$\frac{C}{D} = \frac{5}{\nu} + 0.04\nu^{\frac{1}{4}} + 0.002\nu$$

The bracket procedure gives

ν	5	10	20	40	80
C/D	1.06981	0.59113	0.37459	0.30559	0.34213

The quadratic approximation method follows:

	$\nu 1$	$\nu 2$	$\nu 3$	ν^*
ν	20	40	80	53.72005
C/D	0.37459	0.30559	0.34213	0.30881
ν	20	40	53.72005	45.78858
C/D	0.37459	0.30559	0.30881	0.30483
ν	40	45.78858	53.72005	44.3291
C/D	0.30559	0.30483	0.30881	0.30466
ν	40	44.3291	45.78858	44.06968
C/D	0.30559	0.30466	0.30483	0.30466

The optimum speed is about 44 mph.

■ **16** The gradient and Hessian matrices are calculated as

$$\mathbf{G} = \begin{bmatrix} 2x + 2(x-y) + \frac{1}{4}(x+y+1)^3 \\ \\ -2(x-y) + \frac{1}{4}(x+y+1)^3 \end{bmatrix}$$

and

$$\mathbf{J} = \begin{bmatrix} 4 + \frac{3}{4}(x+y+1)^2 & -2 + \frac{3}{4}(x+y+1)^2 \\ \\ -2 + \frac{3}{4}(x+y+1)^2 & 2 + \frac{3}{4}(x+y+1)^2 \end{bmatrix}$$

(a) Steepest descent

At $\mathbf{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then $f = 0.0625$ and $\mathbf{G} = \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$ so the first search takes place in the direction

$$\mathbf{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \lambda \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} -0.25\lambda \\ -0.25\lambda \end{bmatrix}$$

Note the minus sign since a minimum is required. Putting these values into the function and minimizing gives $\lambda = 0.4582$ and hence the new point at $x = -0.1145, y = -0.1145$ and $f = 0.0352$.

Further steps follow in a similar manner.

(b) Newton method

The use of a package like MATLAB is essential to make progress. The function, gradient and Jacobian are computed in the M-file

```
function [f,G,J]=fnrev16(z)
f=z(1)^2+(z(1)-z(2))^ 2+(z(1)+z(2)+1)^4/16;
G(1)=2*(z(1)-z(2))+2*z(1)+(z(1)+z(2)+1)^3/4;
G(2)=-2*(z(1)-z(2))+(z(1)+z(2)+1)^3/4;
J(1,1)=4+(z(1)+z(2)+1)^2*3/4;
J(1,2)=-2+(z(1)+z(2)+1)^2*3/4;
J(2,1)=J(1,2); J(2,2)=2+(z(1)+z(2)+1)^2*3/4;
```

and the calculation follows

```
a=[0;0];[f,G,J]=fnrev16(a)
% gives   f = 0.0625   G = 0.2500  0.2500
          J = 4.7500     -1.2500
              -1.2500      2.7500
a=a-J\G'

% gives   a=-0.0870     -0.1304
[f,G,J]=fnrev16(a)
% gives   f = 0.0329    G = 0.0329   0.0329
          J = 4.4594       -1.5406
             -1.5406        2.4594
a=a-J\G'
% gives   a=-0.1023     -0.1534 and G very small
```

(c) DFP

Two M-files for the function make the computations straightforward

```
function [f,G]=frev16(z)
f=z(1)^2+(z(1)-z(2))^2+(z(1)+z(2)+1)^4/16;
G(1)=2*(z(1)-z(2))+2*z(1)+(z(1)+z(2)+1)^3/4;
G(2)=-2*(z(1)-z(2))+(z(1)+z(2)+1)^3/4;
```

and

```
function f=fcrev16(x)
global a D
c=a-x*D;
f=(c(1)-c(2))^2+c(1)^2 + (c(1)+c(2)+1)^4/16;
```

The DFP algorithm then iterates to the minimum

```
global a D
a=[0;0];H=eye(2);[f,G]=frev16(a)
% gives   f = 0.0625  G = 0.2500 0.2500
for n=1:1
D=H*G';lamda=fminbnd(@fcrev16,0,2);
aold=a;Gold=G;
a=aold-lamda*D;h=a-aold;
[f,G]=frev16(a);y=G'-Gold';
H=H-H*y*y'*H/(y'*H*y)+h*h'/(h'*y);
end
a,gg=G',H,f
% gives   a = -0.1145 -0.1145     f = 0.0352
         gg = -0.1145 0.1145
          H = 0.3504  -0.0974
             -0.0974   1.1078
```

Repeating the lines of code, a further two iterations give,

```
a = -0.1027 -0.1540 f = 0.0323 gg = 0 0
H = 0.2959   0.1940
    0.1940   0.5404
```

The built in function *fminunc* gives the solution in 5 iterations

```
options=optimset('GradObj','on','display','iter','HessUpdate','DFP',
        'LargeScale','off');
x=[0;0];
[x,fval]=fminunc(@frev16,x,options)
```

■ **17** The problem is not so straightforward since the variables X and Y are constrained and the search is not over the whole plane. It is reasonably simple to evaluate the function on a spreadsheet for X increasing from -0.25 by steps of 0.05 and 0.25 and Y increasing from 0 by steps of 0.1 to 2. It may be observed that

Maximum value of 1.055 at	$X = 0,\ Y = 0.45$	in the interior of the region
Minimum value of 0.528 at	$X = \pm 0.25,\ Y = 2$	on the boundary of the region

A more accurate value of 1.0557 at $X = 0, Y = 0.4736$ for the maximum can be found by any of the methods in the text. Typically one of the extreme points, in this case the minimum, is on the boundary.
The MATLAB routine *fmincon* is set up to deal with exactly this type of problem and can be accessed most easily through the Optimization Toolbox. From the MATLAB window the single instruction gives the minimum immediately (similarly for the maximum, with the sign `fmincon(@(x)sqrt((.5+x(2))^2+x(1)^2)/(x(1)^2+x(2)^2+.6979),[.1;1.5],[1,0;-1,0;0,1;0,-1],[.25;.25;2;0]) of the function changed)`

```
fmincon(@(x)sqrt((.5+x(2))^2+x(1)^2)/(x(1)^2+x(2)^2+.6979),[.1;1.5],
[1,0;-1,0;0,1;0,-1],[.25;.25;2;0])
```

■ **18** The sketch of the progress over the first steps of Partan is shown in Figure 10.1.
The calculations of the first few steps are straightforward and give
Steepest descent

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ f_1 = 1;\ \mathbf{x}_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix},\ f_2 = \frac{1}{2};\ \mathbf{x}_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, f_3 = \frac{1}{4};\ \mathbf{x}_4 = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix},\ f_4 = \frac{1}{8}$$

Partan

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ f_1 = 1;\ \mathbf{x}_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix},\ f_2 = \frac{1}{2};\ \mathbf{z}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix},\ f = \frac{1}{4};\ \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, f_3 = 0$$

Note that Partan obtains the minimum after one complete cycle. This method was an improvement on steepest descent but has been superseded by DFP and conjugate gradient methods which have been found to be superior in performance.

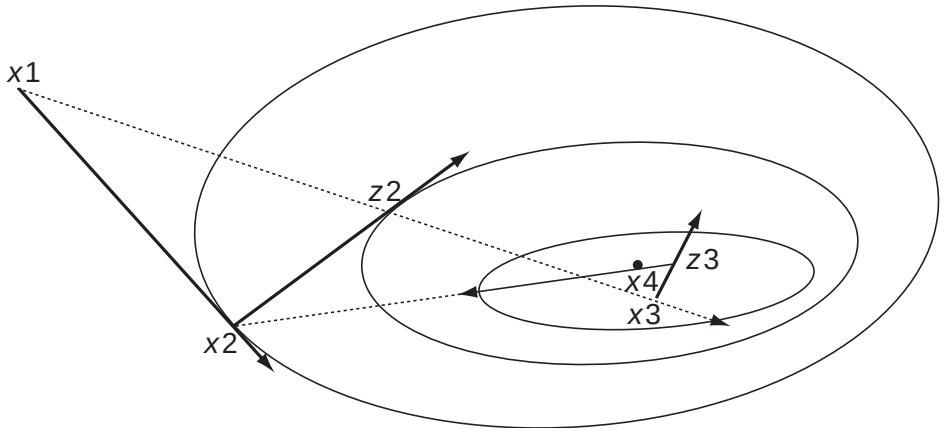


Figure 10.1: Illustration of the first steps of Partan in Exercise 18

■ **19** The problem is to minimize the approximating quadratic

$$f(a+h) = f(a) + \mathbf{h}^T\mathbf{G} + \frac{1}{2}\mathbf{h}^T\mathbf{J}\mathbf{h}$$

subject to $\mathbf{h}^T\mathbf{h} = L^2$. A Lagrange multiplier is required, so extending the argument in section 10.4.3 gives the modified gradient as

$$0 = \mathbf{G} + \mathbf{J}\mathbf{h} + \lambda\mathbf{h} = \mathbf{G} + (\mathbf{J} + \lambda\mathbf{I})\mathbf{h}$$

and hence

$$\mathbf{h} = -(\mathbf{J} + \lambda\mathbf{I})^{-1}\mathbf{G}$$

The implementation suggested can be computed straightforwardly for this exercise. The derivatives are obtained and hence

$$\underline{\lambda = 0} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; f = 1 \; \mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; \mathbf{J} = \begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix}$$

so

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{-4}\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \quad \text{and} \quad f = 1$$

Since there is no improvement, the next step is to take

$$\underline{\lambda = 1} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad f = 1 \quad \mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix}$$

so

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{-1} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad f = -5$$

and the method is ready to proceed to the next iteration.

■ **20(a)** Here

$$\mathbf{f} = \begin{bmatrix} x - y \\ \frac{1}{4}(x + y + 1)^2 \end{bmatrix} \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} 1 & -1 \\ \frac{1}{2}(x + y + 1) & \frac{1}{2}(x + y + 1) \end{bmatrix}$$

so the sequence of calculations is

$$\begin{aligned}
&\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad F = 0.0625 \quad \mathbf{f} = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 1 & -1 \\ 0.5 & 0.5 \end{bmatrix} \\
&\mathbf{x} = \begin{bmatrix} -0.25 \\ -0.25 \end{bmatrix} \quad F = 0.0039 \quad \mathbf{f} = \begin{bmatrix} 0 \\ 0.0625 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 1 & -1 \\ 0.25 & 0.25 \end{bmatrix} \\
&\mathbf{x} = \begin{bmatrix} -0.375 \\ -0.375 \end{bmatrix} \quad F = 0.00024 \quad \mathbf{f} = \begin{bmatrix} 0 \\ 0.0156 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 1 & -1 \\ 0.125 & 0.125 \end{bmatrix}
\end{aligned}$$

20(b) The functions in this exercise are more complicated, although the method is identical.

$$\mathbf{f} = \begin{bmatrix} \dfrac{1}{x + y} \\ \dfrac{x}{1 + 2x + y} \end{bmatrix} \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} \dfrac{-1}{(x + y)^2} & \dfrac{-1}{(x + y)^2} \\ \dfrac{1 + y}{(1 + 2x + y)^2} & \dfrac{-x}{(1 + 2x + y)^2} \end{bmatrix}$$

The computations follow in MATLAB

```
x=[1;1];
a=1/(x(1)+x(2)); b=2*x(1)+x(2)+1;f=[a;x(1)/b]
% gives   f= 0.5000 0.2500
J=[-a^2,-a^2;(1+x(2))/b^2,-x(1)/b^2]
% gives   J= -0.2500 -0.2500
              0.1250 -0.0625
```

```
x=x-inv(J'*J)*J'*f,F=f'*f
% gives   x= 0.3333 3.6667 F= 0.3125
```

Repeating these three lines of code gives successive iterations. From the calculations, the iterations gives x(1) approaches zero and x(2) approaches infinity!

■ **21** Fitting data to any curve, except for a straight line, is quite a tricky job. Optimization gives a method of finding a least squares fit of given data to a known curve. The problem needs the method of Exercise 20 so the vector of functions and the matrix of derivatives are required.

$$\mathbf{f} = \begin{bmatrix} 1 - \dfrac{1}{a} \\ 0.6 - \dfrac{1}{a+b} \\ 0.3 - \dfrac{1}{a+2b} \\ 0.2 - \dfrac{1}{a+3b} \end{bmatrix} \quad \text{and} \quad \mathbf{J} = \begin{bmatrix} \dfrac{1}{a^2} & 0 \\ \dfrac{1}{(a+b)^2} & \dfrac{1}{(a+b)^2} \\ \dfrac{1}{(a+2b)^2} & \dfrac{2}{(a+2b)^2} \\ \dfrac{1}{(a+3b)^2} & \dfrac{3}{(a+3b)^2} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad F = 1.29 \quad \mathbf{f} = \begin{bmatrix} 0 \\ -0.4 \\ -0.7 \\ -0.8 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1.07 \\ 0.27 \end{bmatrix} \quad F = 0.239 \quad \mathbf{f} = \begin{bmatrix} 0.0654 \\ -0.1463 \\ -0.3211 \\ -0.3319 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} 0.8734 & 0 \\ 0.5569 & 0.5569 \\ 0.3858 & 0.7716 \\ 0.2829 & 0.8488 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 0.9810 \\ 0.6922 \end{bmatrix} F = 0.0316 \quad \mathbf{f} = \begin{bmatrix} -0.0194 \\ 0.0023 \\ -0.1228 \\ -0.1271 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} 1.0391 & 0 \\ 0.3572 & 0.3572 \\ 0.1787 & 0.3575 \\ 0.1070 & 0.3209 \end{bmatrix}$$

The computations continue until the values of a and b do not change significantly. MATLAB has built in programs for various versions of this problem, *lscurvefit, lsqlin, lsqnonlin, lsqnonneg.*

■ **22** It is assumed, without loss of generality, that $\mathbf{A}$ is a symmetric matrix. Putting the search direction in the quadratic gives, using the symmetry of $\mathbf{A}$,

$$\mathbf{f} + \mathbf{c} + \mathbf{b}^T\mathbf{a} + \lambda\mathbf{b}^T\mathbf{d} + \frac{1}{2}(\mathbf{a}^T\mathbf{A}\mathbf{a} + 2\lambda\mathbf{a}^T\mathbf{A}\mathbf{d} + \lambda^2\mathbf{d}^T\mathbf{A}\mathbf{d})$$

and differentiating

$$0 = \frac{\partial \mathbf{f}}{\partial \lambda} = \mathbf{b}^T\mathbf{d} + \mathbf{a}^T\mathbf{A}\mathbf{d} + \lambda\mathbf{d}^T\mathbf{A}\mathbf{d}$$

Collecting up the terms

$$\lambda_{\min} = \frac{-(\mathbf{b} + \mathbf{A}\mathbf{a})^T\mathbf{d}}{\mathbf{d}^T\mathbf{A}\mathbf{d}}$$

gives the required result.

■ **23** To use the results of Exercise 22 it is necessary to write the function in the form

$$\mathbf{f} = 1 + [-2 \quad 0]\begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}[x \quad y]\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and the gradient is evaluated as

$$\mathbf{G} = \begin{bmatrix} -2 + 4x - 2y \\ -2x + 2y \end{bmatrix}$$

The calculations proceed as

$$\mathbf{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} f = 1 \; \mathbf{G} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \text{ so } \mathbf{d} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

and

$$\lambda_{\min} = -\frac{[-2 \quad 0]\begin{bmatrix} 2 \\ 0 \end{bmatrix}}{[2 \quad 0]\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}} = \frac{1}{4}$$

The second iteration follows

$$\mathbf{a} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} f = 0.5 \; \mathbf{G} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ so } \mathbf{d} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and

$$\lambda_{\min} = -\frac{[0 \quad -1]\begin{bmatrix}0\\1\end{bmatrix}}{[0 \quad 1]\begin{bmatrix}4 & -2\\-2 & 2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}} = \frac{1}{2}$$

The third iteration commences within the data

$$\mathbf{a} = \begin{bmatrix}0.5\\0.5\end{bmatrix} \quad f = 0.25 \ \mathbf{G} = \begin{bmatrix}-1\\0\end{bmatrix}$$

Because the function is a quadratic, the Newton method must converge in a single iteration.

■ **24** It is necessary to check that the solution given is appropriate.

$$y' = \frac{(1+b)}{b}e^{\frac{x}{b}} \text{ and } y'' = \frac{(1+b)}{b^2}e^{\frac{x}{b}}$$

$$yy'' - y'^2 + y' = \left(\frac{1+b}{b^2}\right)e^{\frac{x}{b}}\left[(1+b)e^{\frac{x}{b}} - b\right] - \left(\frac{1+b}{b}\right)^2 e^{\frac{2x}{b}} + \left(\frac{1+b}{b}\right)e^x$$

and all the terms in the right hand side cancel to zero, so the differential equation is satisfied. Also the point $x = 0, y = 1$ satisfies the equation. To evaluate the derivative at $x = 0$

$$y'(0) = \alpha = \frac{1+b}{b} \Rightarrow \ b = \frac{1}{\alpha - 1}$$

Thus, given α the value b can be computed and

$$y(1) = (1+b)e^{\frac{1}{b}} - b$$

so that

$$f(\alpha) = [\,(1+b)e^{\frac{1}{b}} - b - 3\,]^2$$

Using the bracket $1.4 < \alpha < 1.6$ the values are calculated to be

α	b	f
1.4	2.5	0.0776
1.5	2	0.0029
1.6	1.6667	0.0369

and the quadratic algorithm gives $\alpha^* = 1.5218, b = 1.9278$ and $f = 8.9 \times 10^{-5}$; the final iterated value is $\alpha = 1.523, b = 1.9133$.

- **25 and 26** Exercises 25 and 26 are extended problems that are open ended, so no advice is given on the solution of these two questions.

11

Applied Probability and Statistics

Exercises 11.3.7

■ **1** Let $X =$ lifetime, so $X \sim N(\mu, 2500)$

1(a) $\bar{X} = 780,\ n = 30$

Using $z_{.025} = 1.96$ from the normal table, the 95% confidence interval for μ is

$$780 \pm 1.96 \times 50/\sqrt{30} = (762, 798)$$

1(b) We require n large enough so that

$$\frac{1.96 \times 50}{\sqrt{n}} \leq 10$$

hence, $n \geq \left(\frac{1.96 \times 50}{10}\right)^2 = 96.04$ i.e. $n = 97$.

■ **2** From the data,

$$\bar{X} = 76.06, S_X = 31.85, n = 36$$

Using $z_{0.025} = 1.96$ we have the confidence interval

$$\bar{X} \pm 1.96 S_X/\sqrt{n} = (65.7, 86.5)$$

■ **3** From the data

$$\bar{X} = 8.725,\ S_{X,n-1} = 1.064,\ n = 12$$

Using $t_{.025,11} = 2.201$ from the t-distribution table, the 95% confidence interval for the mean is

$$8.725 \pm 2.201 \times 1.064/\sqrt{12} = (8.05, 9.40)$$

■ **4** Given sample average $\bar{X} = 73.2$, standard deviation 5.4 and size 30, and using $t_{.025,29} = 2.045$, the 95% confidence interval for the mean is

$$73.2 \pm 2.045 \times 5.4/\sqrt{30} = (71.2, 75.2)$$

The value (75) in the hypothesis lies within the interval and the hypothesis is accepted.

■ **5** Given sample average $\bar{X} = 3.42$, standard deviation 0.68 and size 16, and using $t_{.005,15} = 2.947$, the 99% confidence interval for the mean is

$$3.42 \pm 2.947 \times 0.68/\sqrt{16} = (2.92, 3.92)$$

■ **6** Given sample average $\bar{X} = 26.4$, standard deviation 4.28 and size 32, and using $t_{.025,31} \simeq t_{.025,29} = 2.045$ (alternatively the normal distribution figure of 1.96 could be used with a slight loss of accuracy), the 95% confidence interval for the mean is

$$26.4 \pm 2.045 \times 4.28/\sqrt{32} = (24.9, 27.9)$$

■ **7** Given sample average $\bar{X} = 56$, standard deviation 3 and size 10, and using $t_{.025,9} = 2.262$, the 95% confidence interval for the mean is

$$56 \pm 2.262 \times 3/\sqrt{10} = (53.9, 58.1)$$

The value (58%) in the hypothesis lies within the interval and the hypothesis is accepted at the 5% level.

■ **8** Given respective sample averages $\bar{X}_A = 36300$ and $\bar{X}_B = 39100$, standard deviations $S_{A,n-1} = 5000$ and $S_{B,n-1} = 6100$, and sizes $n_A = n_B = 12$, the pooled estimate of standard deviation is

$$S_p = \sqrt{\frac{11 \times (5000^2 + 6100^2)}{22}} = 5577$$

Using $t_{.025,22} = 2.074$ the 95% confidence interval for $\mu_B - \mu_A$ is

$$39100 - 36300 \pm 2.074 \times 5577 \times \sqrt{\frac{2}{12}} = (-1900, 7500)$$

Because zero lies within this interval, the hypothesis that $\mu_B > \mu_A$ is rejected.

■ **9** From the data, respective sample averages $\bar{X}_A = 7281$ and $\bar{X}_B = 6885$, standard deviations $S_{A,n-1} = 419.1$ and $S_{B,n-1} = 402.6$, and sizes $n_A = n_B = 8$, the pooled estimate of standard deviation is

$$S_p = \sqrt{\frac{7 \times (419.1^2 + 402.6^2)}{14}} = 410.9$$

Using $t_{.05,14} = 1.761$ and $t_{.025,14} = 2.145$, confidence intervals for $\mu_A - \mu_B$ are

$$90\%: \ 7281 - 6885 \pm 1.761 \times 410.9 \times \sqrt{\frac{2}{8}} = (34, 758)$$

$$95\%: \ 7281 - 6885 \pm 2.145 \times 410.9 \times \sqrt{\frac{2}{8}} = (-45, 837)$$

The hypothesis that $\mu_A = \mu_B$ is rejected at the 10% level but accepted at the 5% level.

■ **10** The sample proportion $\hat{p} = \frac{38}{540} = 0.0704$ and $n = 540$.
Using $z_{.05} = 1.645$ and $z_{.025} = 1.96$, confidence intervals for the true proportion p are

$$90\%: \ 0.0704 \pm 1.645 \times \sqrt{\frac{0.0704 \times (1 - 0.0704)}{540}} = (0.052, 0.089)$$

$$95\%: \ 0.0704 \pm 1.96 \times \sqrt{\frac{0.0704 \times (1 - 0.0704)}{540}} = (0.049, 0.092)$$

The hypothesis that $p < 0.05$ is rejected at the 10% level but accepted at the 5% level. Alternatively, the test statistic

$$Z = \frac{0.0704 - 0.05}{\sqrt{0.05 \times (1 - 0.05)/540}} = 21.8$$

leads to rejection at both 5% and 10% levels. The test statistic is more accurate (the confidence interval is approximate).

■ **11** Using $z_{.05} = 1.645$, the 90% confidence interval for proportion is

$$\hat{p} \pm 1.645\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Thus,

$$P\left[\mid p - \hat{p} \mid \leq 1.645\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right] = 0.9$$

so with probability 0.9 the maximum error is $1.645\sqrt{\hat{p}(1-\hat{p})/n}$. Although $\hat{p}$ is unknown before the experiment, a figure in the region of 0.25 is expected, hence we require

$$1.645\sqrt{\frac{0.25 \times 0.75}{n}} \leq 0.05$$

from which

$$n \geq \frac{0.25 \times 0.75}{(0.05/1.645)^2} = 203$$

If in fact $n = 200$, the sample proportion is $\hat{p} = \frac{55}{200} = 0.275$, and the 90% confidence interval for p is

$$0.275 \pm 1.645\sqrt{\frac{0.275 \times 0.725}{200}} = (0.223, 0.327)$$

■ **12** Using sample proportions $\hat{p}_1 = 0.31$ and $\hat{p}_2 = \frac{74}{150} = 0.493$, and $z_{.05} = 1.645$ and $z_{.025} = 1.96$, confidence intervals for $p_1 - p_2$ are

$$90\%: \ \hat{p}_1 - \hat{p}_2 \pm 1.645\left[\frac{\hat{p}_1(1-\hat{p}_1)}{100} + \frac{\hat{p}_2(1-\hat{p}_2)}{150}\right]^{1/2} = (-0.28, -0.08)$$

$$95\%: \ \hat{p}_2 - \hat{p}_1 \pm 1.96\left[\frac{\hat{p}_1(1-\hat{p}_1)}{100} + \frac{\hat{p}_2(1-\hat{p}_2)}{150}\right]^{1/2} = (-0.30, -0.06)$$

The hypothesis that $p_1 \leq p_2 - 0.08$ is therefore accepted at the 10% level but rejected at the 5% level.

■ **13** Using sample proportions $\hat{p}_1 = \frac{30}{180} = 0.1667$ and $\hat{p}_2 = \frac{32}{500-180} = 0.1$, and $z_{.025} = 1.96$, the 95% confidence interval for $p_1 - p_2$ is

$$\hat{p}_1 - \hat{p}_2 \pm 1.96\left[\frac{\hat{p}_1(1-\hat{p}_1)}{180} + \frac{\hat{p}_2(1-\hat{p}_2)}{320}\right]^{1/2} = (0.003, 0.130)$$

The hypothesis that $p_1 > p_2$ is therefore accepted at the 5% level. Alternatively, the test statistic

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\left[\hat{p}(1-\hat{p})\left(\frac{1}{180} + \frac{1}{320}\right)\right]^{1/2}} = 2.17 > z_{.025}$$

(where $\hat{p} = \frac{30+32}{500} = 0.124$) again leads to the hypothesis that $p_1 > p_2$ is accepted.

Exercises 11.4.7

■ **14**

		X		
Y	**1**	**2**	**3**	**total**
1	0	0.17	0.08	0.25
2	0.20	0.11	0	0.31
3	0.14	0.25	0.05	0.44
total	0.34	0.53	0.13	1

14(a) Marginal distributions of X and Y (summing rows and columns) are

$$P(X=1) = 0.34, \ P(X=2) = 0.53, \ P(X=3) = 0.13$$
$$P(Y=1) = 0.25, \ P(Y=2) = 0.31, \ P(Y=3) = 0.44$$

14(b)

$$P(Y=3 \mid X=2) = \frac{P(X=2 \cap Y=3)}{P(X=2)} = \frac{0.25}{0.53} = 0.472$$

14(c) The mean and variance of X are given by

$$\begin{aligned} E(X) &= 1 \times 0.34 + 2 \times 0.53 + 3 \times 0.13 = 1.79 \\ E(X^2) &= 1 \times 0.34 + 4 \times 0.53 + 9 \times 0.13 = 3.63 \\ \text{so } \sigma_X^2 &= 3.63 - 1.79^2 = 0.426 \end{aligned}$$

Similarly the mean and variance of Y are given by

$$\begin{aligned} E(Y) &= 1 \times 0.25 + 2 \times 0.31 + 3 \times 0.44 = 2.19 \\ E(Y^2) &= 1 \times 0.25 + 4 \times 0.31 + 9 \times 0.44 = 5.45 \\ \text{so } \sigma_X^2 &= 5.45 - 2.19^2 = 0.654 \end{aligned}$$

The expected value of the product XY is given by

$$\begin{aligned} E(XY) =& 1 \times 0 + 2 \times 0.17 + 3 \times 0.08 + 2 \times 0.2 + 4 \times 0.11 + 9 \times 0 \\ &+ 3 \times 0.14 + 6 \times 0.25 + 9 \times 0.05 = 3.79 \end{aligned}$$

Hence the correlation coefficient is

$$\rho_{X,Y} = \frac{3.79 - 1.79 \times 2.19}{\sqrt{0.426 \times 0.654}} = -0.246$$

■ **15**

$$\begin{aligned} E(X) &= \int_{-1/2}^{1/2} x dx = 0 \\ E(X^3) &= \int_{-1/2}^{1/2} x^3 dx = 0 \\ \text{Cov } (X, X^2) &= E(X^3) - E(X)E(X^2) = 0 \end{aligned}$$

■ **16** From $cx \le y \le cx+1$ we infer $\frac{y-1}{c} \le x \le \frac{y}{c}$, so

$$f_Y(y) = \int f_{X,Y}(x,y)dx$$
$$= \begin{cases} \int_0^{y/c} 1dx = \dfrac{y}{c} & \text{if } 0 \le y \le c \\ \int_0^1 1dx = 1 & \text{if } c \le y \le 1 \\ \int_{(y-1)/c}^1 1dx = 1 - \dfrac{1}{c}(y-1) & \text{if } 1 \le y \le 1+c \end{cases}$$

■ **17(a)**

$$\begin{aligned} P(X>100, Y>100) &= \frac{1}{8}\int_1^\infty \int_1^\infty xe^{-(x+y)/2}dydx \\ &= \frac{1}{8}\int_1^\infty xe^{-x/2}[-2e^{-y/2}]_1^\infty dx \\ &= \frac{1}{4}e^{-1/2}\int_1^\infty xe^{-x/2}dx \\ &= -\frac{1}{2}e^{-1/2}[xe^{-x/2}]_1^\infty + \frac{1}{2}e^{-1/2}\int_1^\infty e^{-x/2}dx \\ &= \frac{1}{2}e^{-1} + \frac{1}{2}e^{-1/2}[-2e^{-x/2}]_1^\infty \\ &= \frac{3}{2e} = 0.552 \end{aligned}$$

17(b)

$$\begin{aligned} f_Y(y) &= \frac{1}{8}\int_0^\infty xe^{-(x+y)/2}dx = \frac{1}{8}e^{-y/2}\int_0^\infty xe^{-x/2}dx \\ &= \frac{e^{-y/2}}{4}\left[-xe^{-x/2}|_0^\infty + \int_0^\infty e^{-x/2}dx\right] = \frac{1}{2}e^{-y/2} \end{aligned}$$

Hence

$$P(Y>2) = \int_2^\infty \frac{1}{2}e^{-y/2}dy = e^{-y/2}|_2^\infty = \frac{1}{e} = 0.368$$

■ **18** From the data, if X denotes height and Y denotes weight, $\bar{X} = 174.26$, $S_X = 7.184$, $\bar{Y} = 75.7$, $S_Y = 11.703$, $\overline{XY} = 13270$; so the sample correlation coefficient is

$$r = \frac{13270 - 174.26 \times 75.7}{7.184 \times 11.703} = 0.934$$

■ **19** From the data,

$$\bar{X} = 14.23,\ S_X = 2.457,\ \bar{Y} = 16.68,\ S_Y = 3.4,\ \overline{XY} = 243.47$$

so the sample correlation coefficient is

$$r = \frac{243.47 - 14.23 \times 16.68}{2.457 \times 3.4} = 0.732$$

■ **20** Given a sample correlation coefficient $r = 0.7$ with $n = 30$, and using $z_{.025} = 1.96$,

$$c = \exp\left(\frac{2 \times 1.96}{\sqrt{27}}\right) = 2.126$$

and the 95% confidence interval for correlation is

$$\left(\frac{1 + r - c(1-r)}{1 + r + c(1-r)}\ ,\ \frac{1 + r - (1-r)/c}{1 + r + (1-r)/c}\right) = (0.45, 0.85)$$

■ **21** From Exercise 18, $r = 0.934$ with $n = 8$. Using $z_{.025} = 1.96$,

$$c = \exp\left(\frac{2 \times 1.96}{\sqrt{5}}\right) = 5.772$$

and the 95% confidence interval for correlation is

$$\left(\frac{1 + r - c(1-r)}{1 + r + c(1-r)}\ ,\ \frac{1 + r - (1-r)/c}{1 + r + (1-r)/c}\right) = (0.67, 0.99)$$

■ **22** From the data, if X and Y denote mathematics and computer studies marks respectively,

$$\bar{X} = 56.80,\ S_X = 8.880,\ \bar{Y} = 54.40,\ S_Y = 12.05,\ \overline{XY} = 3137.4$$

so the sample correlation coefficient is

$$r = \frac{3137.4 - 56.8 \times 54.4}{8.88 \times 12.05} = 0.444$$

Using $z_{.05} = 1.645$,

$$c = \exp\left(\frac{2 \times 1.645}{\sqrt{17}}\right) = 2.221$$

and the 90% confidence interval is

$$\left(\frac{1 + r - c(1-r)}{1 + r + c(1-r)}\ ,\ \frac{1 + r - (1-r)/c}{1 + r + (1-r)/c}\right) = (0.08, 0.70)$$

Similarly using $z_{.025} = 1.96$,

$$c = \exp\left(\frac{2 \times 1.96}{\sqrt{17}}\right) = 2.588$$

and the 95% confidence interval is $(0.00, 0.74)$. This suggests that the correlation coefficient is significant at the 10% level but is marginal at the 5% level. The test statistic

$$Z = \frac{\sqrt{17}}{2} \ln \frac{1 + 0.444}{1 - 0.444} = 1.968$$

leads to a similar conclusion. The ranks of the data are as follows:

Math.	3.5	20	2	16.5	13	16.5	13	7	19	3.5
	10.5	5	18	10.5	7	7	15	13	9	1
Comp.	14	16.5	8	20	4	15	11.5	4	18.5	6.5
	6.5	16.5	9	18.5	10	11.5	2	13	4	1

The rank correlation is $r_s = 0.401$, and the test statistic

$$Z = r_s\sqrt{n-1} = 1.748$$

This is significant at the 10% level. (The approximate formula for rank correlation gives a value 0.405, from which $Z = 1.767$ with the same result.)

■ **23(a)**

$$\begin{aligned}\int_0^1\int_x^1 f_{X,Y}(x,y)dydx &= \int_0^1\int_x^1 c(1-y)dydx\\ &= \int_0^1 c[y-\tfrac{y^2}{2}]_x^1 dx\\ &= \int_0^1 c\left(\frac{1}{2}-x+\frac{1}{2}x^2\right)dx\\ &= c\left[\frac{1}{2}x-\frac{1}{2}x^2+\frac{1}{6}x^3\right]_0^1\\ &= \frac{c}{6}\end{aligned}$$

Since the joint density must integrate to unity, we must have $c = 6$.

23(b)

$$\begin{aligned}P(X<\frac{3}{4}, Y>\frac{1}{2}) &= \int_{1/2}^1\int_0^{3/4} f_{X,Y}(x,y)dxdy\\ &= \int_{1/2}^1\int_0^{\min\{3/4,y\}} c(1-y)dxdy\\ &= \int_{1/2}^{3/4}\int_0^y c(1-y)dxdy + \int_{3/4}^1\int_0^{3/4} c(1-y)dxdy\\ &= \int_{1/2}^{3/4} c(1-y)ydy + \int_{3/4}^1 \frac{3}{4}c(1-y)dy\\ &= c\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_{1/2}^{3/4} + \frac{3}{4}c\left[y-\frac{y^2}{2}\right]_{3/4}^1\\ &= 6\left[\frac{9}{32}-\frac{27}{192}-\frac{1}{8}+\frac{1}{24}\right]+\frac{9}{2}\left[1-\frac{1}{2}-\frac{3}{4}+\frac{9}{32}\right]\\ &= 0.484\end{aligned}$$

23(c)

$$f_X(x) = \int_x^1 c(1-y)dy = 6\left[y - \frac{y^2}{2}\right]_x^1 = 6\left(\frac{1}{2} - x + \frac{x^2}{2}\right) \text{ for } 0 \le x \le 1$$
$$f_Y(y) = \int_0^y c(1-y)dx = 6(1-y)y \text{ for } 0 \le y \le 1$$

■ **24** Individual density functions for X and Y are

$$f_X(x) = \begin{cases} 2 & \text{for } 29.8 < x < 30.3 \\ 0 & \text{otherwise} \end{cases}$$
$$f_Y(y) = \begin{cases} 2 & \text{for } 30.1 < y < 30.6 \\ 0 & \text{otherwise} \end{cases}$$

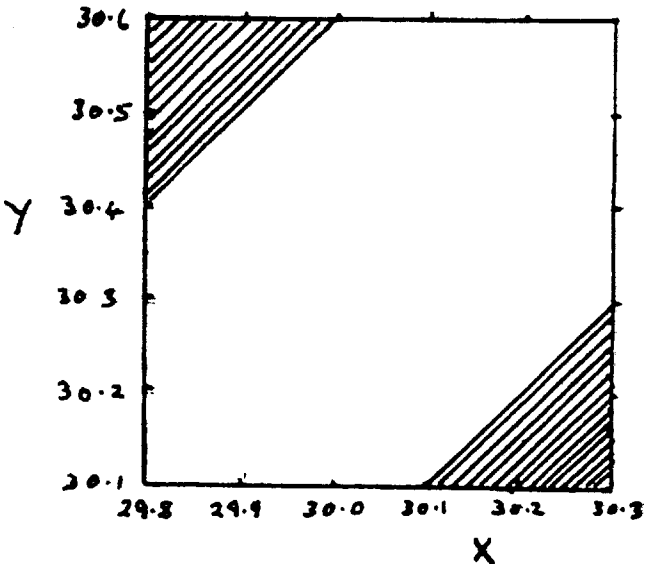
By independence, the joint density function is therefore

$$f_{X,Y}(x,y) = \begin{cases} 4 & \text{for } 29.8 < x < 30.3 \text{ and } 30.1 < y < 30.6 \\ 0 & \text{otherwise} \end{cases}$$

The required probability is therefore

$$\begin{aligned} P(0 \le Y - X \le 0.6) &= \int_{29.8}^{30.3} \int_{\max\{30.1,x\}}^{\min\{30.6,x+0.6\}} 4dydx \\ &= \int_{29.8}^{30.0} \int_{30.1}^{x+0.6} 4dydx \\ &+ \int_{30.0}^{30.1} \int_{30.1}^{30.6} 4dydx + \int_{30.1}^{30.3} \int_{x}^{30.6} 4dydx \\ &= 4\left[\int_{29.8}^{30.0} (x - 29.5)dx + \int_{30.0}^{30.1} 0.5dx \right. \\ &\qquad \left. + \int_{30.1}^{30.3} (30.6 - x)dx\right] \\ &= 4\left[\left[\frac{x^2}{2} - 29.5x\right]_{29.8}^{30.0} + 0.5 \times 0.1 + \left[30.6x - \frac{x^2}{2}\right]_{30.1}^{30.3}\right] \\ &= 0.84 \end{aligned}$$

Alternatively, the shaded area must be excluded from the square. The two parts of shaded area together form a square of side 0.2, so $P(0 \leq Y - X \leq 0.6) = \dfrac{0.5 \times 0.5 - 0.2 \times 0.2}{0.5 \times 0.5} = 0.84$.



Exercises 11.5.5

■ **25** From the data,

$$\bar{X} = 16.370, S_x = 6.789, \bar{Y} = 36.110, S_y = 14.576, \overline{XY} = 689.343$$

Hence

$$\hat{b} = \frac{689.343 - 16.37 \times 36.11}{6.789^2} = 2.13$$
$$\hat{a} = 36.11 - 2.13 \times 16.37 = 1.22$$

■ **26** From the data,

$$\bar{X} = 6.5,\ S_X = 3.452,\ \bar{Y} = 101.5,\ S_Y = 50.74,\ \overline{XY} = 834.25$$

Hence, the regression coefficients are

$$\hat{b} = \frac{834.25 - 6.5 \times 101.5}{3.452^2} = 14.64$$
$$\hat{a} = 101.5 - 14.64 \times 6.5 = 6.315$$

When load (X) is 15 kg, a deflection of 226 mm is predicted.

■ **27** From the results in Example 11.17 we find $\hat{a} = -2.294$ and $\hat{b} = 0.811$, and when 15 V is measured a tension of 9.88 kN is predicted.
Using $t_{.025,12} = 2.179$ and

$$S_E = \sqrt{S_Y^2 - \hat{b}^2 S_X^2} = \sqrt{16.25 - 0.811^2 \times 34.51} = 0.360$$

(remember that X and Y are essentially reversed compared with Example 11.17), the 95% confidence interval for tension when 15 V are measured is

$$9.88 \pm 2.179 \times 0.36 \times \sqrt{\frac{1 + (15 - 12.07)^2/24.51}{12}} = (9.62, 10.14)$$

■ **28(a)** From the data,

$$\bar{X} = 34.17,\ S_X = 11.70,\ \bar{Y} = 453.8,\ S_Y = 59.34,\ \overline{XY} = 15944$$

and the regression coefficients are

$$\hat{b} = \frac{15944 - 34.17 \times 453.8}{11.7^2} = 3.221 \text{ (using unrounded figures)}$$
$$\hat{a} = 453.8 - 3.221 \times 34.17 - 343.7$$

For advertising $x = 6$ (£ 6000), sales of £ 537,000 are predicted.

28(b) $S_E = \sqrt{59.34^2 - 3.221^2 \times 11.7^2} = 45.8$ and using $t_{.025,10} = 2.228$ the 95% confidence interval for regression slope is

$$\hat{b} \pm t_{.025,10} \frac{S_E}{S_X \sqrt{10}} = (0.46, 5.98)$$

The hypothesis that $b = 0$ is rejected at the 5% level.

28(c) The 95% confidence interval for mean sales when $x = 60$ is

$$537 \pm 2.228 \times 45.8 \times \sqrt{\frac{1 + (60 - 34.17)^2/136.8}{10}} = (459, 615)$$

■ **29** From the data,

$$\bar{X} = 11.5,\ S_X = 2.291,\ \bar{Y} = 13.25,\ S_Y = 2.99,\ \overline{XY} = 158.38$$

so the regression coefficients are

$$\hat{b} = \frac{158.38 - 11.5 \times 13.25}{2.291^2} = 1.143$$
$$\hat{a} = 13.25 - 1.143 \times 11.5 = 0.107$$

and $Y = 16.1$ is predicted when $x = 14$. Also using

$$S_E = \sqrt{S_Y^2 - \hat{b}^2 S_X^2} = 1.442$$

and $t_{.05,6} = 1.943$, the 90% confidence interval for mean number of defectives per hour when $x = 14$ is

$$16.1 \pm 1.943 \times 1.442 \times \sqrt{\frac{1 + (14 - 11.5)^2/5.25}{6}} = (14.4, 17.8)$$

■ **30** Given a model (with no constant) of form

$$Y_i = bX_i + E_i$$

and minimizing

$$Q = \sum_{i=1}^{n} [Y_i - bX_i]^2$$

we have

$$\frac{dQ}{db} = -2 \sum_{i=1}^{n} X_i [Y_i - \hat{b} X_i] = 0$$

hence

$$\hat{b} = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}$$

■ **31** If X denotes voltage and $Y = X/R$ denotes current then from the data,

$$\Sigma_i X_i Y_i = 5397,\ \Sigma_i X_i^2 = 650$$

so that (using the result of the previous exercise)

$$\hat{b} = 8.30$$

The estimated resistance is $R = \dfrac{1000}{8.3} = 120\Omega$.

■ **32** Taking logs we have

$$\ln P_i + \lambda \ln V_i = \ln C$$

This is of the form $Y_i = a + bX_i$ with

$$Y_i = \ln P_i,\ a = \ln C,\ b = -\lambda,\ X_i = \ln V_i$$

From the data,

$$\bar{X} = 4.272,\ S_X = 0.254,\ \bar{Y} = 3.423,\ \overline{XY} = 14.452$$

so that

$$\hat{b} = -2.664 \text{ and } \hat{a} = 14.80$$

Hence

$$\hat{C} = e^{\hat{a}} = 2.69 \times 10^6 \text{ and } \hat{\lambda} = -\hat{b} = 2.66$$

When $V = 80\,\text{cm}^3$, a pressure $P = 22.9\,\text{kg/cm}^3$ is predicted.

■ **33** Taking logs we have

$$\ln Y_i = \ln a + b \ln X_i$$
$$\text{or } Y_i' = a' + bX_i'$$

From the data,

$$\bar{X}' = 6.183,\ S_{X'} = 1.515,\ \bar{Y}' = 2.377,\ \overline{X'Y'} = 12.266$$

so

$$\hat{b} = -1.059,\ \hat{a}' = 8.927 \text{ and } \hat{a} = 7533$$

For a lot size $X = 300$, a unit cost of 17.9 is predicted.

Exercises 11.6.3

■ **34** Under the hypothesis, $P(A) = \frac{4}{7}, P(B) = \frac{2}{7}, P(C) = \frac{1}{7}$.

	observed	**probability**	**expected**	χ^2 **contribution**
A	63	4/7	57.14	0.601
B	22	2/7	28.57	1.511
C	15	1/7	14.29	0.035
Total				2.147

No parameters were estimated ($t = 0$), so we compare $\chi^2 = 2.147$ with $\chi^2_{0.05,2} = 5.991$. The hypothesis is accepted.

■ **35** The total number of books is 640, so a uniform number would be 128 per day. Hence

Obs. (f_k):	153	108	120	114	145
Exp. (e_k):	128	128	128	128	128
χ^2 **contribution**:	4.9	3.1	0.5	1.5	2.3

The total chi-square value is

$$\chi^2 = 12.3$$

and this is greater than $\chi^2_{.05,4} = 9.49$ (significant at 5%, but not significant at 1%).

■ **36** The observed mean number of flaws per sample is $(12 + 6 \times 2)/50$ or 0.48. Setting $\lambda = 0.48$, the Poisson probabilities are given by $\lambda^k e^{-\lambda}/k!$ and hence expected values are as follows:

number of flaws	**Obs.** (f_k)	**probability**	**exp.** (e_k)	χ^2 **contribution**
0	32	0.619	30.9	0.036
1	12	0.297	14.9	0.547
≥ 2	6	0.084	4.2	0.761

The total chi-square value is

$$\chi^2 = 1.35$$

which is very small, so the Poisson hypothesis is accepted.

■ **37** The observed average number of α-particles per time interval (taking class > 10 as 11 for this calculation) is

$$(203 + 2 \times 383 + \ldots + 11 \times 6)/(57 + 203 + \ldots + 6) = 3.87$$

Using this value for λ in the Poisson probabilities and proceeding as in the previous exercise a total chi-square value

$$\chi^2 = 12.97$$

is obtained. One parameter has been estimated and used for predicting the expected values, so the comparison is with $\chi^2_{.05,10} = 18.3$, and the Poisson hypothesis is accepted.

■ **38** Using the measured average and standard deviation, probabilities can be obtained from the normal table as follows:

$$P(X < 6.5) = P\big(\frac{X-10}{2} < \frac{6.5-10}{2}\big) = \Phi(-1.75) = 1 - \Phi(1.75) = 0.0401$$
$$P(X < 7.5) = 1 - \Phi(1.25) = 0.1056$$
$$P(X < 8.5) = 1 - \Phi(0.75) = 0.2266$$
$$P(X < 9.5) = 1 - \Phi(0.25) = 0.4013$$
$$P(X < 10.5) = \Phi(0.25) = 0.5987$$
$$P(X < 11.5) = \Phi(0.75) = 0.7734$$
$$P(X < 12.5) = \Phi(1.25) = 0.8944$$
$$P(X < 13.5) = \Phi(1.75) = 0.9599$$
$$P(X < 13.5) = 0.0401$$

The class probabilities can now be inferred by

$$P(6.5 < X < 7.5) = P(X < 7.5) - P(X < 6.5) = 0.0655$$

and so on; hence, the following table:

Class	Probability	Expected	Sample 1	Sample 2
< 6.5	0.0401	4.01	4	3
6.5–7.5	0.0655	6.55	6	6
7.5–8.5	0.1210	12.10	16	16
8.5–9.5	0.1747	17.47	16	13
9.5–10.5	0.1974	19.74	17	26
10.5–11.5	0.1747	17.47	20	7
11.5–12.5	0.1210	12.10	12	19
12.5–13.5	0.0655	6.55	6	5
> 13.5	0.0401	4.01	3	5

For sample 1, $\chi^2 = 2.48$ which is not significant.
For sample 2, $\chi^2 = 15.51$ which exceeds $\chi^2_{.025,6} = 14.45$ (significant at 2.5% level) but does not exceed $\chi^2_{.01,6} = 16.81$ (not significant at 1% level). The second subscript is $m - t - 1$ with $m = 9$ (classes) and $t = 2$ (parameters estimated).

■ **39** The contingency table is as follows:

	Perfect	Intermediate	Unacceptable	Total
A	89 (89.04)	23 (21.44)	12 (13.52)	124
B	62 (58.88)	12 (14.18)	8 (8.94)	82
C	119 (122.07)	30 (29.39)	21 (18.54)	170
Total	270	65	41	376

The expected values are shown in brackets,

$$\frac{270 \times 124}{376} = 89.04$$

and so on. Hence

$$\chi^2 = \frac{(89-89.04)^2}{89.04} + \cdots + \frac{(21-18.54)^2}{18.54}$$
$$= 1.30$$

This is less than $\chi^2_{.05,(3-1)(3-1)} = 9.49$, so there is no significant difference in quality.

■ **40** The contingency table (with the expected values in brackets) is as follows:

o.k.		**defective**		**total**
442	(441.6)	8	(8.38)	450
536	(539.8)	14	(10.25)	550
544	(539.8)	6	(10.25)	550
397	(392.5)	3	(7.45)	400
593	(588.8)	7	(11.18)	600
442	(441.6)	8	(8.38)	450
434	(441.6)	16	(8.38)	450
195	(196.3)	5	(3.73)	200
438	(441.6)	12	(8.38)	450
594	(588.8)	6	(11.18)	600
585	(588.8)	15	(11.18)	600
541	(539.8)	9	(10.25)	550
5741		109		5850

Hence, $\chi^2 = \dfrac{(442-441.6)^2}{441.6} + \cdots + \dfrac{(9-10.25)^2}{10.25} = 20.56$ which exceeds $\chi^2_{.05,11} = 19.68$ but is less than $\chi^2_{.025,11} = 21.92$. The variation is significant at the 5% level. For a $2 \times c$ table of this form, effectively, a comparison of c proportions, it is quicker (and equivalent) to compute

$$\chi^2 = \sum_{j=1}^{c} \frac{(f_j - n_j\hat{p})^2}{n_j\hat{p}(1-\hat{p})}$$

where f_j is the number of defectives in column j (total n_j) and $\hat{p} = \sum_{j=1}^{c} f_j / \sum_{j=1}^{c} n_j$ is the overall sample proportion of defectives.

■ **41** The contigency table (with expected values in brackets and adjusted residuals underneath) is as follows:

Spending level	**Jacket**	**Shirt**	**Trousers**	**Shoes**	**Total**
Low	21(36)	94(92)	57(47)	113(110)	285
	−3.0	0.4	1.8	0.4	
Medium	66(66)	157(169)	94(87)	209(204)	526
	0.0	−1.5	1.0	0.7	
High	58(43)	120(110)	41(57)	125(133)	344
	2.9	1.3	−2.8	−1.1	
Total	145	371	192	447	1155

Chi-square = 20.7, d.f. $= (4-1)(3-1) = 6$, so compare with $\chi^2_{0.005,6} = 18.5$: significant at 0.5% level. High-spending customers are tending to buy more of the jacket and less of the trousers. For low-spending customers, it is the other way round.

■ **42** If p is the proportion requiring adjustments, the number of such sets in a sample of size n is binomial with parameters n, p. With $n = 4$, to test the hypothesis that $p = 0.1$ we have

$$P(0) = 0.9^4 = 0.6561$$
$$P(1) = \binom{4}{1} 0.9^3 0.1 = 0.2916$$
$$P(2) = \binom{4}{2} 0.9^2 0.1^2 = 0.0486$$
$$P(3) = \binom{4}{3} 0.9 \times 0.1^3 = 0.0036$$

Hence the following table:

k	f_k	p_k	$e_k = 200p_k$	χ^2 **contribution**
0	110	0.6561	131.22	3.43
1	73	0.2916	58.32	3.70
2	16	0.0486	9.72	4.06
3	1	0.0036	0.72	0.11

The total chi-square value is $\chi^2 = 11.30$, which is significant at 5% ($\chi^2_{.05,3} = 7.82$) but not quite significant at 1% ($\chi^2_{.01,3} = 11.35$).
Using proportions, the total number of sets requiring adjustments is $73 + 2 \times 16 + 3 \times 1 = 108$, so

$$\hat{p} = \frac{108}{800} = 0.135$$

Using $z_{.025} = 1.96$ and $z_{.005} = 2.576$, confidence intervals for the proportion p of sets requiring adjustments are

$$95\%: \quad \hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{800}} = (0.111, 0.159)$$
$$99\%: \quad \hat{p} \pm 2.576\sqrt{\frac{\hat{p}(1-\hat{p})}{800}} = (0.104, 0.166)$$

This indicates that $p > 0.1$, significant (just) at the 1% level.

Exercises 11.7.4

■ **43** We must have

$$\begin{aligned} 1 = \int_0^\infty f_X(x)dx &= c\int_0^\infty xe^{-2x}dx \\ &= -\frac{1}{2}c[xe^{-2x}]_0^\infty + \frac{1}{2}c\int_0^\infty e^{-2x}dx \\ &= \frac{1}{2}c\big[-\frac{1}{2}e^{-2x}\big]_0^\infty \\ &= \frac{c}{4} \end{aligned}$$

so $c = 4$. The m.g.f. is then

$$\begin{aligned} M_X(t) = \int_0^\infty e^{tx}f_X(x)dx &= 4\int_0^\infty xe^{(t-2)x}dx \\ &= \frac{4}{t-2}[xe^{(t-2)x}]_0^\infty - \frac{4}{t-2}\int_0^\infty e^{(t-2)x}dx \\ &= \frac{4}{(t-2)^2} \text{ provided } t < 2 \end{aligned}$$

Hence

$$E(X) = M'_X(0) = -\frac{8}{(t-2)^3}\bigg|_{t=0} = 1$$
$$E(X^2) = M''_X(0) = \frac{24}{(t-2)^4}\bigg|_{t=0} = \frac{3}{2}$$
$$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{2}$$

■ **44** If $X_1, \ldots, X_n$ are independent Poisson random variables with parameters $\lambda_1, \ldots, \lambda_n$ and if $Y = X_1 + \ldots + X_n$ then

$$\begin{aligned} M_Y(t) &= M_{X_1}(t) \ldots M_{X_n}(t) \\ &= \exp[\lambda_1(e^t - 1)] \ldots \exp[\lambda_n(e^t - 1)] \\ &= \exp[(\lambda_1 + \ldots + \lambda_n)(e^t - 1)] \end{aligned}$$

Hence, Y is another Poisson random variable with parameter $\lambda = \lambda_1 + \ldots + \lambda_n$.

■ **45** Numbers of breakdowns in one hour are separately binomial with parameters $n_1 = 30, p_1 = 0.01$ and $n_2 = 40, p_2 = 0.005$ respectively, and hence approximately Poisson with parameters $\lambda_1 = 0.3$ and $\lambda_2 = 0.2$ respectively. The total number of breakdowns per hour is therefore also approximately Poisson with $\lambda = \lambda_1 + \lambda_2 = 0.5$, and

$$P\,(\text{three or more}) \simeq 1 - e^{-\lambda}\big(1 + \lambda + \frac{\lambda^2}{2!}\big) = 0.014$$

■ **46** Let the proportion defective be p. By the Poisson approximation,

$$P(k \text{ defective in } 100) \simeq \frac{\lambda^k e^{-\lambda}}{k!}$$

where $\lambda = 100p$. The requirement is

$$P(k \leq 1) \simeq e^{-\lambda}(1 + \lambda) > 0.9$$

from which $\lambda < 0.531$ (solving this as a nonlinear equation) and hence $p < \frac{0.531}{100} = 0.0053$. Therefore, at least 99.47% of servomechanisms must be satisfactory.

■ **47** With

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},\ -\infty < z < \infty$$

the m.g.f. is

$$\begin{aligned} M_Z(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} e^{-z^2/2} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2tz+t^2)} e^{t^2/2} dz \\ &= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} dz \end{aligned}$$

The integrand is the p.d.f. of a normal random variable with mean t and standard deviation equalling one, hence

$$M_Z(t) = e^{t^2/2}$$

Exercises 11.9.7

■ **48** From the table (Figure 11.29), with $n = 50, p = 0.1$ so that $np = 5$, we read off the Shewhart warning limit as 9.5 and the action limit as 13.5. For the given data, the warning limit is exceeded at samples 3, 9 and 11, and the action limit is exceeded at sample 12. The upper control limit is given by

$$\text{UCL } = np + 3\sqrt{np(1-p)} = 11.4$$

and this is first exceeded at sample 9.

■ **49** From Figure 11.29, with $n = 100,\ p = 0.02$ so that $np = 2$, we have Shewhart warning and action limits 5.5 and 7.5 respectively. The warning limit is exceeded three times (samples 20, 22 and 25) before the action limit is exceeded at sample 28. The upper control limit is given by

$$\text{UCL } = np + 3\sqrt{np(1-p)} = 6.2$$

and this is first exceeded at sample 25.

■ **50** Using $\sigma = 3$ and $n = 10$, the Shewhart warning and action limits are

$$c_W = 1.96\sigma/\sqrt{n} = 1.86$$
$$c_A = 3.09\sigma/\sqrt{n} = 2.93$$

(above and below the mean). Relative to the mean, the warning limit is exceeded at samples 3 and 9, and the action limit at sample 12.

■ **51** Using $\sigma = 3$ and $n = 10$, the Shewhart warning and action limits above the design mean $\mu = 12$ are

$$\mu + 1.96\sigma/\sqrt{n} = 13.86$$
$$\mu + 3.09\sigma/\sqrt{n} = 14.93$$

The warning limit is exceeded several times (at samples 6, 12, 15, 17 and 18) before the action limit is crossed at sample 19.

■ **52(a)** Using $\sigma = 3$ and $n = 10$, the cusum parameters are

$$r = \frac{\sigma}{2\sqrt{n}} = 0.474 \text{ (relative to the mean)}$$
$$h = 5\frac{\sigma}{\sqrt{n}} = 4.74$$

The chart is built up in the following table:

value	-0.2	1.3	2.1	0.3	-0.8	1.7	1.3	0.6	2.5	1.4	1.6	3.0
cusum	0	0.83	2.45	2.28	1.00	2.23	3.05	3.18	5.21	6.13	7.26	9.78

The decision interval (h) is exceeded at sample 9.

52(b) Using $r = 0.3$ and σ, n as above, the GMA action limits are

$$\pm 3.09\sqrt{\frac{r}{2-r}}\frac{\sigma}{\sqrt{n}} = \pm 1.23$$

(relative to the mean). The chart is built up in the following table:

value	-0.2	1.3	2.1	0.3	-0.8	1.7	1.3	0.6	2.5	1.4	1.6	3.0
GMA	-0.06	0.35	0.87	0.70	0.25	0.69	0.87	0.79	1.30	1.33	1.41	1.89

The action limit is exceeded at sample 9.

■ **53** For the cusum chart we have $\mu = 12, \sigma = 3$ and $n = 10$, so

$$r = \mu + \frac{\sigma}{2\sqrt{n}} = 12.47$$
$$h = 5\frac{\sigma}{\sqrt{n}} = 4.74$$

For the GMA chart with $r = 0.3$ we have action limits at

$$\mu \pm 3.09\sqrt{\frac{r}{2-r}}\frac{\sigma}{\sqrt{n}} = 10.77 \text{ and } 13.23$$

The control charts are built up in the following table:

$\bar{\boldsymbol{X}}_{\boldsymbol{m}}$	12.8	11.2	13.4	12.1	13.6	13.9	12.3	12.9	13.8	13.1
cusum	0.33	0	0.93	0.55	1.68	3.10	2.93	3.35	4.68	5.31
GMA	12.24	11.93	12.37	12.29	12.68	13.05	12.82	12.85	13.13	13.12
$\bar{\boldsymbol{X}}_{\boldsymbol{m}}$	12.9	14.0	13.7	13.4	14.2	13.1	14.0	14.0	15.1	14.3
cusum	5.73	7.26	8.48	9.41	11.13	11.76	13.28	14.81	17.44	19.26
GMA	13.06	13.34	13.45	13.43	13.66	13.49	13.65	13.75	14.16	14.20

The cusum chart indicates action at sample 10, the GMA chart at sample 12.

■ **54** Using $n = 50, p = 0.1$ so that $np = 5$, the cusum parameters from Figure 11.31 are $r = 7, h = 8.5$ (nearest values).

count	5	8	11	5	6	4	9	7	12	9	10	14
cusum	0	1	5	3	2	0	2	2	7	9	12	19

The chart indicates action at sample 10.

■ **55** Using $n = 100, p = 0.02$ so that $np = 2$, the cusum parameters from Figure 11.31 are $r = 3, h = 5.5$ (nearest values).

count	3	3	5	3	5	0	3	1	3	5	4	2	4	3	5	4
cusum	0	0	2	2	4	1	1	0	0	2	3	2	3	3	5	6
count	3	4	5	6	5	6	4	4	7	5	4	8	5	6	6	7
cusum	6	7	9	12	14	17	18	19	23	25	26	31	33	36	39	43

The chart indicates action at sample 16.

■ **56** For the Shewhart chart we have $n = 12, \sigma = 1$, and hence warning and action limits given by

$$c_W = 1.96\sigma/\sqrt{n} = 0.57$$
$$c_A = 3.09\sigma/\sqrt{n} = 0.89$$

For the cusum chart we have

$$r = \frac{\sigma}{2\sqrt{n}} = 0.144 \text{ (relative to the mean)}$$
$$h = 5\frac{\sigma}{\sqrt{n}} = 1.443$$

For the GMA chart with $r = 0.2$, the action limit is

$$3.09\sqrt{\frac{r}{2-r}}\frac{\sigma}{\sqrt{n}} = 0.297$$

$\bar{\bar{X}}_m$	0.1	0.3	-0.2	0.4	0.1	0	0.2	-0.1	0.2	0.4	0.5
cusum	0	.156	0	.256	.211	.067	.123	0	.056	.311	.667
GMA	.020	.076	.021	.097	.097	.078	.102	.062	.089	.152	.221
$\bar{\bar{X}}_m$	0.1	0.4	0.6	0.3	0.4	0.3	0.6	0.5	0.4	0.2	0.3
cusum	.623	.878	1.334	1.490	1.745	1.901	2.357	2.712	2.968	3.024	3.179
GMA	.197	.238	.310	.308	.326	.321	.377	.402	.401	.361	.349
$\bar{\bar{X}}_m$	0.5	0.7	0.3	0.1	0.6	0.5	0.6	0.7	0.4	0.5	
cusum	3.535	4.091	4.246	4.202	4.658	5.013	5.469	6.025	6.280	6.636	
GMA	.379	.443	.415	.352	.401	.421	.457	.505	.484	.488	

For the Shewhart chart there are several warnings but no action indicated. For the cusum and GMA charts, action is indicated at samples 15 and 14 respectively.

■ **57** The GMA S_m is defined recursively by

$$S_0 = \mu_X$$
$$S_m = r\bar{X}_m + (1-r)S_{m-1} \text{ for } m \geq 1$$

Substituting for S_{m-1}, then S_{m-2} and so on gives

$$\begin{aligned} S_m &= r\bar{X}_m + (1-r)[r\bar{X}_{m-1} + (1-r)S_{m-2}] \\ &= r[\bar{X}_m + (1-r)\bar{X}_{m-1}] + (1-r)^2[rX_{m-2} + (1-r)S_{m-3}] \end{aligned}$$

and eventually

$$S_m = r\sum_{i=0}^{m-1}(1-r)^i\bar{X}_{m-i} + (1-r)^m\mu_X$$

But $E(\bar{X}_{m-i}) = \mu_X$ and $Var(\bar{X}_{m-i}) = \dfrac{\sigma_X^2}{n}$ for all i, m.
Using the result

$$\sum_{i=0}^{m-1} x^i = \frac{1-x^m}{1-x} \text{ for } |x| < 1$$

we have

$$\begin{aligned} E(S_m) &= r\mu_X\sum_{i=0}^{m-1}(1-r)^i + (1-r)^m\mu_X \\ &= r\mu_X\frac{1-(1-r)^m}{r} + (1-r)^m\mu_X \\ &= \mu_X \\ Var(S_m) &= \frac{\sigma_X^2}{n}r^2\sum_{i=0}^{m-1}[(1-r)^2]^i = \frac{\sigma_X^2 r^2}{n}\frac{1-(1-r)^{2m}}{1-(1-r)^2} \\ &= \frac{\sigma_X^2}{n}[1-(1-r)^{2m}]\frac{r}{2-r} \\ &\longrightarrow \left(\frac{r}{2-r}\right)\frac{\sigma_X^2}{n} \text{ as } m \to \infty \end{aligned}$$

■ **58** Let X_m = count of defectives for sample m, n = sample size, p = probability of defective. Define

$$S_0 = np$$
$$S_m = rX_m + (1-r)S_{m-1} \text{ for } m \geq 1$$

Substituting for S_{m-1}, S_{m-2} and so on (as in the previous exercise) leads to

$$S_m = r\sum_{i=0}^{m-1}(1-r)^i X_{m-i} + (1-r)^m np$$

From the mean and variance of the binomial distribution,

$$E(X_{m-i}) = np$$
$$Var(X_{m-i}) = np(1-p)$$

hence

$$\begin{aligned} E(S_m) &= rnp\sum_{i=0}^{m-1}(1-r)^i + (1-r)^m np \\ &= np[1-(1-r)^m + (1-r)^m] = np \\ Var(S_m) &= np(1-p)r^2\sum_{i=0}^{m-1}[(1-r)^2]^i \\ &= \left(\frac{r}{2-r}\right)[1-(1-r)^{2m}]np(1-p) \\ &\longrightarrow \left(\frac{r}{2-r}\right)np(1-p) \text{ as } m\to\infty \end{aligned}$$

The upper control limit for $n = 50,\ p = 0.05,\ r = 0.2$ is

$$\text{UCL } = np + 3\sqrt{\left(\frac{r}{2-r}\right)np(1-p)} = 4.04$$

$\boldsymbol{X_m}$	3	5	2	2	1	6	4	4	2	6
$\boldsymbol{S_m}$	2.6	3.08	2.86	2.69	2.35	3.08	3.27	3.41	3.13	3.70
$\boldsymbol{X_m}$	7	4	5	5	8	6	5	9	7	8
$\boldsymbol{S_m}$	4.36	4.29	4.43	4.55	5.24	5.39	5.31	6.05	6.24	6.59

The chart indicates action after 11 samples.

■ **59** For the Shewhart chart we have $n = 10,\ \mu = 6,\ \sigma = 0.2$ and hence warning and action limits given by

$$c_W = \mu \pm 1.96\sigma/\sqrt{n} = 5.88 \text{ and } 6.12$$
$$c_A = \mu \pm 3.09\sigma/\sqrt{n} = 5.80 \text{ and } 6.20$$

For the cusum chart we have

$$r = \mu + \frac{\sigma}{2\sqrt{n}} = 6.032$$
$$h = 5\frac{\sigma}{\sqrt{n}} = 0.316$$

For the GMA chart with $r = 0.2$, the action limits are

$$\mu \pm 3.09\sqrt{\frac{r}{2-r}}\frac{\sigma}{\sqrt{n}} = 5.935 \text{ and } 6.065$$

$\bar{X}_m$	6.04	6.12	5.99	6.02	6.04	6.11	5.97	6.06	6.05	6.06
cusum	0.01	0.10	0.06	0.04	0.05	0.13	0.07	0.10	0.12	0.14
GMA	6.01	6.03	6.02	6.02	6.03	6.04	6.03	6.03	6.04	6.04
$\bar{X}_m$	6.17	6.03	6.13	6.05	6.17	5.97	6.07	6.14	6.03	5.99
cusum	0.28	0.28	0.38	0.40	0.54	0.47	0.51	0.62	0.62	0.58
GMA	6.07	6.06	6.07	6.07	6.09	6.07	6.07	6.08	6.07	6.05
$\bar{X}_m$	6.10	6.01	5.96	6.12	6.02	6.20	6.11	5.98	6.02	6.12
cusum	0.65	0.62	0.55	0.64	0.63	0.80	0.88	0.82	0.81	0.90
GMA	6.06	6.05	6.03	6.05	6.05	6.08	6.08	6.06	6.05	6.07

The Shewhart chart indicates action after 26 samples, the cusum chart after 13 samples and the GMA chart after 11 samples.

Exercises 11.10.6

■ **60** If gales occur at rate $\frac{15}{12} = 1.25$ per month, and occur independently, then the number of gales in any one month has a Poisson distribution, so

$$\begin{aligned} P(\text{more than two in one month}) &= 1 - P(0) - P(1) - P(2) \\ &= 1 - e^{-\lambda T}\left(1 + \lambda T + \frac{\lambda^2 T^2}{2!}\right) \\ &= 0.132 \end{aligned}$$

(with $\lambda = 1.25,\ T = 1$).

■ **61** If $\lambda = 30$ calls per hour (on average) arrive at the switchboard, and do so independently, then the number of calls has a Poisson distribution. Hence

$$P\,(\text{no calls in } \ T = 3 \text{ min}) \ = e^{-\lambda T/60} = 0.223$$

$$P\,(\text{more than five calls in } T = 5 \text{ min})$$

$$= 1 - e^{-\lambda T/60}\left(1 + \frac{\lambda T}{60} + \ldots + \frac{(\lambda T)^5}{60^5 5!}\right)$$

$$= 0.042$$

■ **62** The steady-state distribution for the number (N) in the system is

$$P(n \text{ in system}) \ = p_n = (1-\rho)\rho^n, \ n \geq 0$$

Now

$$\frac{d}{d\rho}\sum_{n=0}^{\infty}\rho^n = \sum_{n=0}^{\infty} n\rho^{n-1} = \frac{d}{d\rho}(1-\rho)^{-1} = (1-\rho)^{-2}$$

Hence, the mean number in the system is

$$N_S = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \rho(1-\rho)\sum_{n=0}^{\infty} n\rho^{n-1}$$

$$= \frac{\rho(1-\rho)}{(1-\rho)^2} = \frac{\rho}{1-\rho}$$

If there are $n > 0$ customers in the queue then there are $n+1$ in the system, so

$$P(n > 0 \text{ in queue}) \ = (1-\rho)\rho^{n+1}, \ n \geq 1$$

(we do not need the probability for $n = 0$). Mean number in queue is

$$N_Q = (1-\rho)\sum_{n=1}^{\infty} n\rho^{n+1} = \rho^2(1-\rho)\sum_{n=0}^{\infty} n\rho^{n-1}$$

$$= \frac{\rho^2(1-\rho)}{(1-\rho)^2} = \frac{\rho^2}{1-\rho}$$

■ **63** For a single-channel queue with $\lambda = 3$ arrivals per hour and $\mu = 4$ patients treated per hour,

63(a) $P(0 \text{ in system}) = 1 - \dfrac{\lambda}{\mu} = \dfrac{1}{4}$

63(b) $N_Q = \dfrac{(\lambda/\mu)^2}{1 - \lambda/\mu} = \dfrac{9}{4}$

63(c)

$$\begin{aligned} P(>3 \text{ in queue}) &= P(>4 \text{ in system}) \\ &= 1 - P(0) - P(1) - P(2) - P(3) - P(4) \\ &= 1 - \left(1 - \frac{\lambda}{\mu}\right)\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \left(\frac{\lambda}{\mu}\right)^4\right) \\ &= 0.237 \end{aligned}$$

63(d) $W_Q = \dfrac{\lambda/\mu}{\mu - \lambda} = \dfrac{3}{4}$ hour.

63(e) P (wait more than one hour) $= \dfrac{\lambda}{\mu} e^{-(\mu - \lambda)} = 0.276$

■ **64** Mean number of aircraft on ground is

$$N_S = \frac{\lambda}{\mu - \lambda}$$

so, total mean cost per hour (waiting time plus servicing) is

$$E\,[\text{total cost per hour}] = \frac{c_1 \lambda}{\mu - \lambda} + c_2 \mu$$

Minimizing this with respect to μ:

$$\frac{d}{d\mu}E\,[\text{total cost per hour}] = -\frac{c_1\lambda}{(\mu-\lambda)^2} + c_2 = 0$$

from which

$$(\mu-\lambda)^2 = \frac{c_1\lambda}{c_2}$$

so $\mu = \lambda + \sqrt{c_1\lambda/c_2}$

■ **65** Breakdown rate is $\lambda = 3$ per hour, and machine idle time is costed at £ 60 per hour per machine. For option A, service rate is $\mu = 4$ per hour at £ 20 per hour, so mean hourly cost is

$$60N_S + 20 = \frac{60\lambda}{\mu-\lambda} + 20 = 200$$

For option B, service rate is $\mu = 5$ at £ 40 per hour, so mean hourly cost is

$$60N_S + 40 = \frac{60\lambda}{\mu-\lambda} + 40 = 130$$

Option B is preferred.

■ **66** Ship arrival rate is $\lambda = \frac{1}{3}$ per hour, and service rate per berth is $\mu = \frac{1}{12}$, so $\rho = \lambda/\mu = 4$. Mean waiting time in the queue is

$$W_Q = \frac{1}{\lambda}\left[\frac{\rho^{c+1}}{(c-1)!(c-\rho)^2}\right]\left[\sum_{n=0}^{c-1}\frac{\rho^n}{n!} + \frac{\rho^c}{(c-1)!(c-\rho)}\right]^{-1}$$

where c is the number of berths. For $c = 5$ berths we find $W_Q = 6.65$ hours, which exceeds the required minimum, so $c = 6$ berths are needed $(W_Q = 1.71)$.

■ **67** Arrival rate is $\lambda = 2$ per minute, and basic service rate per cashier is $\mu = \frac{5}{4}$ per minute. If this service rate is doubled (by providing a packer) then mean queueing time is

$$W_Q = \frac{\rho}{\mu-\lambda} = \frac{4/5}{5/2-2} = 1.6 \text{ min}$$

Alternatively, if an additional cash desk is provided then (using W_Q as in the previous exercise, and $\rho = 8/5$)

$$W_Q = \frac{1}{\lambda}\left[\frac{\rho^3}{(2-\rho)^2}\right]\left[1+\rho+\frac{\rho^2}{2-\rho}\right]^{-1} = 0.076 \text{ min}$$

Clearly, a second cash desk is preferable.

Exercises 11.11.3

■ **68** We have

$$\begin{aligned} P(\text{agent}) &= P(\text{agent}|\text{option 1})P(\text{option 1}) + P(\text{agent}|\text{option 2})P(\text{option 2}) \\ &\quad + P(\text{agent}|\text{option 3})P(\text{option 3}) \\ &= 0.28 \times 0.45 + 0.41 \times 0.32 + 0.16 \times 0.23 \\ &= 0.294 \end{aligned}$$

■ **69** Total probability of explosion is

$$\begin{aligned} P(E) &= P(E \mid (a))P((a)) + P(E \mid (b))P((b)) \\ &\quad + P(E \mid (c))P((c)) + P(E \mid (d))P((d)) \\ &= 0.25 \times 0.2 + 0.2 \times 0.4 + 0.4 \times 0.25 + 0.75 \times 0.15 \\ &= 0.3425 \end{aligned}$$

Hence, by Bayes' Theorem,

$$\begin{aligned} P((a) \mid E) &= P(E \mid (a))P((a))/P(E) = 0.146 \\ P((b) \mid E) &= 0.234 \\ P((c) \mid E) &= 0.292 \\ P((d) \mid E) &= 0.328 \end{aligned}$$

and sabotage is therefore the most likely cause of the explosion.

■ **70** If two bullets hit the target then they could be fired by each possible pair of marksmen, so

$$P\,(\text{two hits}) = P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C)$$

(where A denotes 'bullet from A hits target', etc)

$$= 0.6 \times 0.5 \times 0.6 + 0.6 \times 0.5 \times 0.4 + 0.4 \times 0.5 \times 0.4$$
$$= 0.38$$

Hence

$$\begin{aligned} P(C \mid \text{ two hits}) &= \frac{P(C \cap \text{ two hits})}{P\,(\text{two hits})} \\ &= \frac{P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C)}{P\,(\text{ two hits})} \\ &= \frac{0.6 \times 0.5 \times 0.4 + 0.4 \times 0.5 \times 0.4}{0.38} \\ &= 0.526 \end{aligned}$$

Thus, it is more probable than not that C hit the target.

■ **71** Prior probabilities are $P(A) = \frac{1}{3},\ P(B) = \frac{2}{3}$. Also $P(\text{Smith} \mid A) = 0.1$ and $P(\text{Smith} \mid B) = 0.05$. Hence

$$\begin{aligned} P(A \mid \text{Smith}) &= \frac{P\,(\text{Smith} \mid A)P(A)}{P\,(\text{Smith} \mid A)P(A) + P\,(\text{Smith} \mid B)P(B)} \\ &= \frac{0.1 \times \frac{1}{3}}{0.1 \times \frac{1}{3} + 0.05 \times \frac{2}{3}} \\ &= \frac{1}{2} \end{aligned}$$

■ **72** Let D denote 'has disease' and $+$ denote 'positive diagnosis', so that $P(D) = 0.08, P(+ \mid D) = 0.95$ and $P(+ \mid \bar{D}) = 0.02$

72(a) $P(+) = P(+ \mid D)P(D) + P(+ \mid \bar{D})P(\bar{D})$
$= 0.95 \times 0.08 + 0.02 \times 0.92 = 0.0944$

72(b) $$P(D \mid +) = \frac{P(+ \mid D)P(D)}{P(+)} = \frac{0.95 \times 0.08}{0.0944} = 0.81$$

■ **73** Let G denote 'good stock', $B = \bar{G}$ denote 'bad stock',
g denote 'stockbroker says good',
b denote 'stockbroker says bad',
so that $P(G) = 0.5,\ P(g \mid G) = 0.6,\ P(b \mid B) = 0.8$.

73(a)

$$\begin{aligned} P(G \mid g) &= \frac{P(g \mid G)P(G)}{P(g \mid G)P(G) + P(g \mid B)P(B)} \\ &= \frac{0.6 \times 0.5}{0.6 \times 0.5 + 0.2 \times 0.5} = \frac{3}{4} \end{aligned}$$

73(b) Let E denote 'k out of n stockbrokers say good'. Since the stockbrokers are independent, by the binomial distribution

$$P(E \mid G) = \binom{n}{k}[P(g \mid G)]^k[P(b \mid G)]^{n-k}$$
$$P(E \mid B) = \binom{n}{k}[P(g \mid B)]^k[P(b \mid B)]^{n-k}$$

Hence

$$\begin{aligned} P(G \mid E) &= \frac{P(E \mid G)P(G)}{P(E \mid G)P(G) + P(E \mid B)P(B)} \\ &= \frac{\binom{n}{k}[P(g \mid G)]^k[P(b \mid G)]^{n-k}P(G)}{\binom{n}{k}[P(g \mid G)]^k[P(b \mid G)]^{n-k}P(G) + \binom{n}{k}[P(g \mid B)]^k[P(b \mid B)]^{n-k}P(B)} \\ &= \frac{0.6^k \times 0.4^{n-k} \times 0.5}{0.6^k \times 0.4^{n-k} \times 0.5 + 0.2^k \times 0.8^{n-k} \times 0.5} \\ &= \left[1 + \left(\frac{1}{3}\right)^k 2^{n-k}\right]^{-1} \end{aligned}$$

■ **74** Given that the probability of correct reception of a letter is 0.6, and the error probabilities are 0.2 for the two alternatives, we have

$$P(ABCA \text{ received } \mid AAAA \text{ transmitted}) = 0.6 \times 0.2 \times 0.2 \times 0.6 = 0.0144$$
$$P(ABCA \text{ received } \mid BBBB \text{ transmitted}) = 0.2 \times 0.6 \times 0.2 \times 0.2 = 0.0048$$
$$P(ABCA \text{ received } \mid CCCC \text{ transmitted}) = 0.2 \times 0.2 \times 0.6 \times 0.2 = 0.0048$$

Also $P(AAAA \text{ transmitted}) = 0.3$ etc. Hence

$$\begin{aligned} P(ABCA \text{ received}) &= 0.0144 \times 0.3 + 0.0048 \times (0.4 + 0.3) \\ &= 0.00768 \end{aligned}$$

and

$$P(AAAA \text{ transmitted } \mid ABCA \text{ received}) = \frac{0.0144 \times 0.3}{0.00768} = 0.5625$$
$$P(BBBB \text{ transmitted } \mid ABCA \text{ received}) = 0.25$$
$$P(CCCC \text{ transmitted } \mid ABCA \text{ received}) = 0.1875$$

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75 Average number of accidents per day $= \frac{1\times12+2\times4}{100} = \frac{1}{5}$
First hypothesis (H_1) is for a Poisson distribution, so set $\lambda = \frac{1}{5}$ and probabilities

$$p_i = P(i \text{ accidents in one day}) = \frac{\lambda^i e^{-\lambda}}{i!}$$

Hence, $p_0 = 0.8187,\ p_1 = 0.1637,\ p_2 = 0.0164$. Second hypothesis (H_2) is for a binomial distribution with $n = 3$, so set

$$np = \frac{1}{5} \text{ (hence } p = \frac{1}{15}\text{) and probabilities}$$
$$q_i = P(i \text{ accidents in one day}) = \binom{3}{i} p^i (1-p)^{3-i}$$

Hence, $q_0 = 0.8130,\ q_1 = 0.1742,\ q_2 = 0.0124$. If E denotes the evidence then the odds are updated by

$$\ln \frac{P(H_1 \mid E)}{P(H_2 \mid E)} = \ln \frac{P(E \mid H_1)}{P(E \mid H_2)} + \ln \frac{P(H_1)}{P(H_2)}$$

where

$$P(E \mid H_1) = p_0^{84} p_1^{12} p_2^4$$
$$P(E \mid H_2) = q_0^{84} q_1^{12} q_2^4$$
$$P(H_1)/P(H_2) = \frac{1}{2} \text{ (initial odds)}$$

Hence

$$\ln \frac{P(H_1 \mid E)}{P(H_2 \mid E)} = 84 \ln \frac{p_0}{q_0} + 12 \ln \frac{p_1}{q_1} + 4 \ln \frac{p_2}{q_2} + \ln \frac{1}{2} = 0.247$$

and the updated odds are therefore 1.28 to 1 in favour of the Poisson distribution.

■ **76** If the probabilities of the evidence (E) under H_1 and H_2 are

$$P(E \mid H_1) = \frac{n!}{n_1! \ldots n_k!} p_1^{n_1} \ldots p_k^{n_k}$$
$$P(E \mid H_2) = \frac{n!}{n_1! \ldots n_k!} q_1^{n_1} \ldots q_k^{n_k}$$

then the log-likelihood ratio becomes

$$\ln \frac{P(E \mid H_1)}{P(E \mid H_2)} = \ln \left[\left(\frac{p_1}{q_1} \right)^{n_1} \cdots \left(\frac{p_k}{q_k} \right)^{n_k} \right] = \sum_{i=1}^{k} n_i \ln \frac{p_i}{q_i}$$

■ **77** Under hypothesis H_1 we have

$$p_1 = 0.92,\ p_2 = 0.05,\ p_3 = 0.02,\ p_4 = 0.01$$

and under H_2

$$q_1 = 1 - 0.05 - q_3 - q_4,\ q_2 = 0.05,\ q_3,\ q_4 \text{ unknown}$$

(where $q_3 = p_B$ and $q_4 = P_{AB}$). The likelihood of the evidence E under H_2 is

$$P(E \mid H_2) = \frac{n!}{n_1! \ldots n_4!} (0.95 - q_3 - q_4)^{n_1} 0.05^{n_2} q_3^{n_3} q_4^{n_4}$$

where $n_1 = 912,\ n_2 = 45,\ n_3 = 27,\ n_4 = 16$. Thus

$$\begin{aligned}
\ln P(E \mid H_2) &= n_1 \ln(0.95 - q_3 - q_4) + n_3 \ln q_3 + n_4 \ln q_4 + \text{ constant} \\
\frac{\partial \ln P(E \mid H_2)}{\partial q_3} &= -\frac{n_1}{0.95 - q_3 - q_4} + \frac{n_3}{q_3} = \frac{n_3(0.95 - q_3 - q_4) - n_1 q_3}{(0.95 - q_3 - q_4)q_3} \\
&= 0 \text{ if } (n_1 + n_3)q_3 + n_3 q_4 = 0.95 n_3 \\
\frac{\partial \ln P(E \mid H_2)}{\partial q_4} &= -\frac{n_1}{0.95 - q_3 - q_4} + \frac{n_4}{q_4} = \frac{n_4(0.95 - q_3 - q_4) - n_1 q_4}{(0.95 - q_3 - q_4)q_4} \\
&= 0 \text{ if } n_4 q_3 + (n_1 + n_4)q_4 = 0.95 n_4
\end{aligned}$$

From the simultaneous equations

$$\begin{aligned}
939q_3 + 27q_4 &= 25.65 \\
16q_3 + 928q_4 &= 15.2
\end{aligned}$$

we find $q_3 = 0.0269,\ q_4 = 0.0159$ and therefore $q_1 = 0.9072$.
It follows that (using the result of the previous exercise)

$$\begin{aligned}
\ln \frac{P(E \mid H_1)}{P(E \mid H_2)} &= \sum_{i=1}^{4} n_i \ln \frac{p_i}{q_i} \\
&= 912 \ln \frac{0.92}{0.9072} + 45 \ln \frac{0.05}{0.05} + 27 \ln \frac{0.02}{0.0269} + 16 \ln \frac{0.01}{0.0159} \\
&= -2.645
\end{aligned}$$

If initial odds are $P(H_1)/P(H_2) = 5$ then updated odds are

$$\frac{P(H_1 \mid E)}{P(H_2 \mid E)} = e^{-2.645} \times 5 = 0.355$$

that is, 2.8 to 1 in favour of H_2.

■ **78** Under hypothesis H_1 we have separate estimates as follows:

$$\begin{aligned}
\lambda_A &= \text{ mean number of defects for } A = \frac{24}{6} = 4.0 \\
\lambda_B &= \text{ mean number of defects for } B = \frac{36}{5} = 7.2
\end{aligned}$$

Under hypothesis H_2 we have a single estimate as follows:

$$\lambda = \text{ overall mean number of defects } = \frac{60}{11} = 5.455$$

The evidence E can be expressed as follows:

$$E = \{A : 2 \text{ with 3 defects, 2 with 4 defects, 2 with 5 defects;}$$
$$B : 1 \text{ with 5 defects, 1 with 6 defects, 2 with 8 defects, 1 with 9 defects}\}$$

The log-likelihood ratio is

$$\begin{aligned}\ln\frac{P(E \mid H_1)}{P(E \mid H_2)} &= 2\ln\frac{\lambda_A^3 e^{-\lambda_A}}{\lambda^3 e^{-\lambda}} + 2\ln\frac{\lambda_A^4 e^{-\lambda_A}}{\lambda^4 e^{-\lambda}} + 2\ln\frac{\lambda_A^5 e^{-\lambda_A}}{\lambda^5 e^{-\lambda}} \\ &\quad + \ln\frac{\lambda_B^5 e^{-\lambda_B}}{\lambda^5 e^{-\lambda}} + \ln\frac{\lambda_B^6 e^{-\lambda_B}}{\lambda^6 e^{-\lambda}} + 2\ln\frac{\lambda_B^8 e^{-\lambda_B}}{\lambda^8 e^{-\lambda}} + \ln\frac{\lambda_B^9 e^{-\lambda_B}}{\lambda^9 e^{-\lambda}} \\ &= (6+8+10)\ln\frac{\lambda_A}{\lambda} + (5+6+16+9)\ln\frac{\lambda_B}{\lambda} \\ &\quad + (2+2+2)(\lambda-\lambda_A) + (1+1+2+1)(\lambda-\lambda_B) \\ &= 2.551\end{aligned}$$

and the updated odds (with no initial preference) are

$$\frac{P(H_1 \mid E)}{P(H_2 \mid E)} = e^{2.551}$$

or 12.8 to 1 in favour of H_1.

Review Exercises 11.12

■ **1** For the standard corks, sample proportion oxidized is $\hat{p}_1 = \frac{6}{60} = 0.1$, whereas for the plastic bungs, sample proportion oxidized is $\hat{p}_2 = \frac{3}{36} = 0.0833$. Overall proportion oxidized is $\hat{p} = \frac{9}{96} = 0.0938$. Test statistic

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})(\frac{1}{60}) + (\frac{1}{36})}} = 0.271$$

The hypothesis is accepted.

■ **2** The model is

$$d = d_0 e^{-\lambda t}$$

or equivalently

$$\ln\, d = \ln\, d_0 - \lambda t$$

which is of the form

$$Y = a + bX$$

From the data,

$$\bar{X} = 5.746,\ S_X = 3.036,\ \bar{Y} = -0.2811,\ S_Y = 0.721,\ \overline{XY} = -3.775$$

so

$$\hat{b} = \frac{\overline{XY} - \bar{X}\bar{Y}}{S_X^2} = -0.234$$
$$\hat{a} = \bar{Y} - \hat{b}\bar{X} = 1.065$$

From these we infer

$$\hat{\lambda} = -\hat{b} = 0.234$$
$$\hat{d}_0 = e^{\hat{a}} = 2.90$$

Also the error variance is (using unrounded results)

$$S_E^2 = S_Y^2 - \hat{b}^2 S_X^2 = 0.01418$$

and the 95% confidence interval for λ is

$$\hat{\lambda} \pm t_{.025,8} \frac{S_E}{S_X\sqrt{8}} = (0.202, 0.266)$$

■ **3** If position P and load X are related by

$$P = a + bX$$

then by linear regression on the data we find

$$\hat{a} = 6.129$$
$$\hat{b} = 0.0624$$

But extension Y and load X are related by

$$\hat{b} = \frac{Y}{X} = \frac{L}{\hat{E}A}$$

where $L = 101.4$ and $A = 1.62 \times 10^{-5}$. The estimate of Young's modulus is therefore

$$\hat{E} = \frac{L}{\hat{b}A} = 1.003 \times 10^8$$

Also from the linear regression the error standard deviation is $S_E = 0.00624$, so the 95% confidence interval for b is

$$\hat{b} \pm t_{.025,6} \frac{S_E}{S_X \sqrt{6}} = (0.0597, 0.0651)$$

We infer the 95% confidence interval for $E = L/bA$ as

$$(96.1 \times 10^6, 104.9 \times 10^6)$$

■ **4** From the data, the mean time between arrivals is estimated as 9.422 hours, and this is $1/\lambda$ for the exponential distribution. If we form a histogram of the data, the expected probability of a class (a, b) under the exponential distribution with parameter λ is

$$P(a < X < b) = F_X(b) - F_X(a) = 1 - e^{-\lambda b} - (1 - e^{-\lambda a}) = e^{-\lambda a} - e^{-\lambda b}$$

Using class intervals of five hours we obtain the table as follows:

Class (k)	**Observations (f_k)**	**Probability**	**Expected (e_k)**
0-5	48	0.4118	43.24
5-10	22	0.2422	25.43
10-15	13	0.1425	14.96
15-20	12	0.0838	8.80
20-25	3	0.0493	5.18
25-30	3	0.0290	3.04
>30	4	0.0414	4.35

The value of $\chi^2 = 3.35$ is less than $\chi^2_{.05,5} = 11.07$ (seven classes with one parameter estimated) so the fit to the exponential distribution is good.

■ **5** The maximum value is $X_{\max} = 72$ and the total is $\sum_i X_i = 989.3$, so

$$y = \frac{X_{\max}}{\Sigma_i X_i} = 0.0728$$

and $[1/y] = 13$. Hence

$$P(Y \leq 0.0728) = \sum_{k=0}^{13} (-1)^k \binom{105}{k} (1 - 0.0728k)^{104}$$
$$= 0.9599$$

The probability of there occurring such a large value is therefore around 4%, so the value 72 can be regarded as an outlier at the 5% significance level.
With the outlier included we have $\bar{X} = 9.422,\ S_X = 10.77$ so the 95% confidence interval for mean inter-arrival time is

$$\bar{X} \pm 1.96 \frac{S_X}{\sqrt{105}} = (7.36, 11.48)$$

With the outlier excluded we have $\bar{X} = 8.820,\ S_X = 8.90$ so the confidence interval is (7.11, 10.53).

■ **6** The contingency table (with expected values in brackets and adjusted residuals underneath) is as follows:

Grade	**French**	**German**	**Spanish**	**Total**
Very satisfied	16 (15)	6 (7)	22 (22)	44
	0.5	−0.5	−0.1	
Fairly satisfied	63 (50)	13 (24)	76 (77)	152
	2.8	−3.2	−0.2	
Neutral	40 (42)	27 (20)	60 (64)	127
	−0.5	1.9	−1.0	
Fairly dissatisfied	10 (18)	13 (9)	32 (28)	55
	−2.5	1.6	1.2	
Very dissatisfied	3 (7)	5 (3)	12 (10)	20
	−1.8		0.8	
Total	132	64	202	398

Chi-square = 20.0, d.f. $= (5-1)(3-1) = 8$, and so compare with $\chi^2_{0.025,8} = 17.54$: significant at 2.5% level. The French course scores highest, followed by Spanish and then German.

■ **7** Let D denote 'has disease' and O denote 'operation performed', then

$$P(\text{survive} \mid D \cap O) = \frac{1}{2}$$
$$P(\text{survive} \mid D \cap \bar{O}) = \frac{1}{20}$$
$$P(\text{survive} \mid \bar{D} \cap O) = \frac{4}{5}$$

and we can assume that $P(\text{survive} \mid \bar{D} \cap \bar{O}) = 1$. If the operation is performed, then using the hint we have

$$\begin{aligned} P(\text{survive} \mid O) &= P(\text{survive} \mid D \cap O)P(D) + P(\text{survive} \mid \bar{D} \cap O)P(\bar{D}) \\ &= \frac{1}{2}p + \frac{4}{5}(1-p) = \frac{4}{5} - \frac{3}{10}p \end{aligned}$$

(where $p = P(D)$). If the operation is not performed,

$$\begin{aligned} P(\text{survive} \mid \bar{O}) &= P(\text{survive} \mid D \cap \bar{O})P(D) + P(\text{survive} \mid \bar{D} \cap \bar{O})P(\bar{D}) \\ &= \frac{1}{20}p + (1-p) = 1 - \frac{19}{20}p \end{aligned}$$

These probabilities are equal when

$$\frac{4}{5} - \frac{3}{10}p = 1 - \frac{19}{20}p$$

from which $p = \frac{4}{13}$. The surgeon will operate if the assessment of $P(D)$ exceeds this value.

■ **8** With 200 machines each becoming misaligned every 200 hours on average, the rate at which machines become misaligned is $\lambda_0 = \frac{200}{200} =$ one per hour on average. The total cost per hour for each option is the sum of three components: the fixed cost per hour, the cost of correcting the output and the cost of lost production.
For option A, the fixed cost is £1 per hour per machine, hence £200 per hour. The average run length ARL_0 for a misaligned machine is 20 hours, and this amount of output must be corrected, so the cost per hour of correcting the output is $\lambda_0 \times ARL_0 \times 10 =$ £200. Lost production occurs while a machine is in the queue and being serviced, and this occurs whether the machine is actually misaligned or

not (false alarm). Actual misalignments occur at the rate λ_0 and are detected by the control chart. False alarms occur at the rate

$$\lambda_1 = \frac{200}{ARL_1} = 0.2 \text{ per hour}$$

where $ARL_1 = 1000$ is the mean time between false alarms for a well-adjusted machine. These two kinds of action are independent, so the total rate of actions is $\lambda = \lambda_0 + \lambda_1 = 1.2$. Also, the service rate $\mu = 2$ per hour so $\rho = \lambda/\mu = 0.6$. With $\sigma_s = \frac{1}{4}$, the mean number of machines out of production is

$$N_S = \rho + \frac{(\lambda\sigma_s)^2 + \rho^2}{2(1-\rho)} = 1.163$$

and the cost per hour of lost production is $200N_S = £\ 232.5$. The total cost for option A is therefore $£\ 200 + £\ 200 + £\ 232.5 = £\ 632.5$ per hour.

For option B, the fixed cost is $£\ 1.50$ per hour per machine, hence $£\ 300$ per hour. With $ARL_0 = 4$, the cost per hour of correcting the output is $\lambda_0 \times ARL_0 \times 10 = £\ 40$. With $ARL_1 = 750$, false alarms occur at the rate $\lambda_1 = 200/750 = 0.267$ per hour, so machines are taken out of production at the total rate $\lambda = \lambda_0 + \lambda_1 = 1.267$, hence $\rho = \lambda/\mu = 0.633$. The mean number of machines out of production is therefore $N_S = 1.317$ at a cost per hour $200N_S = £\ 263.4$. The total cost for option B is therefore $£\ 300 + £\ 40 + £\ 263.4 = £\ 603.4$ per hour. This is less than for option A.

■ **9** For the source, $P(\text{in} = 0) = \alpha$ and $P(\text{in} = 1) = 1 - \alpha$. For the channel, $P(\text{out} = 0 \mid \text{in} = 1) = P(\text{out} = 1 \mid \text{in} = 0) = p$.

9(a)

$$\begin{aligned} P(\text{out } = 0) &= P(\text{out } = 0 \mid \text{ in } = 0)P(\text{in } = 0) + P(\text{out } = 0 \mid \text{ in } = 1)P(\text{in } = 1) \\ &= (1-p)\alpha + p(1-\alpha) = \bar{p}\alpha + p\bar{\alpha} \end{aligned}$$

(where $\bar{p} = 1 - p,\ \bar{\alpha} = 1 - \alpha$). Hence

$$P(\text{in } = 0 \mid \text{ out } = 0) = \frac{P(\text{out } = 0 \mid \text{ in } = 0)P(\text{in } = 0)}{P(\text{out } = 0)} = \frac{\bar{p}\alpha}{\bar{p}\alpha + p\bar{\alpha}}$$

$$P(\text{in } = 1 \mid \text{ out } = 0) = \frac{p\bar{\alpha}}{\bar{p}\alpha + p\bar{\alpha}}$$

$$P(\text{in } = 0 \mid \text{ out } = 1) = \frac{p\alpha}{p\alpha + \bar{p}\bar{\alpha}}$$

$$P(\text{in } = 1 \mid \text{ out } = 1) = \frac{\bar{p}\bar{\alpha}}{p\alpha + \bar{p}\bar{\alpha}}$$

9(b) $P(\text{in } = 0 \mid \text{ out } = 0) > P(\text{in } = 1 \mid \text{ out } = 0)$ if

$$\bar{p}\alpha > p\bar{\alpha}$$

from which

$$\bar{p}\alpha > p(1 - \alpha)$$

hence

$$(\bar{p} + p)\alpha = \alpha > p$$

Similarly, $P(\text{in } = 1 \mid \text{ out } = 1) > P(\text{in } = 0 \mid \text{ out } = 1)$ if

$$\bar{p}\bar{\alpha} > p\alpha$$

from which

$$\bar{p}(1 - \alpha) > p\alpha$$

hence

$$(p + \bar{p})\alpha = \alpha < \bar{p}$$

The source symbol is assumed to be the same as the received symbol if $p < \alpha < \bar{p}$.

■ **10** For the binary symmetric channel, $X = \{0, 1\}$ with $P(X = 0) = \alpha$, and $Y = \{0, 1\}$ with $P(Y = 0) = \bar{p}\alpha + p\bar{\alpha},\ P(Y = 1) = p\alpha + \bar{p}\bar{\alpha} (\bar{p} = 1 - p,\ \bar{\alpha} = 1 - \alpha$, using the results of the previous exercise). Also

$$P(X = 0 \cap Y = 0) = P(Y = 0 \mid X = 0)P(X = 0) = \bar{p}\alpha$$

$$P(X = 0 \cap Y = 1) = P(Y = 1 \mid X = 0)P(X = 0) = p\alpha$$

$$P(X=1\cap Y=0)=P(Y=0\mid X=1)P(X=1)=p\bar{\alpha}$$
$$P(X=1\cap Y=1)=P(Y=1\mid X=1)P(X=1)=\bar{p}\bar{\alpha}$$

The mutual information between X and Y is as follows:

$$\begin{aligned}I(X;Y)&=\sum_{x=0}^{1}\sum_{y=0}^{1}P(x,y)\log_2\frac{P(x,y)}{P(X=x)P(Y=y)}\\&=\bar{p}\alpha\log_2\frac{\bar{p}\alpha}{\alpha(\bar{p}\alpha+p\bar{\alpha})}+p\alpha\log_2\frac{p\alpha}{\alpha(p\alpha+\bar{p}\bar{\alpha})}\\&+p\bar{\alpha}\log_2\frac{p\bar{\alpha}}{\bar{\alpha}(\bar{p}\alpha+p\bar{\alpha})}+\bar{p}\bar{\alpha}\log_2\frac{\bar{p}\bar{\alpha}}{\bar{\alpha}(p\alpha+\bar{p}\bar{\alpha})}\\&=\bar{p}(\alpha+\bar{\alpha})\log_2\bar{p}+p(\alpha+\bar{\alpha})\log_2 p\\&-(\bar{p}\alpha+p\bar{\alpha})\log(\bar{p}\alpha+p\bar{\alpha})-(p\alpha+\bar{p}\bar{\alpha})\log(p\alpha+\bar{p}\bar{\alpha})\\&=H(p)-H(\bar{p}\alpha+p\bar{\alpha})\end{aligned}$$

where $H(t)=t\log_2 t+(1-t)\log_2(1-t)$ is called the 'entropy function'. In particular, when $\alpha=\frac{1}{2}$ we have

$$\bar{p}\alpha+p\bar{\alpha}=\frac{1}{2}(\bar{p}+p)=\frac{1}{2}$$

and $H(\frac{1}{2})=\frac{1}{2}\log_2\frac{1}{2}+\frac{1}{2}\log_2\frac{1}{2}=-1$
so that

$$I(X;Y)=1+H(p)=1+p\log_2 p+(1-p)\log_2(1-p)$$

When $p=\frac{1}{2},\ I(X;Y)=1+\frac{1}{2}\log_2\frac{1}{2}+\frac{1}{2}\log_2\frac{1}{2}=0.$
When $p\to 0,\ p\log_2 p\to 0$ and $\bar{p}\log_2\bar{p}\to 0$ so that

$$I(X;Y)\to 1$$

and similarly when $p\to 1$. Full information is transmitted through the channel when either every bit is correct $(p=0)$ or every bit is inverted $(p=1)$. No information is transmitted when the bits are uniformly randomized $(p=\frac{1}{2})$.
